arXiv:1901.06224v1 [math.AG] 18 Jan 2019

ON THE TOPOLOGY OF ELLIPTIC SINGULARITIES

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Abstract. For any elliptic normal surface singularity with rational homology sphere link we consider a new elliptic sequence, which differs from the one introduced by Laufer and S. S.-T. Yau. However, we show that their length coincide. Using the properties of both sequences we succeed to connect the common length with the geometric genus and also with several topological invariants, e.g. with the Seiberg–Witten invariant of the link.

Dedicated to Gert–Martin Greuel

1. Introduction

1.1. The most important analytic invariant of a complex normal surface singularity (X, o) is its geometric genus p_g. Even if we fix a topological type — usually identified by the link M of the germ, or by a resolution graph —, and even if we assume that the link M is a rational homology sphere, the geometric genus might vary when we vary the analytic structure. Hence, it is natural to find topological bounds for it. In the literature there are several topological invariants, which are related with p_g in this sense.

One of them is $\mathrm{Path}^{\uparrow}$, cf. [N07, NS16, NO17], see subsection 2.2 below. It is a topological upper bound for p_g, that is, for any analytic structure one has $p_g \leq \mathrm{Path}^{\uparrow}$. However, usually it is hard to verify whether the inequality is optimal or not for a certain topological type, that is, whether a special analytic structure realizes the equality. (One knows topological types when the inequality is not sharp, see Example 2.2.4.)

Another topological invariant is the (modified) Seiberg–Witten invariant $\overline{\mathfrak{sw}}_0(M))$ of the link (associated with the canonical *spin*^{c}–structure). It is related with the geometric genus via the Seiberg–Witten Invariant Conjecture (SWIC) $p_g(X, o) = \overline{\mathfrak{sw}}_0(M))$, cf. [N07, N12, NO08, NO09, NS16, NW90], which is expected to be true for certain special analytic structures. But, again, the verification of this identity usually is hard (and in some cases it is not even true).

In the case of elliptic singularities there is another topological numerical invariant, the length of the elliptic sequence introduced by Laufer and S. S.-T. Yau [Y79, Y80]. In the numerically Gorenstein case (when a Gorenstein structure exist) it is easier to connect with p_g and $\mathrm{Path}^{\uparrow}$, however in the general case the Yau's elliptic sequence is rather complicated (and it is also hard to connect with possible analytic realizations).

In order to eliminate these difficulties, we consider a new elliptic sequence, which in the non–numerically Gorenstein case is different than the one studied by Yau, and which fits much better in such comparisons. It was motivated (and introduced) in the author's study of the Abel map of surface singularities [NN18], and it has several advantages compared with the earlier approaches. E.g., it identifies the support of a numerically Gorenstein subgraph with the following property. If

2010 *Mathematics Subject Classification.* Primary. 32S05, 32S25, 32S50, 57M27 Secondary. 14Bxx, 14J80.
Key words and phrases. normal surface singularity, resolution graph, rational homology sphere, elliptic singularities, elliptic sequence, Seiberg–Witten invariant, surgery formula, Poincaré series, geometric genus, periodic constant.

the analytic type supported on this subgraph is Gorenstein, that $p_g(X,o)$ is maximal, and it satisfies the identity $p_g(X,o) = \mathrm{Path}^{\uparrow}$ (and the statement (3) from below).

In this note first we prove that the length $\ell+1$ of the Yau's elliptic sequence coincides with the length $m+1$ of our elliptic sequence. Then using properties of both sequences we prove the following statements for any elliptic germ with rational homology sphere link:

(1) $m+1=\ell+1$;

(2) $m+1=\mathrm{Path}^{\uparrow}$;

(3) there exists an analytic structure (characterized precisely) supported on the fixed elliptic topological type such that $p_g=m+1$;

(4) $m+1=\mathfrak{sw}_0(M)$), in particular, for any analytic structure from (2) the SWIC holds.

Strictly speaking, in the proof of (2) we use an additional assumption, namely that the minimal resolution is good. The main reason for this assumption is that the elliptic sequences are defined (and have nice properties) in the minimal resolution, while the invariant $\mathrm{Path}^{\uparrow}$ is defined in via good resolutions. We expect that the statement remains valid in any case, but in this note we did not check the compatibility of the two resolutions (the minimal one and the minimal good one) from the point of view of these two set of invariants (and we didn't carry out the pathological cases either).

1.2. The structure of the article is the following. In section 2 we review the standard notations related with resolution of normal surface singularities, and we recall some facts regarding $\mathrm{Path}^{\uparrow}$. In the next section we discuss elliptic singularities (we always assume that the link is a rational homology sphere). We recall the definition of the elliptic sequence according to Yau, we establish several properties which will be needed later. Then we discuss the special case of numerically Gorenstein graphs, and finally we provide the definition and several properties of the 'new' elliptic sequence. Finally in Theorem 3.4.6 we prove (1) and (2).

Section 4 reviews several results regarding surgery properties of the Seiberg–Witten invariants (based on some coefficient counting of the topological Poincaré series), and in the last section we prove (3) via such a surgery formula.

2. Preliminaries and notations

2.1. **Notations regarding a resolution.** [N99b, N07, N12, L13, NN02] Let (X,o) be the germ of a complex analytic normal surface singularity. We denote by p_g the geometric genus of (X,o). *We will assume that the link M of (X,o) is a rational homology sphere.*

Let $\phi:\widetilde{X}\to X$ be a resolution of (X,o) with exceptional curve $E:=\phi^{-1}(0)$, and let $\cup_{v\in\mathcal{V}}E_v$ be the irreducible decomposition of E.

$L:=H_2(\widetilde{X},\mathbb{Z})$, endowed with a negative definite intersection form $(\,,\,)$, is a lattice. It is freely generated by the classes of $\{E_v\}_{v\in\mathcal{V}}$. The dual lattice is $L'=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})=\{l'\in L\otimes\mathbb{Q}\,:\,(l',L)\in\mathbb{Z}\}$. It is generated by the (anti)dual classes $\{E^*_v\}_{v\in\mathcal{V}}$ defined by $(E^*_v,E_w)=-\delta_{vw}$ (where δ_{vw} stays for the Kronecker symbol). L' is also identified with $H^2(\widetilde{X},\mathbb{Z})$.

All the E_v–coordinates of any E^*_u are strict positive. We define the Lipman cone as $\mathcal{S}':=\{l'\in L'\,:\,(l',E_v)\le 0 \text{ for all } v\}$. As a monoid it is generated over $\mathbb{Z}_{\ge0}$ by $\{E^*_v\}_v$. Write also $\mathcal{S}:=\mathcal{S}'\cap L$.

L embeds into L' with $L'/L\simeq H_1(M,\mathbb{Z})$, which is abridged by H. The class of l' in H is denoted by $[l']$.

There is a natural (partial) ordering of L' and L: we write $l'_1\ge l'_2$ if $l'_1-l'_2=\sum_v r_vE_v$ with all $r_v\ge0$. We set $L_{\ge0}=\{l\in L\,:\,l\ge0\}$ and $L_{>0}=L_{\ge0}\setminus\{0\}$.

The support of a cycle $l=\sum n_vE_v$ is defined as $|l|=\cup_{n_v\neq 0}E_v$.

Since $H_1(M,\mathbb{Q})=0$, each E_v is rational, and the dual graph of any good resolution is a tree.

2.1.1. Minimal cycles in $L'_{\geq 0}$ and in $\mathcal{S}'$. Consider the semi-open cube $\{\sum_v l'_vE_v\in L' \mid 0\leq l'_v<1\}$. It contains a unique representative r_h for every $h\in H$ so that $[r_h]=h$. Similarly, for any $h\in H$ there is a unique minimal element of $\{l'\in L' \mid [l']=h\}\cap\mathcal{S}'$, which will be denoted by s_h (cf. Lemma 2.1.3 below). One has $s_h\geq r_h$; in general, $s_h\neq r_h$.

2.1.2. A 'Laufer–type' computation sequence targeting $\mathcal{S}'$. Recall the following fact:

Lemma 2.1.3. [La72], [N05, Lemma 7.4] *Fix any $l'\in L'$.*

(1) There exists a unique minimal element $s(l')$ of $(l'+L_{\geq 0})\cap\mathcal{S}'$.

