# On L(2, 1)-labelings of some products of oriented cycles

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#### Abstract

We refine two results of Jiang, Shao and Vesel on the L(2, 1)-labeling number  $\lambda$  of the Cartesian and the strong product of two oriented cycles. For the Cartesian product, we compute the exact value of  $\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n})$  for  $m, n \geq 40$ ; in the case of strong product, we either compute the exact value or establish a gap of size one for  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n})$  for  $m, n \geq 48$ .

#### 1 Introduction

A L(p,q)-labeling, or L(p,q)-coloring, of a graph G is a function  $f: V(G) \to \{0, \ldots, k\}$  such that  $|f(u) - f(v)| \ge p$ , if  $e = uv \in E(G)$ ; and  $|f(u) - f(v)| \ge q$ , if there is a path of length two in G joining u and v. To take into account the number of colors used, we say that f is a k-L(p,q)-labeling of G (note that, for historical reasons, the colorings are assumed to start with the label 0). The minimum value of k such that G admits a k-L(p,q)-labeling is denoted by  $\lambda_{p,q}(G)$ , and it is called the L(p,q)-labeling number of G.

The particular case of L(p, q)-labelings that attracted the most attention is p = 2 and q = 1, the L(2, 1)-labeling. It was introduced by Yeh [6], and it traces back to the frequency assignment problem of wireless networks introduced by Hale [3]. In this case, we write  $\lambda(G)$  instead of  $\lambda_{2,1}(G)$  for short.

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The definitions above can be extended to oriented graphs (a directed graph whose underlying graph is simple), namely: if G is an oriented graph, a L(p,q)labeling of G is a function  $f: V(G) \to \{0, \ldots, k\}$  such that  $|f(u) - f(v)| \ge p$ , if  $uv \in E(G)$ ; and  $|f(u) - f(v)| \ge q$ , if there is a *directed* path of length two in G joining u and v. The corresponding L(p,q)-labeling number is again denoted by  $\lambda_{p,q}(G)$  (in some papers, the notation  $\overrightarrow{\lambda}_{p,q}(G)$  is used instead). The L(2, 1)-labelings of oriented graphs were first studied by Chang and Liaw [2], and the L(p,q)-labeling problem has been extensively studied since then in both undirected and directed versions. We refer the interested reader to the excellent surveys of Calamoneri [1] and Yeh [7].

In this paper, we study the L(2, 1)-labeling number of the Cartesian and the strong product of two oriented cycles, improving results of Jiang, Shao and Vesel [4]. We use the notation  $\overrightarrow{C_n}$  to represent the oriented cycle on n vertices, i.e., the digraph such that  $V(\overrightarrow{C_n}) = \{1, 2, \ldots, n\}$  and  $E(\overrightarrow{C_n}) = \{(1, 2), (2, 3), \ldots, (n - 1, n), (n, 1)\}, n \geq 3$ . In the case of Cartesian product, we compute the exact value of  $\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n})$  for  $m, n \geq 40$ ; in the case of strong product, we either compute the exact value or establish a gap of size one for  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n})$  for  $m, n \geq 48$ .

### 2 Cartesian product

The Cartesian product of two graphs (resp. digraphs) G and H is the graph (resp. digraph)  $G \Box H$  such that  $V(G \Box H) = V(G) \times V(H)$ , and where there is an edge joining (a, x) and (b, y) if  $ab \in E(G)$  and x = y, or if a = b and  $xy \in E(H)$  (resp. there is an edge pointing from (a, x) to (b, y) if  $ab \in E(G)$  and x = y, or if a = b and  $xy \in E(H)$  (resp. there is an edge pointing from (a, x) to (b, y) if  $ab \in E(G)$  and x = y, or if a = b and  $xy \in E(H)$ ).

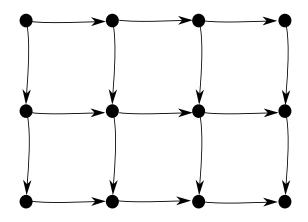


Figure 1: The Cartesian product of  $\overrightarrow{P_3}$  and  $\overrightarrow{P_4}$ 

Let  $S(m,n) = \{am + bn : a, b \ge 0 \text{ integers not both zero}\}$ . A classical result of Sylvester [5] states that  $t \in S(m, n)$  for all integers  $t \ge (m-1)(n-1)$  that are divisible by gcd(m, n), the greatest common divisor of m and n.

