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ON EXTREMAL PROBLEMS IN GRAPH THEORY

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In this talk I am going to speak on results and problems in connection with the investigations concerning distance distribution in a point set A, discussed by P. Turán in his talk.

I shall use the following notations:

$G(P;E)$ is the graph with the set of vertices P and set of edges E

$C_k = C_k(P;E_c)$ is the complete graph, where $|P| = k$.

$G^r(P;G_1,G_2,\ldots,G_r)$ is a so-called r-class graph.

$G_i = G_i(P,E_i)$, where $E_i \cap E_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{r} E_i = E_c$.

$H(k_1,\ldots,k_r)$ is the class of G^r r-class graphs, with the property, that $C_{k_\nu} \not\subseteq G_\nu$ for every $1 \leq \nu \leq r$.

$H_n(k_1,\ldots,k_r)$ is the subclass of it containing G^r with $|P| = n$.

Let n have the form $n = (k-1)m + \ell$ (n,k,m,ℓ non negative integers) and

$$t(n,k) \overset{\text{def}}{=} \frac{k-2}{2(k-1)}(n^2-\ell^2) + \binom{\ell}{2}$$

the Turán-numbers.

Let $R(k_1,\ldots,k_r)$ the Ramsey-number, defined as the smallest number having the property, that if

$$G^r(P;G_1,\ldots,G_r) \subset H(k_1,\ldots,k_r)$$

then $|P| \leq R(k_1,\ldots,k_r)$.

In order to obtain *lower* bound for the number of large distances in a point set, we recall the definition of the k'th covering-constant of a point-set A on the plane.

Let $K(r,P)$ be the disk with radius r and center P. We call $K_1(r,P_1),\dots,K_k(r,P_k)$ a *covering-system* of A, if $P_i \in A$ for $1 \leq i \leq k$ and $A \subset \bigcup_{i=1}^{k} K_i(r,P_i)$. The *covering-constant* c_k of a set A is defined usually as the infimum of the r's with the above property.

From our point of view it's more appropriate to use the equivalent definition:

$$c_k \overset{\text{def}}{=} \inf_{(P_1,\dots,P_k)} \sup_{Q} \min_{1\leq i\leq k} \overline{QP_i}$$

where $Q \in A$, $P_i \in A$.

<u>Theorem</u>. *If $A = \{Q_1,\dots,Q_n\}$ having k^{th} covering constant c_k, then at least*

$$e(n,k) \overset{\text{def}}{=} (k-1)(n-1) + \left[\frac{n-k+2}{2}\right]$$

among the distances $\overline{Q_iQ_j}$ $(1 \leq i < j \leq n)$ are not less, than c_k.

The result is sharp, as it is shown by the following example:

Let $n - k + 1$ even, $Q_1,\dots,Q_{n-k+1}$ the vertices of a regular n-k+1-gon inscribed into $K(r,0)$, and let $Q_{n-k+2},\dots,Q_n$ additional points with the property

$$\begin{aligned} \overline{Q_\nu 0} &> 2r \\ \overline{Q_\nu Q_\mu} &> r \end{aligned} \qquad \text{for } n-k+2 \leq \mu,\ \nu \leq n.$$

For this point set $c_k = r$ and the number of distances, which are $\geq r$ is exactly $e(n,k)$.

The proof follows easily using the following theorem of P. Erdös – L. Moser:

<u>Theorem</u>. (P. Erdös-L. Moser): *If the graph $G(P,E)$ has the property, that for every $\{P_1,\dots,P_k\} \in P$ there exists a $P^* \in P$ such that the edges $P_iP^* \in E$ for $1 \leq i \leq k$, then*

$$|E| \geq (k-1)(n-1) + \left[\frac{n-k+2}{2}\right].$$

The unique extreme graph has the property, that it has k-1 points with degree n-1.

From the point of view of application the following question arises: how fast the minimal possible value of $|E|$ increases, if we restrict the maximal degree. More exactly, if the graph $G(P,E)$ has the above property and the maximal degree in G is δ, what is the minimal possible value for $|E|$ depending on δ (and on k,n).