(2) $s(l')$ can be found via the following computation sequence $\{z_i\}_i$ connecting l' and $s(l')$: set $z_0:=l'$, and assume that z_i $(i\geq 0)$ is already constructed. If $(z_i,E_{v(i)})>0$ for some $v(i)\in\mathcal{V}$ then set $z_{i+1}=z_i+E_{v(i)}$. Otherwise $z_i\in\mathcal{S}'$ and necessarily $z_i=s(l')$.

In general the choice of the individual vertex $v(i)$ might not be unique, nevertheless the final output $s(l')$ is unique.

If we start with an arbitrarily chosen $l'=E_v$ then $s(l')$ is the minimal (fundamental) cycle Z_{min} of L, that is, the minimal element of $\mathcal{S}\setminus\{0\}$ [A62, A66, La72]. In this case, the sequence from part (2) usually is called the '*Laufer's computation sequence of Z_{min}*'.

Similarly, for any $h\in H$, if $l'=r_h$ then $s(l')=s_h$.

2.1.1. The *(anti)canonical cycle* $Z_K\in L'$ is defined by the *adjunction formulae* $(Z_K,E_v)=(E_v,E_v)+2$ for all $v\in\mathcal{V}$. (It is the first Chern class of the dual of the line bundle $\Omega^2_{\widetilde{X}}$.) We write $\chi:L'\to\mathbb{Q}$ for the combinatorial expression $\chi(l'):=-(l',l'-Z_K)/2$.

The singularity (or, its topological type) is called numerically Gorenstein if $Z_K\in L$. (Since $Z_K\in L$ if and only if the line bundle $\Omega^2_{X\setminus\{o\}}$ of holomorphic 2–forms on $X\setminus\{o\}$ is topologically trivial, see e.g. [Du78], the $Z_K\in L$ property is independent of the resolution). (X,o) is called Gorenstein if $Z_K\in L$ and $\Omega^2_{\widetilde{X}}$ (the sheaf of holomorphic 2–forms) is isomorphic to $\mathcal{O}_{\widetilde{X}}(-Z_K)$ (or, equivalently, if the line bundle $\Omega^2_{X\setminus\{o\}}$ is holomorphically trivial).

Recall that if $\widetilde{X}$ is a minimal resolution then (by the adjunction formulae) $Z_K\in\mathcal{S}'$. In particular, $Z_K-s_{[Z_K]}\in L_{\geq 0}$.

Lemma 2.1.4. [NN18, Lemma 2.1.4] *Consider the minimal resolution $\widetilde{X}$ of (X,o). Then $p_g=0$ whenever $Z_K=s_{[Z_K]}$. If $Z_K>s_{[Z_K]}$ then $p_g=h^1(\mathcal{O}_{Z_K-s_{[Z_K]}})$. More generally, $h^1(\widetilde{X},\mathcal{L})=h^1(Z_K-s_{[Z_K]},\mathcal{L})$ for any $\mathcal{L}\in\mathrm{Pic}(\widetilde{X})$ with $c_1(\mathcal{L})\in-\mathcal{S}'$.*

Proof. By generalized Kodaira or Grauert–Riemenschneider vanishing $h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-\lfloor Z_K\rfloor))=0)$. Hence, if $\lfloor Z_K\rfloor)=0$ then $p_g=0$. Otherwise, using the exact sequence $0\to\mathcal{O}_{\widetilde{X}}(-\lfloor Z_K\rfloor)\to\mathcal{O}_{\widetilde{X}}\to\mathcal{O}_{\lfloor Z_K\rfloor}\to 0$ we get $h^1(\mathcal{O}_{\lfloor Z_K\rfloor})=p_g$. Next, consider the computation sequence from Lemma 2.1.3 applied for $l'=r_h$ and show by induction that $h^1(\mathcal{O}_{Z_K-z_i})=p_g$.

More generally, $h^1(\widetilde{X},\mathcal{L})=h^1(Z_K-z_i,\mathcal{L})$ for any i by similar argument. □

2.2. **The invariant** $\mathrm{Path}^{\uparrow}$**.** Assume that at this time $\widetilde{X}$ is the minimal good resolution. In this case, $Z_K\geq 0$, see e.g. [V04, PP11, NBook]. In particular, $\lfloor Z_K\rfloor\in L_{\geq 0}$. Let $\mathcal{K}$ be the (topologically defined) set of cycles $\lfloor Z_K\rfloor+L_{\geq 0}$. Note that by a generalized Grauert–Riemenschneider vanishing [GrRie70] $h^1(\mathcal{O}_Z)=p_g$ for any $Z\in\mathcal{K}$.

An *increasing path* is a sequence of integral cycles $\gamma := \{l_i\}_{i=0}^t$, $l_i \in L$ such that $l_0 = 0$, $l_t \in \mathcal{K}$, and for any $i < t$ one has $l_{i+1} = l_i + E_{v(i)}$ for some $v(i) \in \mathcal{V}$. Denote by $\mathcal{P}^{\uparrow}$ the set of increasing paths. Moreover, for any $\gamma \in \mathcal{P}^{\uparrow}$ and $i < t$ define

$$p_i = \max\{0, \chi(l_i) - \chi(l_{i+1})\} = \max\{0, (E_{v(i)}, l_i) - 1\}, \tag{2.2.1}$$

and set $S(\gamma) := \sum_{i<t} p_i$ for any $\gamma \in \mathcal{P}^{\uparrow}$. Furthermore, set $\mathrm{Path}^{\uparrow} := \min_{\gamma \in \mathcal{P}^{\uparrow}} S(\gamma)$ as well.

The definition is mostly motivated by comparison of the geometric genus with path lattice cohomology [N08b], see also [NS16, NO17].

2.2.2. Upper bounds for the geometric genus. If $\gamma \in \mathcal{P}^{\uparrow}$ with $l_t = Z$ then $p_g = h^1(\mathcal{O}_Z)$. Furthermore, from the exact sequence $0 \to \mathcal{O}_{E_{v(i)}}(-l_i) \to \mathcal{O}_{l_{i+1}} \to \mathcal{O}_{l_i} \to 0$ we get

$$h^1(\mathcal{O}_{l_{i+1}}) - h^1(\mathcal{O}_{l_i}) \leq h^1(\mathcal{O}_{E_{v(i)}}(-l_i)) = p_i \qquad (0 \leq i < t).$$

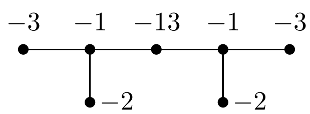
In particular, for any analytic structure with the fixed resolution graph Γ one has

$$p_g \leq \mathrm{Path}^{\uparrow}. \tag{2.2.3}$$

Equality holds if *for some* $\gamma \in \mathcal{P}^{\uparrow}$ the above cohomology exact sequences split for all i. The above inequality $p_g \leq \mathrm{Path}^{\uparrow}$ looks slightly artificial, even naive; nevertheless, for rather important analytic structures along a well–chosen increasing path all the cohomology exact sequences split, and the equality $p_g = \mathrm{Path}^{\uparrow}$ holds. The equality $p_g = \mathrm{Path}^{\uparrow}$ is realized by the following analytic families (with rational homology sphere link): (a) weighted homogeneous singularities; (b) superisolated singularities; (c) Newton non-deg hypersurfaces; (d) rational singularities; (e) Gorenstein elliptic singularities. (For details and further references see [NS16, NBook]).

In Theorem 3.4.6 we will show that the equality in $p_g \leq \mathrm{Path}^{\uparrow}$ can be realized in the case of any non–numerically Gorenstein elliptic topological type as well.

Example 2.2.4. [NO17] On the other hand, one can find topological types of singularities (even with integral homology sphere link) such that for *any* analytic structure the strict inequality $p_g < \mathrm{Path}^{\uparrow}$ holds: For the next graph $\mathrm{Path}^{\uparrow} = 4$, nevertheless for all analytic structures $2 \leq p_g \leq 3$.