In [4], Jiang, Shao and Vesel prove the following theorem:

**Theorem 1** ([4]). For all  $m, n \in S(5, 11), 4 \leq \lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \leq 5$ . In particular, the result holds for every  $m, n \ge 40$ .

Our result in this section determines the exact value of  $\lambda$  in the range above. We start with a lemma which is a slightly stronger version of Lemma 5 from [4] that can be obtained with the same proof, which we include here for the sake of completeness.

**Lemma 1.** For every  $m, n \geq 3$  and every  $4 \cdot L(2,1)$ -labeling f of  $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ , the following periodicity condition holds:

$$f(i,j) = f(i+1 \mod m, j-1 \mod n) \text{ for all } i \in [m], j \in [n].$$
(1)

*Proof.* Let f be a 4-L(2,1)-labeling of  $G = \overrightarrow{C_m} \Box \overrightarrow{C_n}$ . We write f(i,j) for the value of f at the vertex (i, j). By the symmetry of the graph, it is enough to prove that f(2,2) = f(1,3). Suppose, for the sake of contradiction, that this is not the case.

Consider the  $\overrightarrow{P_2} \square \overrightarrow{P_2}$  subgraph spanned by  $\{(1,2), (1,3), (2,2), (2,3)\}$ . Every vertex is at distance at most 2 from each other, expect for the pair (2, 2), (1, 3). This implies, together with our assumption that  $f(2,2) \neq f(1,3)$ , that every vertex of this subgraph gets a distinct color. It is clear that the color 2 cannot be used in any vertex v of this subgraph, since otherwise the two neighbours of v must receive colors 0 and 4, and there is no color left for the fourth vertex. Thus, the colors used on the vertices of this subgraph are exactly 0, 1, 3 and 4, in some order.

Using the fact that 4 - f is a 4-L(2, 1)-labeling of a graph whenever f is, we may assume without loss of generality that  $f(2,2) \in \{0,1\}$ . This implies that  $f(1,3) \in \{0,1\}$  and  $\{f(1,2), f(2,3)\} = \{3,4\}$ . If f(2,3) = 3, there is no color for (3,3), since its within distance two from (2,2) and (1,3), colored with 0 and 1, and it is neighbor of a vertex of color 3. If f(1,2) = 3, the same argument applies for the vertex (1, 1). 

We call labelings with the property of Lemma 1 diagonal.

The following lemma from [4] combined with the result of Sylvester will also help us:

**Lemma 2.** (Lemmas 2 and 3 in [4]) Let  $m, n, p \ge 3$  and  $t, k \ge 1$  be integers. If  $\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \le k$  and  $\lambda(\overrightarrow{C_p} \Box \overrightarrow{C_n}) \le k$ , then  $\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \le k$ . In particular, if m and n are such that  $\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n}) \le k$ ,  $\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_m}) \le k$ and  $\lambda(\overrightarrow{C_n} \Box \overrightarrow{C_n}) \le k$ , then  $\lambda(\overrightarrow{C_a} \Box \overrightarrow{C_b}) \le k$  for all  $a, b \in S(m, n)$ , and hence for all  $a, b \ge (m-1)(n-1)$  divisible by gcd(m, n).

**Theorem 2.** Let  $m, n \ge 40$ . Then:

$$\lambda(\overrightarrow{C_m} \Box \overrightarrow{C_n}) = \begin{cases} 4, & \text{if } \gcd(m, n) \ge 3; \\ 5, & \text{otherwise.} \end{cases}$$

*Proof.* For  $m, n \geq 3$ , let G denote the graph  $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ , i.e.,  $V(G) = [m] \times [n]$  and the directed edges of G point from (i, j) to  $(i + 1 \mod m, j)$  and to  $(i, j + 1 \mod n)$ , for every  $i \in [m], j \in [n]$ . For a labeling f, we write f(i, j) instead of f((i, j)) for short.

Let  $d = \operatorname{gcd}(m, n)$  and assume first that  $d \notin \{1, 2\}$ . According to Lemma 2, it is enough to prove that  $\lambda(\overrightarrow{C_d} \Box \overrightarrow{C_d}) = 4$ . Any 4-L(2, 1)-labeling f of  $\overrightarrow{C_d}$  can be extended to a 4-L(2, 1)-labeling f' of  $\lambda(\overrightarrow{C_d} \Box \overrightarrow{C_d})$  by setting  $f'(i, j) = f(i+j \mod d)$ . It suffices to show, then, that  $\lambda(\overrightarrow{C_d}) = 4$ .

