

# Shattered matchings in intersecting hypergraphs

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## Abstract

Let  $X$  be an  $n$ -element set, where  $n$  is even. We refute a conjecture of J. Gordon and Y. Teplitskaya, according to which, for every maximal intersecting family  $\mathcal{F}$  of  $\frac{n}{2}$ -element subsets of  $X$ , one can partition  $X$  into  $\frac{n}{2}$  disjoint pairs in such a way that no matter how we pick one element from each of the first  $\frac{n}{2} - 1$  pairs, the set formed by them can always be completed to a member of  $\mathcal{F}$  by adding an element of the last pair.

The above problem is related to classical questions in extremal set theory. For any  $t \geq 2$ , we call a family of sets  $\mathcal{F} \subset 2^X$  *t-separable* if for any ordered pair of elements  $(x, y)$  of  $X$ , there exists  $F \in \mathcal{F}$  such that  $F \cap \{x, y\} = \{x\}$ . For a fixed  $t$ ,  $2 \leq t \leq 5$  and  $n \rightarrow \infty$ , we establish asymptotically tight estimates for the smallest integer  $s = s(n, t)$  such that every family  $\mathcal{F}$  with  $|\mathcal{F}| \geq s$  is *t-separable*.

## 1 Introduction

Given an  $n$ -element set  $X$ , a family  $\mathcal{F} \subset 2^X$  is called *intersecting* if any two members of  $\mathcal{F}$  have nonempty intersection. In their seminal work [6], P. Erdős, C. Ko, and R. Rado determined the maximum size of an intersecting family  $\mathcal{F}$  of  $k$ -element subsets of  $X$ , for all  $k \leq \frac{n}{2}$ . In particular, if  $n$  is even and  $k = \frac{n}{2}$ , they proved that  $|\mathcal{F}| \leq \frac{1}{2} \binom{n}{\frac{n}{2}}$ , where equality holds for every *maximal* (that is, non-extendable) intersecting family of  $\frac{n}{2}$ -element subsets of  $X$ .

Motivated by a problem from mathematical finance, J. Gordon and Y. Teplitskaya [14] made the following conjectures:

**Conjecture A.** If  $n$  is *even*, then for any *maximal intersecting* family  $\mathcal{F}$  of  $\frac{n}{2}$ -element subsets of  $X$ , there exists a *perfect matching*  $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{n-1}, x_n\}$  with the property that no matter how we select one element from each of the first  $\frac{n}{2} - 1$  pairs, together with  $x_{n-1}$  or  $x_n$ , they always form a member of  $\mathcal{F}$ .

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**Conjecture B.** If  $n$  is odd, then for any *maximal intersecting* family  $\mathcal{F}$  consisting of  $\frac{n-1}{2}$ -element and  $\frac{n+1}{2}$ -element subsets of  $X$ , there exists a matching  $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{n-2}, x_{n-1}\}$  with the property that no matter how we select one element from each pair, together with the last element  $x_n \in X$ , they always form a member of  $\mathcal{F}$ .

Gordon and Teplitskaya verified these conjectures for  $n \leq 6$ .

In this note, we disprove the above conjectures for all  $n \geq 14$ . For  $n$  even, we will establish a more general result which contradicts Conjecture A in a strong way. For odd  $n$ , the problem will be settled using the even case. To formulate our first result, we need to agree on some terminology.

An unordered collection  $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{2k-1}, x_{2k}\}$  of pairwise disjoint 2-element subsets of  $X$  is called a *matching* of size  $k$  ( $2k \leq n = |X|$ ). If we pick one element from each pair, the  $k$ -element set formed by them is called a *snake* with respect to this matching. A family of subsets of  $X$  is said to be *intersecting* if any two of its members have nonempty intersection.

**Definition 1** A matching  $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{2k-1}, x_{2k}\} \subset X$  is said to be *shattered* by a family  $\mathcal{F} \subset 2^X$  if for every snake  $S$  with respect to this matching, there exists  $F \in \mathcal{F}$  such that  $F \cap \{x_1, x_2, \dots, x_{2k}\} = S$ .

Our main result is the following.

**Theorem 2** Let  $X$  be an  $n$ -element set, where  $n \geq 28$  is even. Let  $k(n)$  denote the largest integer  $k$  such that for every maximal intersecting family of  $\frac{n}{2}$ -element subsets of  $X$ , there exists a shattered matching of size  $k$ . Then we have

$$\frac{n}{4} \leq k(n) \leq \frac{n}{2} - \frac{1}{2} \log_2 n + 1.$$

The fact that the upper bound is smaller than  $\frac{n}{2} - 1$  shows that the Gordon-Teplitskaya conjecture is not true if  $n$  is large enough.