2.3. **Paths with fixed end–cycles.** We fix an arbitrary $Z \in L_{>0}$. We extend the above definition by taking paths γ with end–cycle l_t exactly Z. Accordingly, for any such fixed Z, we set $\mathrm{Path}^{\uparrow}(Z) := \min_{\gamma} S(\gamma)$, where γ runs over all increasing pathes with $l_0 = 0$ and $l_t = Z$. By similar argument as in subsection 2.2.2 one obtains for any $Z > 0$

$$h^1(\mathcal{O}_Z) \leq \mathrm{Path}^{\uparrow}(Z). \tag{2.3.1}$$

Lemma 2.3.2. *The following facts hold:*

(a) If $Z_1 \leq Z_2$ *then* $\mathrm{Path}^{\uparrow}(Z_1) \leq \mathrm{Path}^{\uparrow}(Z_2)$ *(this is true for any good resolution graph).*

(b) Assume additionally that $\lfloor Z_K \rfloor > 0$ *(e.g., when the resolution graph is minimal good). If* $\lfloor Z_K \rfloor \leq Z$ *then* $\mathrm{Path}^{\uparrow}(\lfloor Z_K \rfloor) = \mathrm{Path}^{\uparrow}(Z)$. *In particular,* $\mathrm{Path}^{\uparrow}(\lfloor Z_K \rfloor) = \max_{Z>0} \mathrm{Path}^{\uparrow}(Z) = \mathrm{Path}^{\uparrow}$.

Proof. (a) Take $Z_2 = Z_1 + E_v$. Fix a path $\gamma = \{l_i\}_i$ with $l_t = Z_2$. Let k be the largest index with $v(k) = v$. For any $i \in \{k+1, \ldots, t\}$ write $l_i = \bar{l}_i + E_v$. Then $\bar{l}_{k+1} = l_k$ and $\bar{l}_t = Z_1$. We will replace the path γ by the path $\bar{\gamma}$ consisting of $l_0, \ldots, l_k, \bar{l}_{k+2}, \ldots, \bar{l}_t$. Note that for $k+1 \leq i < t$ one has

$$\chi(l_i) - \chi(l_{i+1}) = \chi(\bar{l}_i) - \chi(\bar{l}_{i+1}) + (E_v, E_{v(i)}) \geq \chi(\bar{l}_i) - \chi(\bar{l}_{i+1}).$$

Hence $S(\bar{\gamma}) \leq S(\gamma)$. Then use induction.

(b) For any $Z > \lfloor Z_K \rfloor$ there exists $E_v \subset |Z - \lfloor Z_K \rfloor|$ such that $\chi(Z - E_v) \leq \chi(Z)$. Indeed, if not, then $(Z - Z_K, E_v) \geq 0$ (†) for any $E_v \subset |Z - \lfloor Z_K \rfloor|$. Let $\{Z_K\}_1$ be the part of $\{Z_K\}$ supported on $|Z - \lfloor Z_K \rfloor|$, and $\{Z_K\}_2 = \{Z_K\} - \{Z_K\}_1$. Hence, from (†), $(Z - \lfloor Z_K \rfloor - \{Z_K\}_1, E_v) \geq (\{Z_K\}_2, E_v) \geq 0$. This can happen only if $Z - \lfloor Z_K \rfloor - \{Z_K\}_1 = 0$. But Z is integral, hence $Z = \lfloor Z_K \rfloor$, a contradiction. Therefore, the path γ which realizes $\mathrm{Path}^{\uparrow}(\lfloor Z_K \rfloor)$ can be completed to a longer path from 0 to Z with the same $S(\gamma)$ (construct inductively a decreasing path from Z to $\lfloor Z_K \rfloor$ via the previous statement). Hence $\mathrm{Path}^{\uparrow}(Z) \leq \mathrm{Path}^{\uparrow}(\lfloor Z_K \rfloor)$. Then use (a). □

3. Elliptic singularities. The elliptic sequences.

3.1. **Elliptic singularities.** Let $Z_{min} \in L$ be the minimal cycle. Recall that (X, o) is called elliptic if $\chi(Z_{min}) = 0$, or equivalently, $\min_{l \in L_{>0}} \chi(l) = 0$ [La77, Wa70]. It is known that if we decrease the decorations (Euler numbers), or we take a full subgraph of an elliptic graph, then we get either elliptic or a rational graph.

Let C be the minimally elliptic cycle [La77, N99], that is, $\chi(C) = 0$ and $\chi(l) > 0$ for any $0 < l < C$. There is a unique cycle with this property, and if $\chi(D) = 0$ ($D \in L$) then necessarily $C \leq D$. In particular, $C \leq Z_{min}$. In the sequel we assume that the resolution is minimal. Then $Z_K \in \mathcal{S}'$, hence in the numerically Gorenstein case $Z_{min} \leq Z_K$ by the minimality of Z_{min} in $\mathcal{S} \setminus 0$.

The minimally elliptic singularities were introduced by Laufer in [La77]. In a minimal resolution they are characterized (topologically) by $Z_{min} = Z_K = C$. Moreover, (X, o) is minimally elliptic if and only if $p_g(X, o) = 1$ and (X, o) is Gorenstein. For details see [La77, N99, N99b].

For an arbitrary elliptic singularity the minimally elliptic cycle C supports a minimally elliptic singularity (resolution graph). One has the following lemma of Laufer.

Lemma 3.1.1. [La77] *Consider the minimal resolution of a minimally elliptic singularity.*

(a) Let $\{z_i\}_{i=1}^t$ be a computation sequence of Z_{min} with $z_1 = E_v$ for some v. Then $\chi(z_i) = 1$ for all $i < t$, $(z_i, E_{v(i)}) = 1$ for all $i < t-1$, and in the last step $(z_{t-1}, E_{v(t-1)}) = 2$.

(b) Fix any pair E_0 and E_1 ($E_0 \neq E_1$) of irreducible exceptional divisors. Then there exists a computation sequence for Z_{min} which starts with E_1 (i.e. $z_1 = E_1$) and ends with E_0 (i.e. $E_{v(t-1)} = E_0$). Moreover, let E_0 be an irreducible component whose coefficient in Z_{min} is greater than one. Then there exists a computation sequence for Z_{min} which starts and ends with E_0.

3.1.1. **Elliptic sequences.** One of the most important tools in the study of elliptic singularities are the *elliptic sequences*. It is defined from the combinatorics of the resolution graph. It can be regarded also as a sequence of cycles with decreasing supports, or also as resolution graphs of a sequence of singularities obtained by contracting the exceptional divisors supported in the corresponding cycles. They were introduced by Laufer and S. S.-T. Yau, for the definition in the general (non–Gorenstein) case see [Y79, Y80]. In the numerically Gorenstein case the construction is simpler, see additionally [N99, N99b, O05] as well.

First we recall the construction of the sequence in the general (not necessarily numerically Gorenstein) case according to S. S.-T. Yau, and we list several properties what we will need. Later we will provide another elliptic sequence in the non–numerically Gorenstein case, which was introduced in [NN18], whose definition 'adapts' the numerically Gorenstein case. The length of both sequences serve as upper bounds for the geometric genus of any analytic structure supported on the topological type identified by the graph. The sequence from [NN18] differs from the one introduced by Yau, however, our goal is to prove that their length is the same.

3.2. **The elliptic sequence, the general case, according to S. S.-T. Yau.** For any non–zero reduced effective cycle $D\in L_{>0}$ we write Z_D for the minimal cycle of the full subgraph determined by $|D|=D$.

Definition 3.2.1. [Y79], [Y80, Def. 3.3] Let E be the exceptional set of the minimal resolution $\phi:\widetilde{X}\to X$ of an elliptic singularity. Let C be the minimally elliptic cycle.

If $(C,Z_{min})<0$ then the elliptic sequence consists of one element, namely $\{Z_{min}\}$.

If $(C,Z_{min})=0$, let D_1 be the maximal connected subvariety (reduced effective cycle) of E containing the support $|C|$, such that $(E_v,Z_{min})=0$ for all $E_v\subset D_1$. Since $Z^2_{min}<0$, $D_1\neq E$.

Assume that the term D_{i-1} of the elliptic sequence is already defined. If $(C,Z_{D_{i-1}})=0$, let D_i be the maximal connected subvariety of D_{i-1}, containing $|C|$, such that $(E_v,Z_{D_{i-1}})=0$ for all $E_v\subset D_i$. Again, D_i is properly contained in D_{i-1}. This process stops after finitely many steps, say with D_ℓ, which has the property $(C,Z_{D_\ell})<0$.