If  $d \equiv 0 \pmod{3}$ , then we can label  $\overrightarrow{C_d}$  with d/3 blocks 024. If  $d \equiv 1 \pmod{3}$ , we label  $\overrightarrow{C_d}$  with (d-4)/3 consecutive blocks 024 and then one block 0314. Finally, if  $d \equiv 2 \pmod{3}$ , then we label  $\overrightarrow{C_d}$  with (d-2)/3 consecutive blocks 024 and then a block 13.

On the other hand, assume for the sake of contradiction that  $d \in \{1, 2\}$  and there is a 4-L(2, 1)-labeling f of  $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ . In particular,  $m \neq n$ , so let us assume that m > n.

It is easy to check that, if  $m \ge n+3$ , f induces a valid 4-L(2, 1)-labeling of  $\overrightarrow{C_{m-n}} \Box \overrightarrow{C_n}$ . In fact, let g(i,j) = f(i,j) for all  $1 \le i \le m-n$  and  $1 \le j \le n$ . We claim that g is a 4-L(2, 1)-labeling of  $\overrightarrow{C_{m-n}} \Box \overrightarrow{C_n}$ , which, in particular, satisfies (1) as well.

Indeed, all we have to check is that the following conditions hold for g, since the other restrictions are inherited by  $f: |g(m-n-1,j) - g(1,j)| \ge 1, |g(m-n,j) - g(1,j)| \ge 2, |g(m-n,j) - g(2,j)| \ge 1, |g(m-n,j) - g(1,j+1 \mod n)| \ge 2,$ for every  $j \in [n]$ . All these conditions follow from  $g(m-n-1,j) = f(m-n-1,j) = f(m-n-1,j) = f(m-1,j+n \mod n) = f(m-1,j)$  and  $g(m-n,j) = f(m-n,j) = f(m,j+n \mod n) = f(m,j)$ , which result from the application of (1) n times, together with the fact that f is a L(2,1)-labeling of  $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ .

Applying this argument consecutively, using the fact that  $d = \gcd(m, n)$  and by the symmetry of the factors of the product, we conclude that f induces a 4-L(2, 1)-labeling c of either  $\overrightarrow{C}_{k+1} \Box \overrightarrow{C}_k$  or  $\overrightarrow{C}_{k+2} \Box \overrightarrow{C}_k$ , for some  $k \ge 3$ . This is a contradiction, since in this case we would have, by Lemma 1, c(1, 1) = c(2, k) = $\cdots = c(k+1, 1)$  and (k+1, 1) and (1, 1) are joined by and edge or by a directed path of length two, respectively.

# **3** Strong product

The strong product of two graphs (resp. digraphs) G and H is the graph (resp. digraph)  $G \boxtimes H$  such that  $V(G \boxtimes H) = V(G) \times V(H)$ , and where there is an edge joining (a, x) and (b, y) if either  $ab \in E(G)$  and x = y, or if a = b and  $xy \in E(H)$ , or if  $ab \in E(G)$  and  $xy \in E(H)$ . (resp. either there is an edge pointing from (a, x) to (b, y) if  $ab \in E(G)$  and x = y, or if a = b and  $xy \in E(H)$ , or  $ab \in E(G)$  and  $xy \in E(H)$ .

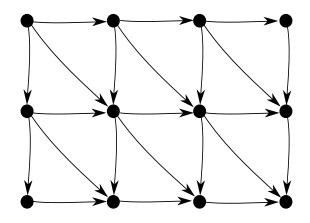


Figure 2: The strong product of  $\overrightarrow{P_3}$  and  $\overrightarrow{P_4}$ 

In the same paper, Jiang, Shao and Vesel prove the following result for the strong product of two directed cycles:

**Theorem 3** ([4]). If  $m, n \ge 48$ , then  $6 \le \lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) \le 8$ .

In this section, we refine this theorem in the following way:

**Theorem 4.** If  $m, n \ge 48$ , then

$$\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = \begin{cases} 6, & \text{if } m \equiv n \equiv 0 \pmod{7}; \\ 7 \text{ or } 8, & \text{otherwise.} \end{cases}$$

The key lemma of in the proof of Theorem 4 is analogous to Lemma 1:

**Lemma 3.** Let  $m, n \ge 4$  be integers. Any 6-L(2,1)-labeling f of  $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$  is diagonal, i.e., the following condition holds:

$$f(i,j) = f(i+1 \mod m, j-1 \mod n) \text{ for all } i \in [m], j \in [n].$$
(2)

Proof of Lemma 3. Let G be the graph  $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$ . For every vertex (i, j) of G, there is a  $\overrightarrow{P_4} \boxtimes \overrightarrow{P_4}$  subgraph of G as in Figure 3 such that  $v_{23}$  is the vertex (i, j).