The above question is closely related to a classical result from extremal set theory [8, 12, 13]. A family  $\mathcal{F}$  of subsets of  $X$  is said to *shatter* a set  $A$  if for every subset  $B \subset A$ , there is  $F \in \mathcal{F}$  with  $F \cap A = B$ . It was shown by Vapnik and Chervonenkis and a little later, independently, by Sauer and Shelah that if  $\mathcal{F}$  is large, then there is a large subset  $A \subset X$  shattered by  $\mathcal{F}$ . The size of the largest shattered subset of  $X$  is called the *Vapnik-Chervonenkis dimension* of  $\mathcal{F}$ , and is denoted by  $\text{VC-dim}(\mathcal{F})$ . This notion plays a central role in statistics, learning theory, discrete and computational geometry, and elsewhere.

More precisely, the following is true.

**Theorem 3** (Sauer [19], Shelah [20], Vapnik-Chervonenkis [23]) Let  $|X| = n$  and let  $\mathcal{F}$  be a family of subsets of  $X$  with  $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$ .

Then there is a  $k$ -element set  $A \subset X$  shattered by  $\mathcal{F}$ , i.e.,  $\text{VC-dim}(\mathcal{F}) \geq k$ . This bound is tight.

Obviously, if  $A$  is shattered and  $|A|$  is even, then any perfect matching of  $A$  (that is, any partition of  $A$  into 2-element subsets) is a shattered matching. Using this idea, one can easily obtain the lower bound  $k(n) \geq (\frac{1}{4} - o(1))n$ , which is only slightly weaker than the bound in Theorem 2. However, this proof only uses that  $\mathcal{F}$  has many members, without taking into account the assumption that  $\mathcal{F}$  is a maximal intersecting family.

In the spirit of Theorem 3, we can ask how large  $\mathcal{F}$  must be in order to guarantee the existence of a shattered matching of size  $k$ .

**Problem 4** *Let  $n, k$  be positive integers,  $n \geq 2k$ . Determine or estimate the smallest number  $p = p(n, k)$  such that for every family  $\mathcal{F}$  of at least  $p$  subsets of an  $n$ -element set, there is a matching of size  $k$  shattered by  $\mathcal{F}$ .*

Obviously, if  $\mathcal{F}$  shatters a  $2k$ -element set  $A$ , then any partition of  $A$  into 2-element sets is a shattered matching of size  $k$ . Thus, Theorem 3 immediately implies that

$$p(n, k) \leq 1 + \sum_{i=0}^{2k-1} \binom{n}{i} \quad \text{for every } k \geq 1.$$

This bound is tight for  $k = 1$ , and we will see that its order of magnitude is best possible for any  $k$ , as  $n \rightarrow \infty$ . See Corollary 11.

**Definition 5** *A family  $\mathcal{F} \subset 2^X$  is said to be  $t$ -separable if there is a  $t$ -element subset  $T \subset X$  such that for every ordered pair  $x, y \in T$ ,  $x \neq y$ , there exists  $F \in \mathcal{F}$  such that  $F \cap \{x, y\} = \{x\}$ .*

*For any  $n \geq t \geq 2$ , let  $s(n, t)$  denote the smallest number  $s$  with the property that every family  $\mathcal{F}$  of at least  $s$  subsets of an  $n$ -element set is  $t$ -separable.*

If  $\{x_1, x_2\}, \dots, \{x_{2k-1}, x_{2k}\}$  is a matching of size  $k$  shattered by  $\mathcal{F}$ , then  $\mathcal{F}$  is  $2k$ -separable, as the set  $T = \{x_1, \dots, x_{2k}\}$  satisfies the above requirements. Therefore, we have

$$p(n, k) \geq s(n, 2k).$$

The problem of determining or estimating  $s(n, t)$  appears to be a nontrivial task of independent interest. We prove the following.

**Theorem 6** *Let  $n \geq t \geq 2$ , and let  $X$  be an  $n$ -element set. Let  $s(n, t)$  denote the smallest number  $s$  with the property that every family  $\mathcal{F} \subset 2^X$  with  $|\mathcal{F}| \geq s$  is  $t$ -separable.*

- (i) *For  $t = 2$ , we have  $s(n, 2) = n + 2$ .*
- (ii) *For  $t = 3$ , we have  $s(n, 3) = \lfloor \frac{n^2}{4} \rfloor + n + 2$ .*
- (iii) *For  $t = 4$  or  $5$ , we have  $s(n, t) = (\frac{n}{t-1})^{t-1} + \Theta(n^{t-2})$ .*
- (iv) *For  $t \geq 6$ , we have  $(\frac{n}{t-1})^{t-1} < s(n, t) \leq 1 + \sum_{i=0}^{t-1} \binom{n}{i}$ .*

A more precise form of parts (iii) and (iv) is stated and proved as Theorem 6' in Section 3.

This note is organized as follows. In the next section, we prove Theorem 2. In Section 3, we study the function  $s(n, t)$ , and we establish Theorem 6 (and Theorem 6'). The last section contains some open problems and related results.

## 2 Shattered matchings—Proof of Theorem 2

Throughout this note, let  $A \sqcup B$  denote the *disjoint union* of the sets  $A$  and  $B$ . For any  $F, Y \subset X$ , we call  $F \cap Y$  the *trace of  $F$  on  $Y$* . For a family  $\mathcal{F} \subset 2^X$  and  $Y \subset X$ , the set of traces  $F \cap Y$  over all  $F \in \mathcal{F}$  is denoted by  $\mathcal{F}|_Y$ .