Write $D_0:=E$, $Z_{D_0}:=Z_{min}$. Then the elliptic sequence is $\{Z_{D_0},\dots,Z_{D_\ell}\}$. Its length is $\ell+1$ ($\ell\geq 0$).

3.2.1. The sequence satisfies several properties, see e.g. [Y79, Y80]. E.g., the next ones are immediate. From the construction $|C|\subseteq D_\ell\subsetneq\cdots\subsetneq D_0$. Note also that $C=Z_{|C|}$ (valid for minimally elliptic singularities). Hence $C\leq Z_{D_\ell}<\cdots<Z_{D_0}$ too.

Moreover, by a general property of the minimal cycles, $h^0(\mathcal{O}_{Z_{D_j}})=1$. On the other hand $h^1(\mathcal{O}_{Z_{D_j}})\geq h^1(\mathcal{O}_C)=1$ and $\chi(Z_{D_j})\geq 0$ (ellipticity), hence $\chi(Z_{D_j})=0$ for all $0\leq j\leq \ell$. Furthermore, from the construction, $(Z_{D_k},Z_{D_j})=0$ for any $k\neq j$.

Lemma 3.2.2. *For any $0\leq k\leq \ell$ set $F_k:=\sum_{i=0}^k Z_{D_i}$. Then $\chi(F_k)=0$ and $F_k\in\mathcal{S}$.*

Proof. $\chi(F_k)=0$ follows from the above discussions. Next we prove $F_k\in\mathcal{S}$.

If $E_v\subset|D_j|$ for all $j\leq k$ then $(E_v,Z_{D_j})\leq 0$ for all j, hence $(F_k,E_v)\leq 0$.

Assume that $E_v\subset D_{j-1}$ but $E_v\not\subset D_j$ for some $1\leq j\leq k$. Then $(E_v,Z_{D_i})\leq 0$ for $i\leq j-1$ since $D_{j-1}\subset D_i$. If $(E_v,Z_{D_j})=0$, that is, E_v does not intersect the support D_j, then E_v does not intersect the smaller supports $\{D_i\}_{k\geq i\geq j}$ either, hence $(E_v,Z_{D_i})=0$ for all $i\geq j$. Hence we are done again.

Next, assume that $(E_v,Z_{D_j})>0$, hence E_v intersects D_j, say along the component E_u. Then we observe two facts. First, (†) $(E_v,Z_{D_{j-1}})<0$ since otherwise E_v would be in D_j. Second, Z_{D_j} can be completed by a computation sequence to Z_{D_0} by adding E_v at the first step, hence the multiplicity of E_u in Z_{D_j} should be 1 (by Laufer's algorithm, and from the fact that both $\chi(Z_{D_j})$ and $\chi(Z_{D_0})$ are zero). Therefore, (‡) $(Z_{D_j},E_v)=1$. Then (†) and (‡) imply $(Z_{D_j}+Z_{D_{j-1}},E_v)\leq 0$. If $j=k$ then again we are done.

If $j<k$ then $(Z_{D_j},C)=0$ and $Z_{D_{j+1}}$ exists, and it is a summand of F_k. We show that $(Z_{D_{j+1}},E_v)=0$. This means that E_v does not intersect the support D_{j+1} hence neither the smaller supports $\{D_i\}_{k\geq i\geq j+1}$, hence $(Z_{D_i},E_v)=0$ for all $i\geq j+1$.

Assume the opposite, that is, $(Z_{D_{j+1}},E_v)>0$. Then necessarily $(Z_{D_j},E_v)>0$ too. Then consider the cycle $l:=Z_{D_{j+1}}+Z_{D_j}+E_v$. Then $\chi(l)=\chi(Z_{D_{j+1}}+Z_{D_j})+1-(Z_{D_{j+1}}+Z_{D_j},E_v)=1-(Z_{D_{j+1}}+Z_{D_j},E_v)<0$, a fact which contradicts the ellipticity of the graph. □

Proposition 3.2.3. *Assume that the minimal resolution is good. Then* $\mathrm{Path}^{\uparrow}\leq \ell+1$.

Proof. Since $(C, Z_{D_m}) < 0$ there exists $E_0 \subset |C|$ with $(E_0, Z_{D_m}) < 0$. By Lemma 3.1.1 there exists a computation sequence $\{z_i\}_{i=1}^t$ of $C = Z_{|C|}$ such that $(z_i, E_{v(i)}) = 1$ for $i < t-1$ and $(z_{t-1}, E_{v(t-1)}) = 2$, where $E_{v(t-1)}$ is exactly E_0. We mark this step by (†). (The first cycle z_1 can be any base–cycle E_1 from $|C|$.)

This computation sequence can be completed to a computation sequence of Z_{D_j}, $\{z_i^{(j)}\}_{i=1}^{t(j)}$, such that for $t \le i < t(j)$ one has $(z_i^{(j)}, E_{v(i)}) = 1$ (for $\chi(Z_{D_j}) = \chi(C) = 0$).

If we concatenate these sequences, $\{z_i^{(0)}\}_i$, $\{z_i^{(1)}\}_i$, $\ldots, \{z_i^{(\ell)}\}$, we get $\{z_i^c\}_{i=1}^{\sum_j t(j)}$, which connects 0 (or E_1) to $F_\ell := \sum_{j=0}^{\ell} Z_{D_j}$, and in it exactly $\ell + 1$ times happens that $\chi(z_{i+1}^c) < \chi(z_i^c)$. When this happens then $\chi(z_{i+1}^c) = \chi(z_i^c) - 1$, and they occur exactly when we add the last component of C, namely during steps marked by (†).

Next, we continue the sequence $\{z^c\}_i$ with $F_\ell + \{z^c\}_i$. Note that F_ℓ has two key properties: $(F_\ell, E_0) = (Z_{D_\ell}, E_0) < 0$ and $F_\ell \in \mathcal{S}$, cf. Lemma 3.2.2. Therefore, $(F_\ell + z_i^c, E_{v(i)}) \le (z_i^c, E_{v(i)}) \le 2$, and $(F_\ell + z_i^c, E_{v(i)})$ might be 2 only at steps marked by (†). But at these steps $(F_\ell, E_{v(i)}) = (F_\ell, E_0) = (Z_{D_m}, E_0) < 0$, hence $(F_\ell + z_i^c, E_{v(i)}) < 2$ always, and the χ-values along the sequence $F_\ell + \{z^c\}_i$ are non-decreasing. This remains true for $nF_\ell + \{z^c\}_i$ for any $n \ge 1$, hence we get an infinite sequence $\{\ell_i\}_i$ whose multiplicities tend to infinity, and which satisfies $\sum_i \max\{0, \chi(\ell_i) - \chi(\ell_{i+1})\} = \ell + 1$. This proves the inequality $\mathrm{Path}^\uparrow \le \ell + 1$. □

Corollary 3.2.4. *For any analytic structure supported by an elliptic graph with length* $\ell + 1$ *one has* $p_g \le \ell + 1$.

Proof. Combine (2.2.3) with Proposition 3.2.3. □

Remark 3.2.5. S.S.-T. Yau in [Y79] considered another sequence, the 'Laufer sequence', and he proved that p_g is not greater than the length of the Laufer sequence. On the other hand, J. Stevens in (the first preprint version of) [St84] proved that the elliptic sequence and the Laufer sequence coincide. Hence, these two results imply the inequality of Corollary 3.2.4.

3.3. **The elliptic sequence in the numerically Gorenstein case.** See also [N99, N99b, O05].

The elliptic sequence consists of a sequence of integral cycles $\{Z_{B_j}\}_{j=0}^m$, where Z_{B_j} is the minimal cycle supported on the connected reduced cycle B_j. $\{B_j\}_{j=0}^m$ are defined inductively as follows. For $j = 0$ one takes $B_0 = E$, hence $Z_{B_0} = Z_{min}$. Then $C \le Z_{min} = Z_{B_0} \le Z_K$. If $Z_{B_0} = Z_K$ then we stop, $m = 0$, this situation corresponds to the minimally elliptic case.

