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Highlights

- The aim of the present paper is to shed light on the interaction between capacity constraints and local monopoly power using a standard Hotelling setup.
- Substantial horizontal product differentiation results in a variety of equilibrium firm behavior and it generates at least one pure-strategy equilibrium for any capacity level.
- The existence of pure-strategy equilibria for every capacity pair is in stark contrast with most of the literature on capacity-constrained pricing.

Bertrand-Edgeworth competition with substantial horizontal product differentiation

Robert Somogyi *

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Abstract

Since Kreps and Scheinkman's seminal article (1983) a large number of papers have analyzed capacity constraints' potential to relax price competition. However, the majority of the ensuing literature has assumed that products are either perfect or very close substitutes. Therefore very little is known about the interaction between capacity constraints and local monopoly power. The aim of the present paper is to shed light on this question using a standard Hotelling setup. The high level of product differentiation results in a variety of equilibrium firm behavior and it generates at least one pure-strategy equilibrium for any capacity level.

JEL Classification: D21, D43, L13

Keywords: Duopoly, Bertrand-Edgeworth competition, Hotelling, Capacity constraint

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1 Introduction

The problem of capacity-constrained pricing decision in oligopolies has received considerable attention since Kreps and Scheinkman's seminal article (1983). Most of the work in the field of Bertrand-Edgeworth oligopolies focused on the case of homogeneous goods and the capacities' potential impact of relaxing price competition.[1] However, a large number of real-world industries characterized by capacity constraints offer differentiated products. Examples include the airline industry, where capacities clearly play a central role and different companies tend to include different services in the price of their ticket (checked-in luggage, seat reservation, in-flight meal etc.). In the telecommunication sector, mobile service operators are bound by the size of their 4G and 5G networks, and clearly offer differentiated products (monthly data cap, speed, network coverage etc.). In the hospitality industry, competing hotels tend to be differentiated (breakfast, reservation policy, amenities) and constrained by the number of available rooms. Finally, the co-existence of capacity constraints and physical transportation costs play a crucial role in the cement industry as well. Hunold et al. (2017); and Hunold and Muthers (2019) show this both theoretically and empirically, and discuss its potential relevance for competition policy.

Moreover, taking into account both horizontal product differentiation and the presence of capacity constraints might lead to novel and surprising theoretical results, as first demonstrated by Wauthy (1996). Despite the prevalence of such industries and the theoretical interest they present, the literature on Bertrand-Edgeworth oligopolies with product differentiation remains scarce. As Wauthy (2014) points out in a recent survey of this branch of literature:

> " The minimal core of strategic decisions a firm has to make is threefold: What to produce? At which scale? At what price? A full-fledged theory of oligopolistic competition should be able to embrace these three dimensions jointly. [..] we do not have such a theory at our disposal. [..] it is urgent to devote more efforts to analyze in full depth the class of Bertrand-Edgeworth pricing games with product differentiation. "

This paper aims to make a step in this direction. Specifically, it analyzes Bertrand-Edgeworth competition on markets characterized by a substantial level of product differentiation. By restricting attention to relatively high levels of product differentiation in a standard Hotelling setup, it shows that there exists at least one pure-strategy equilibrium for any capacity-pair. This stands in contrast with most

[1] Recent examples include Acemoglu et al. (2009); de Frutos and Fabra (2011); Lepore (2012); Lemus and Moreno (2017); Cabon-Dhersin and Drouhin (2019).

models of Bertrand-Edgeworth competition that typically find non-existence for intermediate capacity-levels. The main result of the paper is a complete characterization of the pure-strategy equilibria, which reveals a variety of equilibrium firm behavior in this setting. In addition, I show that an even higher level of product differentiation leads to a trivial pure-strategy equilibrium: non-interacting firms acting as local monopolies. Finally, I also demonstrate that lower levels of product differentiation destroy the existence result. In particular, there always exists a non-empty range of capacity pairs of intermediate size for which an equilibrium in pure strategies fails to exist. I believe that the results are most suitable to describe relatively short-run price competition as the fixed capacity sizes are more realistic in the short run.