**Proof of the lower bound.** Let  $\mathcal{F} \subset 2^X$  be a maximal intersecting family of  $\frac{n}{2}$ -element subsets of  $X$ . By the maximality of  $\mathcal{F}$ , if an  $\frac{n}{2}$ -element set  $Y \subset X$  does not belong to  $\mathcal{F}$ , then its complement  $\bar{Y} = X \setminus Y$  does.

Let  $M = \{x_1, x_2\} \sqcup \dots \sqcup \{x_{2k-1}, x_{2k}\}$  be any matching of size  $k$  in  $X$ . Extend it to a perfect matching of  $X$  by adding a perfect matching of the remaining  $n - 2k$  elements:  $M' = \{x_{2k+1}, x_{2k+2}\} \sqcup \dots \sqcup \{x_{n-1}, x_n\}$ . If  $M$  is not a shattered matching, then we can choose a snake  $S$  of  $M$  such that no member  $F \in \mathcal{F}$  intersects  $\{x_1, \dots, x_{2k}\}$  precisely in the elements of  $S$ . Suppose without loss of generality that  $\{x_1, x_3, \dots, x_{2k-1}\}$  is such a snake. This implies that

$$\{x_1, x_3, \dots, x_{2k-1}\} \sqcup \{x_{2k+2-\varepsilon(1)}, x_{2k+4-\varepsilon(2)}, \dots, x_{n-\varepsilon(\frac{n}{2}-k)}\}$$

does not belong to  $\mathcal{F}$  for any  $\varepsilon(i) \in \{0, 1\}$ ,  $1 \leq i \leq \frac{n}{2} - k$ . Hence, the complement of this set,

$$\{x_2, x_4, \dots, x_{2k}\} \sqcup \{x_{2k+1+\varepsilon(1)}, x_{2k+3+\varepsilon(2)}, \dots, x_{n-1+\varepsilon(\frac{n}{2}-k)}\} \in \mathcal{F},$$

for every  $\varepsilon(i) \in \{0, 1\}$ . This means, by definition, that  $M'$  is a shattered matching of size  $\frac{n}{2} - k$ .

Thus, either there is a shattered matching  $M$  of size  $k$ , or a shattered matching  $M'$  of size  $\frac{n}{2} - k$ .

□

**Proof of the upper bound.** Two matchings of the same size in  $X$  are considered identical if they differ only in the order of pairs. We need some simple facts.

**Claim 7** *The number of matchings of size  $k$  in  $X$  is smaller than  $\frac{n!}{k!2^k}$ .*

Indeed, with each permutation  $(x_1, x_2, \dots, x_n)$  associate the matching  $\{x_1, x_2\}, \dots, \{x_{2k-1}, x_{2k}\}$ , and note that we obtain every matching at least  $k!2^k$  times.

Now we *randomly* generate a maximal intersecting family  $\mathcal{F} \subset 2^X$  consisting of  $\frac{n}{2}$ -element subsets of  $X$ , as follows. From each of the  $\frac{1}{2}\binom{n}{n/2}$  unordered pairs  $(Y, X \setminus Y)$  with  $|Y| = n/2$ , we select either  $Y$  or  $X \setminus Y$ , independently with probability  $\frac{1}{2}$ . Let  $\mathcal{F}$  consist of all the selected sets. Obviously, any two members of  $\mathcal{F}$  have nonempty intersection and  $\mathcal{F}$  is maximal with respect to this property.

Fix a matching (partition)  $M = \{x_1, x_2\} \sqcup \dots \sqcup \{x_{2k-1}, x_{2k}\}$  of size  $k$  in  $X$ . Let  $S$  be a snake with respect to  $M$ . We say that  $S$  is *carved out of  $M$*  by a family  $\mathcal{F} \subset 2^X$  if there exists  $F \in \mathcal{F}$  whose trace on  $\{x_1, \dots, x_{2k}\}$  is  $S$ , i.e., if we have  $F \cap \{x_1, \dots, x_{2k}\} = S$ . With a slight abuse of notation, we write  $M \setminus S$  for the set  $\{x_1, \dots, x_{2k}\} \setminus S$  which is also a snake with respect to the matching  $M$ .

In the sequel, for convenience, we write  $2\ell$  for  $n - 2k$ , so that  $k + \ell = \frac{n}{2}$ .

**Claim 8** *Let  $M$  be a fixed matching of size  $k = \frac{n}{2} - \ell$  in  $X$ . For any snake  $S$  with respect to  $M$ , the probability that  $S$  is not carved out of  $M$  by the randomly generated family  $\mathcal{F}$  is equal to  $2^{-\binom{2\ell}{\ell}}$ .*

To see this, it is enough to notice that if  $Y \cap \{x_1, \dots, x_{2k}\}$  is not equal to  $S$ , nor to  $M \setminus S$ , then it does not matter which set we select from the pair  $(Y, X \setminus Y)$ , it can not separate  $S$  from  $M$ . Therefore, it is enough to consider the  $\binom{2\ell}{\ell}$  pairs of complementary sets  $(Y, X \setminus Y)$ , where  $Y = S \sqcup T$  for some  $\ell$ -element subset  $T \subset X \setminus M$ . For each of these pairs, we have to select the set  $X \setminus Y$  to be contained in  $\mathcal{F}$ , otherwise  $S$  will be carved out by  $Y$  and, hence, by  $\mathcal{F}$ . This proves Claim 8. The probability  $2^{-\binom{2\ell}{\ell}}$  may appear to be tiny, but for a fixed  $\ell$  it is bounded away from 0, as  $n \rightarrow \infty$ .