Otherwise one takes $B_1 := |Z_K - Z_{B_0}|$. One verifies that $|C| \subseteq B_1 \subsetneq B_0$, B_1 is connected, and it supports a numerically Gorenstein elliptic topological type with canonical cycle $Z_K - Z_{B_0}$. (Furthermore, $(E_v, Z_{B_0}) = 0$ for any $E_v \subset B_1$. The proof of all these facts are similar to the proof of Lemma 3.4.1 below.) In particular, $C \le Z_{B_1} \le Z_K - Z_{B_0}$. Then we repeat the inductive argument. If $Z_{B_1} = Z_K - Z_{B_0}$, then we stop, $m = 1$. Otherwise, we define $B_2 := |Z_K - Z_{B_0} - Z_{B_1}|$. B_2 again is connected, $|C| \subseteq B_2 \subsetneq B_1$, and supports a numerically Gorenstein elliptic topological type with canonical cycle $Z_K - Z_{B_0} - Z_{B_1}$. After finite steps we get $Z_{B_m} = Z_K - Z_{B_0} - \cdots - Z_{B_{m-1}}$, hence the minimal cycle and the canonical cycle on B_m coincide. This means that B_m supports a minimally elliptic singularity with $Z_{B_m} = C$.

We say that the length of the elliptic sequence $\{Z_{B_j}\}_{j=0}^m$ is $m + 1$.

It is also convenient to introduce the notations

$$C_j = \sum_{i=0}^{j} Z_{B_i} \quad \text{and} \quad C'_j = \sum_{i=j}^{m} Z_{B_i} \quad (0 \le j \le m).$$

By these notations, $C_0 = Z_{min}$, $C'_m = C$, and $C_m = C'_0 = Z_K$.

The next lemma summarizes the immediate properties of the elliptic sequence.

Lemma 3.3.1. *(a)* $B_0 = E$, $B_1 = |Z_K - Z_{B_0}|$, $B_2 = |Z_K - Z_{B_0} - Z_{B_1}|$, $\ldots$, $B_m = |C|$; *each* B_j *is connected and the inclusions* $B_{j+1} \subset B_j$ *are strict. Moreover,* $Z_{min} = Z_{B_0} \supset Z_{B_1} \supset \cdots \supset Z_{B_m} = C$.

(b) If $E_v \subset B_{j+1}$ *then* $(E_v, Z_{B_j}) = 0$ *for all* v *and* j*. In particular,* $(Z_{B_i}, Z_{B_j}) = (C_i, Z_{B_j}) = 0$ *for all* $0 \leq i < j \leq m$.

(c) $Z_K = \sum_{i=0}^{m} Z_{B_i}$.

(d) $(E_v, C'_i) = (E_v, Z_K)$ *for any* $E_v \subset |C'_i|$*. In other words,* C'_i *is the canonical cycle of* $|C'_i| = B_i$.

(e) $C_i \in \mathcal{S}$.

Proof. *(a)–(d)* follow from the construction. The proof of *(e)* is as follows. If $E_v \subset B_i$ then $(E_v, Z_{B_j}) \leq 0$ for any $j \leq i$, hence $(E_v, C_i) \leq 0$. If $E_v \not\subset B_i$ then $(E_v, C_i) = (E_v, Z_K - C'_{i+1})$. Now, $(E_v, Z_K) \leq 0$ (by the minimality of the resolution) and $(E_v, C'_{i+1}) \geq 0$ (because $|C'_{i+1}| \subset B_i$). □

Proposition 3.3.2. *Fix a numerically Gorenstein elliptic minimal graph. Consider the elliptic sequence* $\{D_j\}_{j=0}^{\ell}$ *defined in 3.2.1 and* $\{B_j\}_{j=0}^{m}$ *defined in 3.3. Then* $m = \ell$ *and* $D_j = B_j$ *for any* j*. In particular (cf. Corollary 3.2.4),* $p_g \leq m + 1$.

Proof. Clearly, $D_0 = B_0 = E$. Moreover, the continuation of both sequences is decided by the same criterion: by 3.3.1*(b)* one has $(Z_{D_0}, C) = 0 \Leftrightarrow Z_K > Z_{min}$. Next we show that $D_1 = B_1$. From 3.3.1*(b)* we get $B_1 \subset D_1$. Assume that $B_1 \neq D_1$. Since D_1 is connected, then there exists $E_v \subset D_1$, in the support of $D_1 - B_1$, such that $(E_v, B_1) > 0$. Then, $(E_v, Z_{B_0}) = 0$, but $(E_v, Z_K - Z_{B_0}) = (E_v, \sum_{i\geq 1} Z_{B_i}) > 0$. Hence $(E_v, Z_K) > 0$, a fact which contradicts with the minimality of the resolution. Then we proceed by induction. □

Remark 3.3.3. Any numerically Gorenstein topological type admits a Gorenstein analytic structure [PP11]. Hence, any numerically Gorenstein elliptic topological type is realized by a Gorenstein elliptic analytic structure. For analytic characterizations of such structures see [N99]. One of the characterizations is that (X, o) is Gorenstein if and only if $p_g = m + 1$. Hence, the Gorenstein structure are exactly those ones which realizes the maximal $m + 1$.

3.4. **The elliptic sequence in the non–numerically Gorenstein case, according to** [NN18]**.** Assume that $Z_K \notin L$, that is, $r_{[Z_K]} \neq 0$. Since the resolution is minimal, $Z_K \in \mathcal{S}'$, hence $Z_K \geq s_{[Z_K]}$. Since the graph is not rational, by Lemma 2.1.4 $Z_K > s_{[Z_K]}$. We will use the following notations: $B_{-1} := E$, $Z_{B_{-1}} := s_{[Z_K]}$ and $B_0 := |Z_K - s_{[Z_K]}|$. (Note that $Z_{B_{-1}} \in L' \setminus L$.)

Lemma 3.4.1. [NN18] *(a)* B_0 *is connected,* $C \subseteq B_0 \subsetneq E$*, and* $(E_v, Z_{B_{-1}}) = 0$ *for any* $E_v \subset B_0$.

(b) B_0 *supports a numerically Gorenstein elliptic topological type with canonical cycle* $Z_K - s_{[Z_K]}$.

For the convenience of the reader we insert the proof from [NN18] here as well.

Proof. *(a)* Write $l := Z_K - s_{[Z_K]}$. Then $\chi(s_{[Z_K]}) = \chi(Z_K - l) = \chi(l)$. Since (X, o) is elliptic $\chi(s_{[Z_K]}) = \chi(l) \geq 0$ (†). Also, $(s_{[Z_K]}, l) \leq 0$ since $s_{[Z_K]} \in \mathcal{S}'$ (‡). On the other hand, $0 = \chi(Z_K) = \chi(l + s_{[Z_K]}) = \chi(l) + \chi(s_{[Z_K]}) - (l, s_{[Z_K]})$. Then by (†) and (‡) the expressions from the right hand side are ≥ 0, hence necessarily $\chi(l) = (l, s_{[Z_K]}) = 0$. If l has more connected components, say $\cup_i l_i$, then $\chi(l_i) = 0$ for all i, hence each l_i contains/dominates a minimally elliptic cycle (cf. [La77]), a fact which contradicts the uniqueness of the minimally elliptic cycle. Hence $|l| = B_0$ is connected and $|C| \subset B_0$. Furthermore, $(l, s_{[Z_K]}) = 0$ shows that $|l| \neq E$.

(b) $C \subseteq B_0 \subsetneq E$ shows that $\min_{|l|\subset B_0,\, l>0} \chi(l) = 0$, hence B_0 supports an elliptic topological type. Moreover, from $(l, s_{[Z_K]}) = 0$ we read that for any E_v from the support of l one has $(E_v, s_{[Z_K]}) = 0$, hence $(E_v, Z_K - s_{[Z_K]}) = (E_v, Z_K)$, hence $Z_K - s_{[Z_K]} \in L$ is the canonical cycle on B_0. □

Then, as a continuation of the sequence, starting from B_0 and its integral canonical class $Z_K - s_{[Z_K]}$ we construct the sequence $\{Z_{B_j}\}_{j=0}^m$ as in the numerically Gorenstein case.