E.g. in the simplest case, when $\delta = n-2$, the following holds:

<u>Theorem</u>. *If the graph $G(P,E)$ has the property as in theorem of P. Erdős - L. Moser, and every vertex in P has degree $\leq n-2$, then*

$$|E| \geq k(n-k) + \binom{k}{2}.$$

The above graph problems are strongly connected with the following one for (0,1) matrices:

Let $A = \{a_{ij}\}$ an $n \times n$ symmetric (0,1) matrix with $a_{ii} = 0$ and with the property, that for $A^2 = \{b_{ij}\}$ we have $b_{ij} \geq k$. What is the minimal possible value of $\sum_{i=1}^{n} \sum_{i=1}^{n} a_{ij}$? Further, the same question under the additional condition $\sum_{i=1}^{n} a_{ij} \leq \delta$?

In order to get some results for the distribution of distances of a point set A using the packing-constants, refining the ideas of P. Turán, we proceed as follows:

For the case of simplicity we consider a point set A with diameter 1.

We divide the interval [0,1] by the packing-constants*
$1 = d_2 \geq d_3 \geq \dots > d_{r+1} > 0$. We define the r-class graph $G^r = G^r(P,G_1,\dots,G_r)$ with $P = \{1,2,\dots,n\}$ belonging to a point-set $\{Q_1,\dots,Q_n\} \subset A$ by:

$$(i,j) \in E_\nu \Longleftrightarrow \overline{Q_iQ_j} \in [d_{\nu+1},d_{\nu+2}) \quad \text{for } \nu \leq r-1$$
$$\in [d_{r+1},0] \quad \text{for } \nu = r.$$

Now from the definition of the numbers d_ν we get simultaneous structural conditions for the graph $G_1,\dots,G_r$. The general question is, what are the possible values of $|E_1|,\dots,|E_r|$ (under certain condition). This leads to the following simplest special but important graph problem:

Let $k_1,\dots,k_r$ given positive integers. If G^r is an r-class graph having the property, that

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*For the definition of the packing constants $d_2,d_3,\dots$ see P. Turán: Applications of graph th. etc. in this volume.

$$C_{k_\nu} \not\subset G_\nu \quad \text{for } 1 \leq \nu \leq r$$

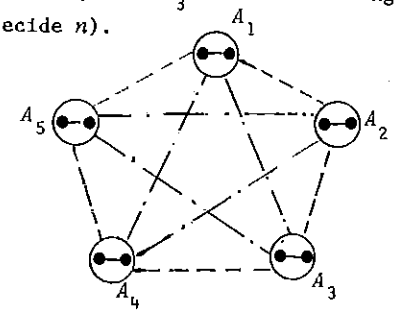
then what is max $(E_1 + \ldots + E_{r-1})$?

In the case, when $k_r = n+1$ (i.e. there is no condition for G_r) it is easy to see, that

$$E_1 + \ldots + E_{r-1} \leq t(n, R(k_1, \ldots, k_{r-1}))$$

and the upper bound is sharp.

E.g. for $r = 3$, $k_1 = k_2 = 3$ the 3-class graph in $H_n(3,3, n+1)$ having the minimal number of edges in G_3 is the following: (for the sake of symplicity let 5 decide n).



Let A_i $(1 \leq i \leq 5)$ be disjoint sets of the n vertices each having $\frac{n}{5}$ points. E_3 consists of all the edges with both vertices in the same set A_ν $(1 \leq \nu \leq 5)$. E_2 consists of all the edges (i,j) for which $i \in A_\nu$ and $j \in A_{\nu+1}$ $(A_6 \equiv A_1)$ and E_1 consists of all the remaining edges.

Already in the case $k = 2$ it seems to be very difficult to determine

$$f_n(k_1,k_2) \overset{\text{def}}{=} \max_{G^2 \subset H_n(k_1,k_2)} E_1 \quad .$$

The remark, that (i) $f_n(k_1,n+1) = t(n,k_1)$ and (ii) $f_n(k_1,k_2) = o(n^2)$ for fixed k_1,k_2 shows the connection between Turán's and Ramsey's theorem. Namely the statement (i) is just Turán's theorem. While statement (ii) implies that for n large the class $H_n(k_1,k_2)$ must be empty, which is just Ramsey's theorem. So having some information on $f_n(k_1,k_2)$ Turán's and Ramsey's theorems are the consequences of the two extreme cases.

For the case, when k_1 is fixed and $k_2 = cn$ or $k_2 = o(n)$ for large enough, we have some results with P. Erdös which will appear later.