It suffices, then, to show that  $f(v_{23}) = f(v_{32})$  for every 6-L(2, 1)-labeling of the graph in Figure 3.

Let f be a 6-L(2, 1)-labeling of G. By the fact that 6 - f is also a 6-L(2, 1)labeling of G, we may assume that  $f(v_{32}) \in \{0, 1, 2, 3\}$ . We will divide the rest of the proof in cases according to the value of  $f(v_{32})$ . In each case, we will assume that  $f(v_{23}) \neq f(v_{32})$  and reach a contradiction by applying the rules of L(2, 1)-labeling and finding a vertex for which there is no available color. We will use the notation v! to mean that there is no color available for the vertex v, and the notation  $\{u, v\} \in S!$  to mean that u and v cannot be colored using the colors in S, where S is the set of possible colors for u and v based on the colors of the previous vertices and the rules of L(2, 1)-labeling. For instance, if u and v are joined by an edge, then  $\{u, v\} \in \{0, 1\}!$ .

Case 1:  $f(v_{32}) = 3$ 

- $f(v_{23}) \in \{5, 6\} \Rightarrow \{v_{22}, v_{33}\} \in \{0, 1\}!$ .
- $f(v_{23}) = 4 \Rightarrow f(v_{22}) = 6$  and  $f(v_{33}) \in \{0,1\}$ , or  $f(v_{22}) \in \{0,1\}$  and  $f(v_{33}) = 6$ . In the first case,  $f(v_{43}) = 5 \Rightarrow f(v_{44}) = 2 \Rightarrow f(v_{33}) = 0 \Rightarrow v_{34}!$ ; in the second,  $f(v_{21}) = 5 \Rightarrow f(v_{11}) = 2 \Rightarrow f(v_{22}) = 0 \Rightarrow v_{12}!$ .
- $f(v_{23}) \in \{0, 1\} \Rightarrow \{v_{12}, v_{23}\} \in \{5, 6\}!$ .
- $f(v_{23}) = 2 \Rightarrow f(v_{22}) = 0$  and  $f(v_{33}) \in \{5, 6\}$ , or  $f(v_{33}) = 0$  and  $f(v_{22}) \in \{5, 6\}$ . In the first case,  $\Rightarrow f(v_{43}) = 1 \Rightarrow f(v_{44}) = 4 \Rightarrow f(v_{33}) = 6 \Rightarrow v_{34}!$ ; in the second,  $\Rightarrow f(v_{21}) = 1 \Rightarrow f(v_{11}) = 4 \Rightarrow f(v_{22}) = 6 \Rightarrow v_{12}!$ .

**Case 2:**  $f(v_{32}) = 0$ 

- $f(v_{23}) = 3 \Rightarrow \{v_{22}, v_{33}\} \in \{5, 6\}!.$
- $f(v_{23}) = 5 \Rightarrow \{v_{22}, v_{33}\} \in \{2, 3\}!.$
- $f(v_{23}) = 2 \Rightarrow f(v_{22}) = 6$  and  $f(v_{33}) = 4$ , or  $f(v_{33}) = 6$  and  $f(v_{22}) = 4$ . In the first case,  $v_{43}$ !; in the second,  $v_{21}$ !.
- $f(v_{23}) = 4 \Rightarrow f(v_{22}) = 6$  and  $f(v_{33}) = 2$ , or  $f(v_{22}) = 2$  and  $f(v_{33}) = 6$ . In the first case,  $v_{34}$ !; in the second,  $v_{12}$ !.
- $f(v_{23}) = 6 \Rightarrow f(v_{22}) = 2$  and  $f(v_{33}) = 4$ , or  $f(v_{22}) = 4$  and  $f(v_{33}) = 2$ . In the first case,  $v_{43}$ !; in the second,  $v_{21}$ !.
- $f(v_{23}) = 1 \Rightarrow f(v_{22}) = 3$  and  $f(v_{33}) = 5$ , or  $f(v_{22}) = 5$  and  $f(v_{33}) = 3$ , or  $f(v_{22}) = 6$  and  $f(v_{33}) = 4$ , or  $f(v_{22}) = 4$  and  $f(v_{33}) = 6$ , or  $f(v_{22}) = 6$  and  $f(v_{33}) = 3$ , or  $f(v_{22}) = 3$  and  $f(v_{33}) = 6$ . In the first and the third cases,  $v_{34}$ !; in the second and the fourth,  $v_{12}$ !; in the fifth,  $f(v_{34}) = 5 \Rightarrow v_{44}$ !; in the sixth,  $f(v_{12}) = 5 \Rightarrow v_{11}$ !.