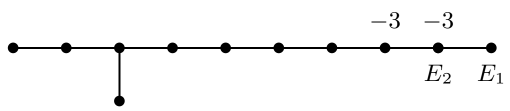
We say that the elliptic sequence $\{Z_{B_j}\}_{j=-1}^m$ has length $m+1$ and 'pre–term' $Z_{B_{-1}} = s_{[Z_k]} \in L'$.

In order to have a uniform notation, in the numerically Gorenstein case we set $Z_{B_{-1}} := 0$ (which, in fact, it is $s_{[Z_k]}$). In any case, from above (see also [N99, 2.11]), for latter references,

$$(E_v, Z_{B_j}) \ \text{ for any } E_v \subset B_{j+1} \quad (-1 \le j < m). \tag{3.4.2}$$

Set $C_t := \sum_{i=-1}^t Z_{B_i}$ and $C'_t := \sum_{i=t}^m Z_{B_i}$, $-1 \le t \le m$. E.g. $C_m = Z_K$ and, in general, C'_j is the canonical cycle of B_j. Furthermore, $\chi(Z_{B_j}) = \chi(C_j) = \chi(C'_j) = 0$.

Example 3.4.3. Consider the next elliptic graph



where the (-2)–vertices are unmarked. Z_K and $s_{[Z_K]}$ are

14/3 28/3 42/3 35/3 28/3 21/3 14/3 7/3 4/3 2/3   2/3 4/3 6/3 5/3 4/3 3/3 2/3 1/3 1/3 2/3

21/3   3/3

B_0 is obtained by deleting E_1 from E, while B_1 by deleting E_1 and E_2, hence $B_1 = |C|$. The length is $m+1 = 2$. Furthermore, $C_{-1} = s_{[Z_K]}$, $C_0 = s_{[Z_K]} + Z_{B_0}$ and $C_1 = Z_K = s_{[Z_K]} + Z_{B_0} + Z_{B_1}$; they are not integral cycles.

On the other hand, $D_0 = E$ and $D_1 = |C|$ (since $(Z_{min}, E_2) < 0$). $F_0 = Z_{min}$ (which equals $Z_{B_0} + E_1$) and $F_1 = Z_{min} + C$. These are integral cycles. The length is $\ell + 1 = 2$.

In the above example E_2 from the support of B_0 satisfies $(Z_{min}, E_2) < 0$. This is a general phenomenon, a fact, which provides the 'starting bridge' between the two elliptic sequences $\{D_j\}_j$ and $\{B_j\}_j$.

Proposition 3.4.4. *(a) There exists E_v in the support of B_0 with $(E_v, Z_{min}) < 0$.*

(b) Any numerically Gorenstein connected subgraph is contained in B_0. In particular, the largest numerically Gorenstein connected subgraph is B_0.

Proof. (a) Though the statement is topological, it is convenient to fix a special analytic structure on (X, o), which produces a very fast and elegant proof. Since $Z_{min} \in \mathcal{S}$, there exists an analytic structure for which this cycle is realized as a divisor of $f \circ \phi$ for a certain function f [P01]. Assume that $(E_v, Z_{min}) = 0$ for any $E_v \subset B_0$. Then the strict transforms of $\{f = 0\}$ do not intersect B_0, hence $\mathcal{O}_{\widetilde{X}}(-Z_{min})|_{Z_K(B_0)}$ is trivialized by f. Therefore, using Lemma 2.1.4 for the structure sheaf, we get that $h^1(Z_k - s_{[Z_K]}, \mathcal{O}_{\widetilde{X}}(-Z_{min})) = h^1(Z_k - s_{[Z_K]}, \mathcal{O}_{Z_k - s_{[Z_K]}}) = p_g$. On the other hand, using Lemma 2.1.4 for $\mathcal{O}_{\widetilde{X}}(-Z_{min})$ we get $h^1(Z_k - s_{[Z_K]}, \mathcal{O}_{\widetilde{X}}(-Z_{min})) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_{min}))$. In particular, $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_{min})) = p_g$.

Now consider the long cohomology exact sequence associated with $0 \to \mathcal{O}_{\widetilde{X}}(-Z_{min}) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{Z_{min}} \to 0$, and using the well–know fact that $H^0(\mathcal{O}_{\widetilde{X}}) \to H^0(\mathcal{O}_{Z_{min}}) = \mathbb{C}$ is onto, we get that $h^1(\mathcal{O}_{\widetilde{X}}(-Z_{min})) = p_g - h^1(\mathcal{O}_{Z_{min}}) = p_g - 1$. This leads to a contradiction.

(b) Let I be a connected support of a numerically Gorenstein subgraph. Let the canonical cycle on I be $Z \in L$. Then $(Z_K - Z, E_v) = 0$ for all $E_v \subset |Z|$. Else, if $E_v \not\subset |Z|$, we have $(Z_K, E_v) \le 0$ and $(Z, E_v) \ge 0$, so $(Z_K - Z, E_v) \le 0$. This means, that $Z_K - Z \in S'$. Therefore, $Z_K - Z \ge s_{[Z_K]}$. This reads as $Z \le Z_K - s_{[Z_K]}$, or $I \subset B_0$. $\square$

Remark 3.4.5. For a non–numerically Gorestein graph one can consider both elliptic sequences, namely $\{D_j\}_{j=0}^{\ell}$ and $\{B_j\}_{j=0}^{m}$. For the first one we know from Corollary 3.2.4 that $p_g \leq \ell+1$. For the second one we know from Lemma 2.1.4 that $p_g = h^1(\mathcal{O}_{B_0})$ and also $h^1(\mathcal{O}_{B_0}) \leq m+1$, cf. [N99]. Furthermore, we know that on $Z_K(B_0)$ the maximal $p_g = m+1$ can be realized by (any) Gorenstein structure, see also Remark 3.3.3. (This is one of the main advantages of the sequence $\{B_j\}_{j=0}^{m}$: it produces numerically Gorenstein supports, and the elliptic length of the numerically Gorenstein support B_0 coincides with the length of Γ.) The Gorenstein analytic structure of B_0 (or of a small tubular neighbourhood of $\cup_{v\in B_0} E_v$ in $\widetilde{X}$) can be extended to an analytic structure of $\widetilde{X}$. This shows that the graph Γ supports an analytic structure with $p_g = m+1$.

The analogous statement for $\{D_j\}_{j=0}^{\ell}$ (which guarantees the existence of any analytic structure with $p_g = \ell+1$) is not clear yet. This will be a consequence of the next theorem.

Theorem 3.4.6. *Fix any (not necessarily numerically Gorenstein) elliptic graph (associated with a minimal resolution). Then*

(a) $m+1 = \ell+1$.

(b) In particular, there exists an analytic type for which $p_g = \ell+1$.

(c) Assume that the minimal resolution is good. Then $m+1 = \ell+1 = \mathrm{Path}^{\uparrow}$. *Therefore, the general topological upper bound* $p_g \leq \mathrm{Path}^{\uparrow}$ *for* p_g *in the case of elliptic singularities is sharp: the equality can be realized by some analytic structure.*

Proof. The direction $m+1 \leq \ell+1$ follows from the discussion from Remark 3.4.5: there exists an analytic structure with $p_g = m+1$; hence from Lemma 3.2.4 one has $m+1 = p_g \leq \ell+1$.

We prove $m+1 \geq \ell+1$ by induction on ℓ. If $\ell = 0$ it is trivial. Next assume that $\ell > 0$ and we know the statement for singularities with 'D–length' ℓ.

By Proposition 3.4.4 $B_0 \not\subset D_1$ (†). Next, denote by $B_0(D_1)$ the 'B_0 term' of the elliptic sequence associated with D_1. Then $B_0(D_1) \subset B_0$. Indeed, $B_0(D_1)$ is included in D_1 and it is a connected numerically Gorenstein support, hence by Proposition 3.4.4*(b)*, $B_0(D_1) \subset B_0$. But this inclusion should be strict. Indeed, $B_0 \neq B_0(D_1)$ contradicts (†). Hence $B_0(D_1) \subsetneq B_0$.

Let us denote by $maxp_g(D)$ the maximum p_g which can be realized by different analytic structures supported on a connected support/subgraph D.