Notice that if  $S$  and  $S'$  are two distinct snakes with respect to  $M$  and  $S' \neq M \setminus S$ , then the events that “ $S$  is carved out of  $M$ ” and “ $S'$  is carved out of  $M$ ” are independent, because they depend on completely different random choices. On the other hand, for  $S' = M \setminus S$ , we have

$$\begin{aligned} & \Pr[\text{at least one of } S \text{ and } M \setminus S \text{ is not carved out of } M] \\ &= \Pr[\text{precisely one of } S \text{ and } M \setminus S \text{ is not carved out of } M] \\ &= 2 \cdot \Pr[S \text{ is not carved out of } M] = 2 \cdot 2^{-\binom{2\ell}{\ell}}. \end{aligned}$$

The number of unordered pairs of snakes  $(S, M \setminus S)$  is  $2^{k-1}$ . Hence,

$$\begin{aligned} & \Pr[M \text{ is shattered}] \\ &= \Pr[\text{every pair of snakes } (S, M \setminus S) \text{ are carved out of } M] \\ &= (1 - 2 \cdot 2^{-\binom{2\ell}{\ell}})^{2^{k-1}} < \exp(-2^{k-1} \cdot 2^{-\binom{2\ell}{\ell}}). \end{aligned}$$

Combining this with Claim 7, we obtain that

$$\begin{aligned} & \Pr[\text{there exists a shattered matching of size } k] \\ &\leq \sum_M \Pr[M \text{ is shattered}] \\ &< \frac{n!}{k! 2^k} \exp(-2^{k-1} \cdot 2^{-\binom{2\ell}{\ell}}) < \exp(n \ln n - 2^{\frac{n}{2}-\ell-\binom{2\ell}{\ell}}). \end{aligned}$$

To conclude, it is enough to show that the right-hand side of this inequality is smaller than 1, that is,  $2^{\frac{n}{2}-\ell-\binom{2\ell}{\ell}} > n \ln n$  holds, provided that  $k \geq \frac{n}{2} - \frac{1}{2} \log_2 n + 1$ .

According to the last condition,  $\ell + 1 \leq \frac{1}{2} \log_2 n$ , which implies that

$$2^{\frac{n}{2}-\ell-\binom{2\ell}{\ell}} \geq 2^{\frac{n}{2}-4\ell} \geq 2^{\frac{n}{4}} > n \ln n,$$

if  $n \geq 28$ . This completes the proof of the upper bound and, hence, Theorem 2.  $\square$

It is easy to verify using the above estimates that the probability that there exists a shattered matching of size  $k = \frac{n}{2} - 1$  is smaller than 1, for every  $n \geq 14$ . Therefore, in these cases, Conjecture A of Gordon and Teplitskaya fails.

Next, we turn to Conjecture B.

**Corollary 9** *Let  $X$  be an  $n$ -element set, where  $n \geq 15$  is odd.*

*There is a maximal intersecting family  $\mathcal{F} \subset 2^X$  such that  $|F| = \frac{n-1}{2}$  or  $\frac{n+1}{2}$  for every  $F \in \mathcal{F}$ , and the following condition is satisfied. There do not exist  $y \in X$  and a perfect matching  $M$  of  $X \setminus \{y\}$  with the property that adding  $y$  to every snake with respect to  $M$ , we get a member of  $\mathcal{F}$ .*

**Proof.** Let  $|X| = n = 2k + 1$ , let  $\mathcal{F} \subset 2^X$  be a maximal intersecting family, and suppose that every member of  $\mathcal{F}$  has  $k$  or  $k + 1$  elements. By the maximality of  $\mathcal{F}$ , for each  $k$ -element subset  $Y \subset X$ , either  $Y$  or  $X \setminus Y$  belongs to  $\mathcal{F}$ . Therefore, we have  $|\mathcal{F}| = \binom{n}{k}$ .

Fix a  $2k$ -element subset  $V \subset X$ , and denote the unique element of  $X \setminus V$  by  $x$ . According to the remark after the proof of Theorem 2, we can choose a maximal intersecting family  $\mathcal{G}$  of  $\frac{1}{2}\binom{2k}{k}$   $k$ -element subsets of  $V$  such that  $\mathcal{G}$  does not shatter any matching of size  $k - 1$  in  $V$ . Let

$$\mathcal{F} = \mathcal{G} \cup \{Y \cup \{x\} \mid Y \in \mathcal{G}\} \cup \{U \mid U \subset V, |U| = k + 1\}.$$

Obviously,  $\mathcal{F}$  is a *maximal intersecting* family consisting of  $k$ -element and  $(k + 1)$ -element subsets of  $X$ .