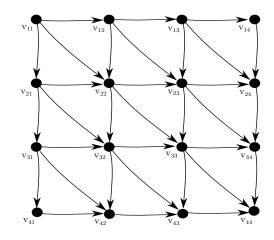


Figure 3: A  $\overrightarrow{P_4} \boxtimes \overrightarrow{P_4}$  subgraph of  $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$ 

**Case 3:**  $f(v_{32}) = 1$ 

- $f(v_{23}) = 3$  or  $f(v_{23}) = 0$ : can be treated similarly as the previous cases  $f(v_{32}) = 3$ ,  $f(v_{23}) = 1$ , and  $f(v_{32}) = 0$ ,  $f(v_{23}) = 1$ .
- $f(v_{23}) = 6 \Rightarrow \{v_{22}, v_{33}\} \in \{3, 4\}!.$
- $f(v_{23}) = 5 \Rightarrow \{v_{22}, v_{33}\} \in \{3\}!.$
- $f(v_{23}) = 4 \Rightarrow \{v_{22}, v_{33}\} \in \{6\}!.$
- $f(v_{23}) = 2 \Rightarrow f(v_{22}) = 4$  and  $f(v_{33}) = 6$ , or  $f(v_{22}) = 6$  and  $f(v_{33}) = 4$ . In the first case,  $f(v_{43}) = 3$  and  $f(v_{34}) = 0 \Rightarrow v_{44}!$ ; in the second,  $f(v_{21}) = 3$  and  $f(v_{12}) = 0 \Rightarrow v_{11}!$ .

**Case 4:**  $f(v_{32}) = 2$ 

- $f(v_{23}) \in \{0, 1, 3\}$ : can be treated similarly as in previous cases with  $f(v_{23})$ and  $f(v_{32})$  swapped.
- $f(v_{23}) = 5 \Rightarrow \{v_{22}, v_{33}\} \in \{0\}!.$
- $f(v_{23}) = 6 \Rightarrow f(v_{22}) = 0$  and  $f(v_{33}) = 4$ , or  $f(v_{22}) = 4$  and  $f(v_{33}) = 0$ . In the first case,  $v_{43}$ !; in the second,  $v_{21}$ !.
- $f(v_{23}) = 4 \Rightarrow f(v_{22}) = 0$  and  $f(v_{33}) = 6$ , or  $f(v_{22}) = 6$  and  $f(v_{33}) = 0$ . In the first case,  $v_{43}$ !; in the second,  $v_{21}$ !.

Proof of Theorem 4. By Theorem 3, it is enough to prove that  $G = \lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 6$  if and only if 7 divides m and n.

If both m and n are divisible by 7, the following periodic labeling is easily checked to be a L(2, 1)-labeling of  $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$ : the pattern 0246135 is repeated along the cycles. More explicitly, f(i, j) = 0, 2, 4, 6, 1, 3, 5 if  $i + j \equiv 2, 3, 4, 5, 6, 0, 1 \pmod{7}$ , respectively.

On the other hand, assume that G admits a 6-L(2, 1)-labeling f. By Lemma 3, f is diagonal. Similarly as in the proof of Theorem 2, it is simple to check that, if  $m \ge n+3$ , then f induces a 6-L(2, 1)-labeling of  $\overrightarrow{C_{m-n}} \boxtimes \overrightarrow{C_n}$ , simply by considering the restriction of the coloring in a  $\overrightarrow{C_{m-n}} \boxtimes \overrightarrow{C_n}$  subgraph of  $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$ . Again, applying this argument consecutively, we are either left with a  $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$ , where  $d = \gcd(m, n)$ , if  $d \ge 3$ , or a  $\overrightarrow{C_{k+1}} \boxtimes \overrightarrow{C_k}$ , or a  $\overrightarrow{C_{k+2}} \boxtimes \overrightarrow{C_k}$ .