Now, $maxp_g(D_1) = maxp_g(B_0(D_1))$ by Lemma 2.1.4. Since $B_0(D_1) \subset B_0$ we have $maxp_g(B_0(D_1)) \leq maxp_g(B_0)$. However, since B_0 is a numerically Gorenstein support, and its maximal p_g is realized by a Gorenstein structure, which has the property that its cohomological cycle is exactly its canonical cycle with support B_0, any smaller support has strict smaller $maxp_g$. Since $B_0(D_1) \subsetneq B_0$, we get that $maxp_g(B_0(D_1)) < maxp_g(B_0) = m+1$.

On the other hand, the D-length of D_1 is ℓ (since the D–elliptic sequence of D_1 is $\{D_1,\ldots,D_\ell\}$). Hence for D_1 the inductive step works. In particular, $maxp_g(D_1) = \ell$. This combined with the statements from the previous paragraph gives $m+1 = maxp_g(B_0) > maxp_g(B_0(D_1)) = maxp_g(D_1) = \ell$. That is, $m+1 \geq \ell+1$.

From $m+1 = \ell+1$ and Remark 3.4.5 we get that there exists an analytic structure with $p_g = \ell+1$. This combined with $p_g \leq \mathrm{Path}^{\uparrow} \leq \ell+1$ (valid for any analytic structure, cf. (2.2.3) and Proposition 3.2.3) we get $maxp_g(E) = \mathrm{Path}^{\uparrow} = \ell+1$. □

Remark 3.4.7. Both elliptic sequences $\{D_j\}_j$ and $\{B_j\}_j$ have some geometric universal properties. For more information (and proofs) the reader is invited to consult the references below. Here we mention only the next chosen ones (they will be not applied in this form in this paper, though some related partial statements were already used).

(a) [NBook] If $l\in\mathcal{S}$ and $\chi(l)=0$ then $l\in\{0,F_0,\dots,F_\ell\}$.
(b) [NN18] If $l'\in\mathcal{S}'$, $[l']=[Z_K]$, and $l'\leq Z_K$ then $l'\in\{C_{-1},C_0,\dots C_m\}$.
(c) [NN18] The support of any numerically Gorenstein connected subgraph belongs to $\{B_i\}_{i=0}^m$.

4. Review of surgery formulae for the Seiberg–Witten invariant

We fix a complex normal surface singularity (X,o) and one of its good resolutions $\phi:\widetilde{X}\to X$. In the sequel we will review some topological invariants associated with the link M and with the resolution graph Γ (or, with the lattice L). We will adopt all the notations of Section 2. In particular, we will assume that M is a rational homology sphere. We will write also $M=M(\Gamma)$, where we think about it as the plumbed manifold associated with Γ. For more information and more details see [CDGZ04, CDGZ08, N11, NN02, BN10, LNN17, LNN18]. For an overview see also [N18, NBook].

4.1. **The Seiberg–Witten invariants of the link.** The smooth oriented 4–manifold $\widetilde{X}$ admits several *spin*^{c}–structures. Let $\widetilde{\sigma}_{can}$ be the *canonical spin*^{c}*–structure on* $\widetilde{X}$ identified by $c_1(\widetilde{\sigma}_{can})=-K$. Furthermore, let $\sigma_{can}\in\mathrm{Spin}^c(M)$ be its restriction to M, called the *canonical spin*^{c}*–structure on* M. $\mathrm{Spin}^c(M)$ is an H–torsor, hence the number of *spin*^{c}–structures supported on the oriented 3–manifold M is $|H|$. In this note we will focus only on the canonical one.

We denote by $\mathfrak{sw}_\sigma(M)\in\mathbb{Q}$ the *Seiberg–Witten invariant* of M indexed by the *spin*^{c}–structures $\sigma\in\mathrm{Spin}^c(M)$ (cf. [Lim00, Nic04]). (We will use the sign convention of [BN10, N11].) Again, in this note we focus merely on the SW–invariant associated with the canonical *spin*^{c}–structure, $\mathfrak{sw}_{can}(M)$.

In fact, it is more convenient (imposed by surgery formulae) to use the *modified Seiberg–Witten invariant* defined by

$$\overline{\mathfrak{sw}}_0(M)=-\frac{K^2+|\mathcal{V}|}{8}-\mathfrak{sw}_{\sigma_{can}}(M). \tag{4.1.1}$$

There are several combinatorial expressions established for the Seiberg–Witten invariants. For rational homology spheres, Nicolaescu [Nic04] showed that $\mathfrak{sw}(M)$ is equal to the Reidemeister–Turaev torsion normalized by the Casson–Walker invariant. In the case when M is a negative definite plumbed rational homology sphere, combinatorial formula for Casson–Walker invariant in terms of the plumbing graph can be found in Lescop [Les96], and the Reidemeister–Turaev torsion is determined by Némethi and Nicolaescu [NN02] using Dedekind–Fourier sums.

A different combinatorial formula of $\{\mathfrak{sw}_\sigma(M)\}_\sigma$ was proved in [N11] using qualitative properties of the coefficients of the topological multivariable series ('zeta function') $Z(\mathbf{t})$. This note also will exploit this connection further.

4.2. **The topological Poincaré series** $Z(\mathbf{t})$**.** The *multivariable topological Poincaré series* is the Taylor expansion $Z(\mathbf{t})=\sum_{l'}z(l')\mathbf{t}^{l'}\in\mathbb{Z}[[L']]$ at the origin of the 'rational function'

$$f(\mathbf{t})=\prod_{v\in\mathcal{V}}(1-\mathbf{t}^{E_v^*})^{\delta_v-2}, \tag{4.2.1}$$

where $\mathbf{t}^{l'}:=\prod_{v\in\mathcal{V}}t_v^{l'_v}$ for any $l'=\sum_{v\in\mathcal{V}}l'_vE_v\in L'$ ($l'_v\in\mathbb{Q}$). It has a natural and unique decomposition according to the elements of $h\in H$ defined by $Z(\mathbf{t})=\sum_{h\in H}Z_h(\mathbf{t})$, where $Z_h(\mathbf{t})=\sum_{[l']=h}z(l')\mathbf{t}^{l'}$. Corresponding to the choice of the canonical *spin*^{c}–structures here we make the choice of the series $Z_0(\mathbf{t})$ associated with $h=0$. In this subseries $Z_0(\mathbf{t})$ of $Z(\mathbf{t})$ all the exponents belong to L (hence, it is a 'genuine' series). The expression (4.2.1) shows that $Z(\mathbf{t})$ is supported in the *Lipman cone* $\mathcal{S}'$, in particular $Z_0(\mathbf{t})$ is supported in $\mathcal{S}=\mathcal{S}'\cap L$.

Recall that all the entries of E_v^* are strict positive, hence for any $x\in L$, $\{l'\in\mathcal{S}'\,:\,l'\not\geq x\}$ is finite. In particular the next 'counting function' of the coefficients of Z_h ($h\in H$) is well–defined:

$$Q_h:\{x\in L'\,:\,[x]=h\}\to\mathbb{Z},\qquad Q_h(x)=\sum_{l'\not\geq x,\,[l']=h} z(l'). \tag{4.2.2}$$

The point is that for x 'sufficiently deeply inside of the Lipman cone' the function $x\mapsto Q_h(x)$ behaves as a quasipolynomial $\mathfrak{Q}_h(x)$. Furthermore, the values $\mathfrak{Q}_h(0)$ (indexed by all $h\in H$) provide the modified Seiberg–Witten invariants of the link (indexed by the *spin*^{c}–structures) [N11]. E.g., $\mathfrak{Q}_0(0)$ is exactly $\overline{\mathfrak{sw}}_0(M)$. The value $\mathfrak{Q}_0(0)$ is called the 'periodic constant' of the series $Z_0(\mathbf{t})$. In this note we try to bypass the theory of periodic constants and the theory of quasipolynomials associated with counting functions, since in the final arguments we will not need them; in this overview we mention them just to show the line of ideas behind the scenes.

The point is that important surgery formulae are also formulated in terms of 'periodic constants' [BN10, LNN17, LNN18]. Here we will recall the most general (and recent) one.