We claim that  $\mathcal{F}$  meets the requirements of Corollary 9. Suppose for contradiction that there are  $y \in X$  and a perfect matching  $M$  of  $X \setminus \{y\}$  such that every snake with respect to  $M$  can be extended to a member of  $\mathcal{F}$  by adding  $y$ . We distinguish two cases.

Suppose first that  $y = x$ . Then  $X \setminus \{y\} = X \setminus \{x\} = V$ , and  $M$  is a partition of  $V$  into 2-element sets:

$$\{x_1, x_2\} \sqcup \{x_3, x_4\} \sqcup \dots \sqcup \{x_{2k-1}, x_{2k}\}.$$

It follows from the definition of  $\mathcal{G}$  that only one of the snakes with respect to  $M$ ,  $\{x_1, x_3, \dots, x_{2k-1}\}$  or  $\{x_2, x_4, \dots, x_{2k}\}$  belongs to  $\mathcal{G}$ . Hence, only one of the “extended” snakes  $\{x_1, x_3, \dots, x_{2k-1}, y\}$  or  $\{x_2, x_4, \dots, x_{2k}, y\}$  belongs to  $\mathcal{F}$ . The other one does not, contradicting our assumption.

Suppose next that  $y \neq x$ , and let  $M$  be the partition (perfect matching) of  $X \setminus \{y\}$  with the above property,

$$X \setminus \{y\} = \{x_1, x_2\} \sqcup \{x_3, x_4\} \sqcup \dots \sqcup \{x_{2k-1}, x_{2k}\}.$$

We can assume, by symmetry, that  $x = x_{2k}$ . It follows from the definition of  $\mathcal{G}$  that the “partial” matching  $M'$  of  $M$ ,

$$\{x_1, x_2\} \sqcup \{x_3, x_4\} \sqcup \dots \sqcup \{x_{2k-3}, x_{2k-2}\},$$

is not shattered by  $\mathcal{G}$ .

We can assume without loss of generality that the snake  $\{x_1, x_3, \dots, x_{2k-3}\}$  with respect to  $M'$  is *not* the trace of any member of  $\mathcal{G}$  on  $V \setminus \{x_{2k-1}, y\}$ . In particular, we have  $\{x_1, x_3, \dots, x_{2k-3}, y\} \notin \mathcal{G}$ . By the definition of  $\mathcal{F}$ , this implies that

$$\{x_1, x_3, \dots, x_{2k-3}, y, x\} \notin \mathcal{F}.$$

However, this means that the snake  $\{x_1, x_3, \dots, x_{2k-3}, x\}$  with respect to the matching  $M$ , cannot be extended to a member of  $\mathcal{F}$  by adding  $y$ , contradiction. This completes the proof of the corollary.

□

### 3 Separable families—Proof of Theorem 6

We start with a construction of non- $t$ -separable families. Let  $x_1, \dots, x_n$  be the elements of a set  $X$ , listed in an arbitrary order. The set-system

$$\mathcal{C} = \{\emptyset, \{x_1\}, \{x_1, x_2\}, \dots, \{x_1, \dots, x_n\}\}$$

is called a *maximal chain* on  $X$ .

**Lemma 10** *For  $n \geq t \geq 2$ , consider a partition of an  $n$ -element set  $X$  into  $t - 1$  parts,  $X = X_1 \sqcup \dots \sqcup X_{t-1}$ . Fix a maximal chain  $\mathcal{C}_i$  on each  $X_i$ .*

*Then the family*

$$\mathcal{D}(X_1, \dots, X_{t-1}) = \{C_1 \sqcup \dots \sqcup C_{t-1} : C_i \in \mathcal{C}_i \text{ for } i = 1, \dots, t-1\}$$

*is not  $t$ -separable.*

**Proof.** Suppose for contradiction that  $\mathcal{D}(X_1, \dots, X_{t-1})$  is  $t$ -separable, that is, there exists a  $t$ -element subset  $T \subset X$  satisfying the conditions in Definition 6. By the pigeonhole principle, there is an  $X_i$ ,  $1 \leq i \leq t - 1$  which contains at least two elements,  $x, y \in T$ . Suppose without loss of generality that  $x$  precedes  $y$  in the order that defines the chain  $\mathcal{C}_i$  on  $X_i$ . Then  $\mathcal{C}_i$  and, hence,  $\mathcal{D}(X_1, \dots, X_{t-1})$  has no member which contains  $y$ , but not  $x$ .  $\square$

Similar constructions involving direct products of *chains* can be found, e.g., in [1, 2].