In the last two cases, the fact that f is diagonal immediately implies that there are either two consecutive vertices or two vertices within distance two which receive the same color, which is a contradiction.

We are done, then, if we prove that  $\lambda(\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}) \geq 7$  if d is not a multiple of 7. Indeed, assume that there is a 6-L(2, 1)-labeling of  $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$ . Again, by Lemma 3, it should be diagonal. In particular, it means that it corresponds to a labeling of the cycle  $C_d$  with the following property: every pair of vertices with distance at most two must receive colors two apart, and every pair of vertices with distance three or four must receive distinct colors: indeed, the vertex (i + 1, j) is a neighbor of (i, j), and the vertex (i + 2, j) has the same color as (i+1, j+1), which is adjacent to (i, j) in  $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$ , so they must receive colors two apart from the color of (i, j); similarly we prove that (i + 3, j) and (i + 4, j) must receive distinct colors from (i, j). Such a coloring is denoted in the literature by L(2, 2, 1, 1)-labeling. Note that, in particular, this implies the statement for d = 3 and d = 4. Let us assume in what follows that  $d \geq 5$ .

If  $C_d$  has such a coloring, it is readily checked that it must use the color 0 or 6. Indeed, otherwise all available colors are 12345, and it is impossible to color a  $P_5$  subgraph of  $C_d$  with only these colors.

By symmetry, we may assume that 0 is used. Let c be the coloring of  $C_d$ , and let the integers modulo d represent its vertices. Let us assume that c(0) = 0.

If c(1) = 3, then 2 must receive a color from  $\{5, 6\}$ . If c(2) = 5, then c(3) = 1, and then c(-1) = 6, c(-2) = 2 or c(-2) = 4. In the first case, c(-3) = 4, and finally there is no available color for -4; in the second,  $c(-3) \in \{1, 2\}$ , and in either case there is no color available for -4. If c(2) = 6, then c(3) = 1, and then c(-1) = 5, c(-2) = 2, and there is no available color for -3.

If c(1) = 4, then c(2) is either 2 or 6. In the first case, c(-1) = 6 and there is no available color for 3. In the second, c(-1) = 2, which implies c(-2) = 5 and then there is no available color for -3.

If c(1) = 6, then  $c(2) \in \{2, 3, 4\}$ . The first implies c(3) = 4 and there is no color for 4. The second implies c(3) = 1, and then c(4) = 5 and there is no color for 5. Finally, the third implies that either c(3) = 1 or c(3) = 2, both of which makes impossible to find a color for 4.

The argument above implies that the neighbors of 0 must have colors 2 and 5. Without loss of generality, we may assume that c(1) = 5 and c(-1) = 2. This implies that c(2) = 3, which in turn implies c(3) = 1, then c(4) = 6 and c(5) = 4. It follows that  $c(6) \in \{0, 2\}$ , but by the paragraphs above, 4 cannot be a neighbor of 0, so c(6) = 2. Then c(7) = 0, and the block 2053164 of size 7 is repeated. The only way the coloring can be completed along the cycle is, then, if 7 divides d.

4 Final remarks

The natural next step would be to close the gap left from Theorem 4, deciding for which m and n we have  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 7$ .

In the proof of Theorem 4, we gave a periodic 6-labeling of  $\lambda(\overrightarrow{C_7} \boxtimes \overrightarrow{C_7})$ , namely that one in which the pattern 0246135 is repeated along the cycles diagonally. In a similar fashion, the following periodic 7-coloring works for  $\lambda(\overrightarrow{C_8} \boxtimes \overrightarrow{C_8})$ : 02461357. Concatenating these two patterns, one can show that  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_m}) = 7$  for every sufficiently large m (namely, for every  $m \in S(7,8)$ ; in particular for  $m \ge 42$ ), and consequently  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 7$  for every m, nsuch that 7 does not divide both m and n and  $gcd(m, n) \ge 42$ .

Finally, we remark that it is simple to check that the proof of Lemma 2 works in the setting of strong product of cycles as well. As we know from the paragraph above that  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_m}) = 7$  for every m in S(7,8) (and, in particular for every  $m \ge 42$ ), to prove that  $\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 7$  for all sufficiently large m and n, it would be enough to find a pair of coprime integers  $a, b \in S(7,8)$  such that  $\lambda(\overrightarrow{C_a} \boxtimes \overrightarrow{C_b}) = 7$ .

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