4.3. **A surgery formula.** [LNN17] Let $\mathcal{I}\subset\mathcal{V}$ be an arbitrary non–empty subset of $\mathcal{V}$, and write $\mathcal{V}\setminus\mathcal{I}$ as the union of full connected subgraphs $\cup_i\Gamma_i$. Then one has the following formula:

$$\overline{\mathfrak{sw}}_0(M(\Gamma))-\sum_i\overline{\mathfrak{sw}}_0(M(\Gamma_i))=\mathrm{pc}(Z_0(\mathbf{t}_{\mathcal{I}})), \tag{4.3.1}$$

where $Z_0(\mathbf{t}_{\mathcal{I}})$ is the series with reduced variables defined as $Z_0(\mathbf{t}_{\mathcal{I}}):=Z_0(\mathbf{t})|_{t_v=1,v\notin\mathcal{I}}$, and $\mathrm{pc}(Z_0(\mathbf{t}_{\mathcal{I}}))$ is its periodic constant.

Since the periodic constant is determined by a complicated regularization procedure using the asymptotic behaviour of the counting function of the coefficients of the corresponding series, usually it is hardly computable. This is the reason why is desired to find a replacement for it. The next formula determines it in terms of a concrete finite sum (precise evaluation of the 'dual' counting function). Behind this result the key ingredients are the H–equivariant multivariable Ehrhart theory of quasipolynomials associated with the above Poincaré series [LN14, L13], and the Ehrhart–Macdonald–Stanley equivariant reciprocities (combined with the duality of L' and the series $Z(\mathbf{t})$ induced by $l\leftrightarrow Z_K-l$). The next identity, proved in [LNN18, Theorem 4.4.1 (b)], shows that $\mathrm{pc}(Z_0(\mathbf{t}_{\mathcal{I}}))$ equals the value of the counting function associated with the coefficients of $Z_{[Z_K]}(\mathbf{t}_{\mathcal{I}})$ evaluated at Z_K:

$$\mathrm{pc}(Z_0(\mathbf{t}_{\mathcal{I}}))=Q_{[Z_K],\mathcal{I}}(Z_K):=\sum_{l'|_{\mathcal{I}}\not\geq Z_K|_{\mathcal{I}},\ [l']=[Z_K]} z(l'), \tag{4.3.2}$$

where $l'|_{\mathcal{I}}$ is the projection of l' to the variables $v\in\mathcal{I}$ (if $l'=\sum_v r_vE_v$ then $l'|_{\mathcal{I}}=\sum_{v\in\mathcal{I}} r_vE_v$).

5. The Seiberg–Witten invariant of links of elliptic singularities

5.1. **The canonical SW invariant of the plumbed manifold of an elliptic graph.**

We fix an elliptic graph Γ as in Section 3 and we will use all the notations of that section.

Above we discussed already two topological invariants of M (or Γ), namely the length of the elliptic sequence (defined in two different ways), $m+1=\ell+1$, and also $\mathrm{Path}^{\uparrow}$. Theorem 3.4.6 established their coincidence. The previous section introduced a third invariant, namely $\overline{\mathfrak{sw}}_0(M(\Gamma))$.

Theorem 5.1.1. $\overline{\mathfrak{sw}}_0(M(\Gamma))=m+1$.

Proof. In the proof we will use an inductice procedure based on the structure of the elliptic sequence $\{B_j\}_{j=-1}^m$ from subsection 3.3 and 3.4. (We also write $B_{m+1}:=\emptyset$.)

The proof is given in two steps separating the numerically and non–numerically Gorenstein cases.

Case 1. Assume that Γ is numerically Gorenstein. We will use induction on $m \geq 0$. If $m = 0$ then the graph is minimally elliptic. In this case any analytic structure supported on Γ is Gorenstein with $p_g = 1$, and they are also splice quotients. Hence for them the Seiberg–Witten Invariant Conjecture from [NN02] holds, that is, $\overline{\mathfrak{sw}}_0(M(\Gamma)) = p_g$ (proved in [NO08, N12]), hence $\overline{\mathfrak{sw}}_0(M(\Gamma)) = 1$. Otherwise, the $m = 0$ case can also be proved by adopting the next inductive argument by comparing the graph supported on B_0 by the empty set.

Next, we run induction. Assume that the statement is already proved for a graph with length m and we fix some elliptic Γ with length $m+1$. We fix $I := B_0 \setminus B_1$. Since $M(\Gamma(B_1))$ is minimally elliptic with length m, we know from the inductive step that $\overline{\mathfrak{sw}}_0(M(\Gamma(B_1))) = m$. On the other hand, from (4.3.1) and (4.3.2) we have

$$\overline{\mathfrak{sw}}_0(M(\Gamma(B_0))) = \overline{\mathfrak{sw}}_0(M(\Gamma(B_1))) + Q_{[Z_K],\mathcal{I}}(Z_K).$$

Hence, we need to show that $Q_{[Z_K],\mathcal{I}}(Z_K) = 1$. In this numerically Gorenstein case $Z_K \in L$, hence $[Z_K] = 0 \in H$, and the expression of $Q_{[Z_K],\mathcal{I}}(Z_K)$ from the right hand side of (4.3.2) becomes $\sum z(l)$, summed over $l \in L$ with $l|_{\mathcal{I}} \ngeq Z_K|_{\mathcal{I}}$.

Since Z_0 is supported in $\mathcal{S}$, any $l \neq 0$ in the support of Z_0 has the property that $l \geq Z_{min}$. On the other hand, along $\mathcal{I} = B_0 \setminus B_1$ we have $Z_K|_{\mathcal{I}} = Z_{min}|_{\mathcal{I}}$, cf. Lemma 3.3.1*(c)*. This means that any $l \neq 0$ form the support of Z_0 satisfies $l|_{\mathcal{I}} \geq Z_K|_{\mathcal{I}}$. In particular, in the sum only one term is non–zero, namely the one corresponding to $l = 0$ with $z(0) = 1$.

Case 2. Assume that Γ is an elliptic graph of length $m+1$ with $Z_K \notin L$. Now we set $\mathcal{I} := B_{-1} \setminus B_0 = E \setminus B_0$. Since B_0 supports a numerically Gorentein graph, from Step 1 we already know that $\overline{\mathfrak{sw}}_0(M(\Gamma(B_0))) = m+1$. We wish to show that $\overline{\mathfrak{sw}}_0(M(\Gamma(B_{-1}))) = m+1$ too. Hence from the surgery formula (4.3.1) we need to verify that the following sum is zero:

$$\sum_{l'|_{\mathcal{I}} \ngeq Z_K|_{\mathcal{I}},\ [l']=[Z_K]} z(l').$$

Now, we know that $z(l') = 0$ unless $l' \in S'$. However, if $l' \in S'$ and $[l'] = [Z_K]$, then $l' \geq s_{[Z_K]}$. But $(s_{[Z_K]})|_{\mathcal{I}} = Z_K|_{\mathcal{I}}$ (cf. 3.4). This reads as $l'|_{\mathcal{I}} \geq Z_K|_{\mathcal{I}}$ for any relevant l', which means that the above summation is summed over the empty set. □

Remark 5.1.2. For an normal surface singularity (X, o) with link M we say that the Seiberg–Witten Invariant Conjecture (SWIC) is satisfied if $p_g(X, o) = \overline{\mathfrak{sw}}_0(M))$. For details and several examples see [N07, N12, NO08, NO09, NS16, NW90]. By our Theorem 3.4.6 and 5.1.1 we obtain that SWIC is satisfied by any elliptic singularity with rational homology sphere link, such that the restriction of the analytic structure to B_0 is Gorenstein. (The identity $\ell + 1 = \overline{\mathfrak{sw}}_0(M))$ can be proved using the techniques of the lattice cohomology and graded roots as well, since the Seiberg–Witten invariant can also be realized as the Euler characteristic of the lattice cohomology, cf. [N05, N07, N08b, N05, N11, NBook]. The identity $p_g = \ell + 1$ was known in the Gorenstein elliptic case, cf. Remark 3.3.3.) The present proof shows the 'power' of the combination of [LNN17] and [LNN18]: they provide a surprisingly short proof for the p_g–formula in this elliptic case.

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