**Corollary 11** *Let  $n \geq t \geq 2$ , and let  $X_1 \sqcup \dots \sqcup X_{t-1}$  be a partition of an  $n$ -element set into  $t - 1$  parts, as equal as possible. That is, we have  $|X_i| = \lfloor \frac{n}{t-1} \rfloor$  or  $\lceil \frac{n}{t-1} \rceil$  for  $1 \leq i \leq t - 1$ . Then the smallest number  $s = s(n, t)$  with the property that every family of at least  $s$  subsets of  $X$  is  $t$ -separable, satisfies*

$$s(n, t) > \prod_{i=1}^{t-1} (|X_i| + 1) > \left(\frac{n}{t-1}\right)^{t-1}.$$

*Consequently, for every  $k \leq \frac{n}{2}$ , we have*

$$p(n, k) \geq s(n, 2k) > \left(\frac{n}{2k-1}\right)^{2k-1}.$$

$\square$

For  $t = 2$ , the first part of Corollary 11 implies that  $s(n, 2) \geq n + 2$ . On the other hand, it follows from Theorem 3 that if  $|\mathcal{F}| > \binom{n}{1} + \binom{n}{0} = n + 1$  for a family  $\mathcal{F}$  of subsets of an  $n$ -element set  $X$ , then  $\mathcal{F}$  shatters a 2-element subset of  $X$ , hence,  $\mathcal{F}$  is 2-separable. Thus,  $s(n, 2) = n + 2$ , which proves part (i) of Theorem 6.

In what follows, we use the “arrow” notation proposed by Hajnal (see [4, 5]). We write

$$(n, m) \rightarrow (a, b)$$

if the following statement is true: For any family of  $m$  subsets  $\mathcal{F} = \{F_1, \dots, F_m\}$  of an  $n$ -element set  $X$ , there is  $T \subset X$  with  $|T| = a$  such that the family of *traces*  $\mathcal{F}|_T = \{F_1 \cap T, \dots, F_m \cap T\}$  has at least  $b$  distinct members.

Using this notation, Theorem 3 can be reformulated as

$$(n, 1 + \sum_{i=0}^{k-1} \binom{n}{i}) \rightarrow (k, 2^k).$$

A family  $\mathcal{F}$  is called *downward closed* if for any  $F \in \mathcal{F}$  and  $G \subset F$ , we have  $G \in \mathcal{F}$ . A crucial property of the arrow relation was established by the first author [7].

**Lemma 12** (Frankl [7]) *The relation  $(n, m) \rightarrow (a, b)$  holds if and only if for any downward closed family  $\mathcal{F}$  of  $m$  subsets of a set  $X$  with  $|X| = n$ , there is  $T \subset X$  with  $|T| = a$  such that  $|\mathcal{F}|_T| \geq b$ .*

In other words, in order to show that there is  $T \subset X$  with  $|T| = a$  and  $|\mathcal{F}|_T| \geq b$ , it is sufficient to verify it for downward closed families.

**Lemma 13** *Suppose that  $(n, m) \rightarrow (t, 2^t - 2^{t-2} + 1)$  holds.*

*Then every family  $\mathcal{F}$  of subsets of an  $n$ -element set with  $|\mathcal{F}| \geq m$  is  $t$ -separable.*

**Proof.** Assume that  $\mathcal{F} \subset 2^X$  satisfies the above condition, and let  $T$  be a  $t$ -element set of  $X$  with  $|\mathcal{F}|_T| > 2^t - 2^{t-2}$ . For any  $x, y \in T, x \neq y$ , there are  $2^{t-2}$  subsets of  $T$  that contain  $x$ , but not  $y$ . At least one of them must belong to  $\mathcal{F}|_T$ , which proves the claim.  $\square$

Now we can settle the case  $t = 3$  in Theorem 6. The first part of Corollary 11 implies that

$$s(n, 3) \geq (\lfloor \frac{n}{2} \rfloor + 1)(\lceil \frac{n}{2} \rceil + 1) + 1 = \lfloor \frac{n^2}{4} \rfloor + n + 2.$$

On the other hand, the first author [7] proved that  $(n, \lfloor \frac{n^2}{4} \rfloor + n + 2) \rightarrow (3, 7)$ . Applying Lemma 13 with  $m = \lfloor \frac{n^2}{4} \rfloor + n + 2$  and  $t = 3$ , we obtain that  $s(n, 3) = \lfloor \frac{n^2}{4} \rfloor + n + 2$ , which proves part (ii) of Theorem 6.

For the rest of the argument, we need some further results from extremal set theory.

Consider again a partition of the  $n$ -element set  $X$  into  $t - 1$  parts,  $X = X_1 \sqcup \dots \sqcup X_{t-1}$ . A family  $\mathcal{G} \subset \binom{X}{t-1}$  of  $(t - 1)$ -element subsets of  $X$  is called a  $(t - 1)$ -*uniform hypergraph* or, simply, a  $(t - 1)$ -*graph*. If every *edge*  $E \in \mathcal{G}$  intersects each  $X_i$  in precisely 1 point, then  $\mathcal{G}$  is said to be  $(t - 1)$ -*partite*.

For graphs, i.e., for  $t = 3$ , Mantel [17] and Turán [22] proved that if a graph (2-graph)  $\mathcal{G}$  has more than  $\lfloor \frac{n^2}{4} \rfloor$  edges (sets), then it contains a *triangle*, i.e., there are  $x, y, z \in X$  with  $\{x, y\}, \{x, z\}, \{y, z\} \in \mathcal{G}$ . This bound is best possible, as is shown by a 2-partite (bipartite) graph whose parts are of size  $\lfloor \frac{n}{2} \rfloor$  and  $\lceil \frac{n}{2} \rceil$ .

For a fixed  $t \geq 3$ , a *generalized triangle* consists of 3 distinct  $(t - 1)$ -elements sets  $E_1, E_2, E_3$  such that  $|E_1 \cap E_2| = t - 2$  and  $E_3 \supseteq (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$ . For  $t = 3$ , the only generalized triangle is the



usual triangle. For  $t \geq 4$ , however, there are  $t - 2$  non-isomorphic generalized triangles, depending on the value of  $|E_1 \cap E_2 \cap E_3|$ , which can be  $0, 1, \dots, t - 3$ . Obviously, none of these generalized triangles is  $(t - 1)$ -partite.

The Mantel-Turán theorem was extended to 3-graphs and 4-graphs by Bollobás and Sidorenko, respectively, as follows.

**Lemma 14** *Let  $X$  be an  $n$ -element set,  $n \geq 4$ . For any  $k \geq 2$ , let  $g(n, k)$  denote the maximum size of a  $k$ -graph  $\mathcal{G}$  which does not contain any generalized triangle. Then we have*

- (i) (Bollobás [3])  $g(n, 3) \leq \lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor$ ;
- (ii) (Sidorenko [21])  $g(n, 4) \leq \lfloor \frac{n}{4} \rfloor \lfloor \frac{n+1}{4} \rfloor \lfloor \frac{n+2}{4} \rfloor \lfloor \frac{n+3}{4} \rfloor$ .

*Both results are best possible as is shown by the complete 3-partite (4-partite) 3-graphs (resp., 4-graphs) whose parts are as equal as possible.*

It is not hard to see that part (i) of Lemma 14 implies the Mantel-Turán theorem for ordinary triangles [15]. It was proved by Frankl and Füredi [9] (see also [16]) that, if  $\mathcal{G} \subset \binom{X}{3}$  is a 3-graph with  $|\mathcal{G}| > g(n, 3)$  and  $X$  is sufficiently large, then  $\mathcal{G}$  also contains a generalized triangle with  $E_1 \cap E_2 \cap E_3 = \emptyset$ .

One might hope that analogous results hold for  $k > 4$ . However, this is not the case. For  $k = 5$  and 6, Frankl and Füredi [10] determined all largest  $k$ -graphs on  $n > n_0$  vertices that do not contain a generalized triangle. These turned out to have substantially more edges than the balanced complete  $k$ -partite  $k$ -graphs, and one can obtain them by “blowing up” certain Steiner systems called Witt designs. For a survey on this fascinating problem, consult [18].

It remains to establish parts (iii) and (iv) of Theorem 6. With the notation of Lemma 14, they can be rephrased in the following form.

**Theorem 6’** *Let  $n \geq t \geq 4$  and  $X = \{1, 2, \dots, n\}$ . Let  $s = s(n, t)$  denote the smallest number with the property that every family  $\mathcal{F} \subset 2^X$  with  $|\mathcal{F}| \geq s$  is  $t$ -separable. Then we have*

$$\left(\frac{n}{t-1}\right)^{t-1} < s(n, t) \leq g(n, t-1) + 1 + \sum_{i=0}^{t-2} \binom{n}{i}.$$

*According to Lemma 14, for  $t = 4$  and 5, the lower bound and the upper bound are asymptotically the same. For  $t > 5$ , the two bounds are asymptotically different, but their order of magnitude is the same,  $\Theta(n^{t-1})$ .*

**Proof.** Let  $\mathcal{F} \subset 2^X$  be a family satisfying

$$|\mathcal{F}| > g(n, t-1) + \sum_{i=0}^{t-2} \binom{n}{i}.$$

We will show that  $\mathcal{F}$  is  $t$ -separable. By Lemma 13, it is sufficient to prove that there is a  $t$ -element subset  $T \subset X$  such that  $|\mathcal{F}_T| \geq 2^t - 2^{t-2} + 1$ . According to Lemma 12, we can assume that  $\mathcal{F}$  is downward closed.

Therefore, if  $\mathcal{F}$  has a member of size at least  $t$ , then it also has a member  $F$  of size precisely  $t$ . In this case, choosing  $T$  to be  $F$ , we have  $|\mathcal{F}_{|T}| = 2^t$ , and we are done.

Thus, we can assume that the number of  $(t-1)$ -element members in  $\mathcal{F}$  is larger than  $g(n, t-1)$ . By the definition of  $g(n, t-1)$ , the  $(t-1)$ -graph  $\mathcal{G} \subset \mathcal{F}$  formed by these edges contains a generalized triangle  $E_1, E_2, E_3$ . We can assume without loss of generality that  $E_1 = \{1, 2, \dots, t-2, t-1\}$ ,  $E_2 = \{1, 2, \dots, t-2, t\}$ , and  $\{t-1, t\} \subset E_3$ . Set  $T = \{1, 2, \dots, t\}$ . Then we have  $|T| = t$  and  $\mathcal{G}_{|T} \supset (2^{E_1} \cup 2^{E_2}) \sqcup \{t-1, t\}$ . As

$$|2^{E_1} \cup 2^{E_2}| = 2^{|E_1|} + 2^{|E_2|} - 2^{|E_1 \cap E_2|} = 2^{t-1} + 2^{t-1} - 2^{t-2} = 2^t - 2^{t-2},$$

we obtain that  $|\mathcal{F}_{|T}| \geq |\mathcal{G}_{|T}| \geq 2^t - 2^{t-2} + 1$ , as required. This completes the proof of the upper bound in Theorem 6'. The lower bound is given by Corollary 11.  $\square$

## 4 Open problems, concluding remarks

It would be interesting to close the gaps between the lower and upper bounds in Theorems 2 and 6.

**4.1.** What happens if, instead of concentrating on maximal intersecting families of  $\frac{n}{2}$ -element subsets of an  $n$  element set, as we did in Theorem 2, we consider *all* maximal intersecting families of subsets of  $X$ , with no restriction on the sizes of the subsets? In particular, we can ask the following.

**Problem 15** *Determine or estimate the largest integer  $k^* = k^*(n)$  such that for every maximal intersecting family  $\mathcal{F}$  of subsets of an  $n$ -element set, one can find a shattered matching of size  $k^*$ .*

**4.2.** Given a family  $\mathcal{F} \subset 2^X$  and a system  $\mathcal{M}$  of pairwise disjoint  $r$ -element subsets of  $X$  for some  $r \geq 3$ , we say that  $\mathcal{M}$  is *shattered* by  $\mathcal{F}$  if no matter how we pick one element from each  $r$ -tuple of  $\mathcal{M}$ , there is a member  $F \in \mathcal{F}$  which carves out precisely these elements of  $\cup \mathcal{M}$ .

**Problem 16** *For any even integer  $n \geq 4$  and  $r \geq 3$ , determine or estimate  $k_r(n)$ , the largest integer  $k$  such that for every maximal intersecting family  $\mathcal{F}$  of  $\frac{n}{2}$ -element subsets of an  $n$ -element set, one can find a system  $\mathcal{M}$  of  $k$  pairwise disjoint  $r$ -element sets which is shattered by  $\mathcal{F}$ .*

We trivially have  $k_r(n) < \frac{n}{2(r-1)}$ , whenever  $n$  is a multiple of  $2(r-1)$ . If we take  $\frac{n}{2(r-1)}$   $r$ -tuples and pick one element from each, then there is a unique way how to add further elements from the remainder to obtain an  $\frac{n}{2}$ -element set.

As in Problem 15, here we can also relax the condition that every member of our maximal intersecting family is of size  $\frac{n}{2}$ . Furthermore, in the spirit of Problem 4, we can completely drop the restriction that  $\mathcal{F}$  is intersecting, and we can ask *how large*  $\mathcal{F}$  needs to be in order to ensure that it shatters some system of  $k$  pairwise disjoint  $r$ -element sets.

**4.3.** Following [11], we call a family of  $t$  sets  $F_1, \dots, F_t \subset X$  *disjointly representable* if there exist  $x_1, \dots, x_t \in X$  with the property that  $x_i \in F_j$  if and only if  $i = j$ . In other words, a family of sets is disjointly representable if and only if none of its members is completely covered by the union of the others.

Modifying the question addressed in Theorem 6, we can ask the following.

**Problem 17** Let  $n \geq t \geq 2$ . Determine or estimate the smallest number  $r = r(n, t)$  with the property that every family  $\mathcal{F}$  of at least  $r$  subsets of an  $n$ -element set has  $t$  disjointly representable members.

It follows from the definition that if a family has  $t$  disjointly representable members, then it is  $t$ -separable. Therefore, we have  $r(n, t) \geq s(n, t)$  for every  $n$  and  $t$ .

The proof of part (i) of Theorem 6 also gives  $r(n, 2) = s(n, 2) = n + 2$ .

**Claim 18**  $r(n, 3) = \binom{n}{2} + n + 2 > s(n, 3)$  for every  $n \geq 3$ .

**Proof.** The upper bound immediately follows from Theorem 3.

To prove the lower bound, consider the following subsets of  $X = \{1, 2, \dots, n\}$ . For any two elements  $a < b$  of the auxiliary set  $A = \{\frac{1}{2}, 1 + \frac{1}{2}, \dots, n + \frac{1}{2}\}$ , let

$$F(a, b) = \{x \in X : x < a\} \cup \{x \in X \mid x > b\}.$$

The family  $\mathcal{F} = \{F(a, b) : a, b \in A \text{ and } a < b\} \cup \{\emptyset\}$  has  $\binom{n+1}{2} + 1 = \binom{n}{2} + n + 1$  members, and it has no 3 disjointly representable members. Indeed, given any 3 elements  $x_1 < x_2 < x_3 \in X$ , there is no  $F \in \mathcal{F}$  with  $F \cap \{x_1, x_2, x_3\} = \{x_2\}$ .  $\square$

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