Most closely related to this paper is Boccard and Wauthy (2010). They investigate the interaction between capacity constraints and Hotelling-type differentiation and find the absence of an equilibrium in pure strategies for intermediate capacity levels. Their main finding is that the support of equilibrium prices consists of a finite number of atoms, and the number of these atoms is decreasing in the level of product differentiation. An important assumption their paper makes is that consumers' valuation for the good is large compared to transportation costs, which results in the market always being covered in equilibrium. While this assumption prevails in the Hotelling literature[2], the present paper shows that it hides an interesting setting, namely the case of substantial product differentiation.

In recent work, Hunold and Muthers (2019) investigate capacity-constrained competition with horizontally differentiated consumers, with the additional key assumption that firms can charge location-specific prices to four different consumer segments. The special structure of horizontal differentiation is the key distinguishing feature of their model in addition to customer-specific pricing. They find that pure-strategy equilibria typically fails to exist for medium capacity levels and provide a characterization of the mixed-strategy equilibria. They find that in the competitive equilibrium, despite overcapacities, firms only serve the customers closest to them. This finding is similar to one of the equilibria of my model (the situation I call secret handshake equilibrium), however, the mechanisms leading to this are distinct. Hunold et al. (2017) build on this theoretical work to compare competitive and collusive outcomes on such markets. They validate the theoretical predictions by studying a cartel breakdown in Germany empirically. In particular, they document the strong relationship between transport distance and capacity

[2]For an exception that discusses this issue in detail, see Economides (1984). For more recent work making the same assumption implicitly or explicitly, see for example Gal-Or (1997), Lyon (1999), and Brekke et al. (2006) for models of the health care market, and Ishibashi and Kaneko (2008) for a model of a mixed duopoly.

levels in the cement industry after the cartel breakdown.

In earlier work, Benassy (1989) and Canoy (1996) also analyze Bertrand-Edgeworth models with horizontal product differentiation. The main difference with the present paper is that both of these papers use non-standard specifications of product differentiation. Specifically, Benassy (1989) captures product differentiation through demand elasticities in a model of monopolistic competition, whereas Canoy (1996) introduces asymmetries between the firms and allows consumers to buy several units of the good. A common finding of all three papers is the existence of pure-strategy equilibrium for sufficiently high levels of product differentiation. The present paper reformulates this result in the more standard Hotelling framework. Furthermore, contrary to the papers above, the simplicity of the model allows for the complete characterization of pure-strategy equilibria for substantial levels of product differentiation.

The paper is organized as follows. Section 2 describes the model, formulates the profit function and identifies the best reply strategies for intermediate levels of product differentiation. Section 3 contains the main result of the paper, the complete characterization of the equilibria for intermediate levels of product differentiation. Section 4 discusses the results in the light of the existing literature, and also discusses the cases of relatively low and high product differentiation. Section 5 examines a vertical product differentiation extension of the baseline model. Section 6 concludes.

2 The model

2.1 Setting

This paper analyzes a duopoly with firms labeled 1 and 2 that produce substitute products. They choose a price p_i ($i \in \{1, 2\}$) for one unit of their product. Assume the firms are located on the two extreme points of a unit-length Hotelling-line (Firm 1 at $x = 0$, Firm 2 at $x = 1$) and transportation cost is linear. Moreover, consumers are uniformly distributed along the line but are otherwise identical. They all seek to buy one unit of the product which provides them a gross surplus v. The value of the outside option of not buying the product is normalized to 0. In addition, the firms face rigid capacity constraints k_1, k_2. For simplicity, assume that marginal costs of production are constant and normalized to zero. The size of the capacities as well as the value of the other parameters of the model are common knowledge. The firms' objective is to maximize their profit by choosing their price.

A consumer located at point x purchasing from Firm 1 has a net surplus of

$$v - p_1 - t \cdot x$$

while purchasing from Firm 2 provides her a net surplus of

$$v - p_2 - t \cdot (1 - x)$$

where t is the per-unit transportation cost.

In most of this article, I assume $v/t \leq 1.5$, i.e. the products of the firms are substantially different from one another. Furthermore, to focus on the arguably most interesting case, I will also assume $1 < v/t \leq 1.5$ and refer to it as *intermediate level of product differentiation.*

Boccard and Wauthy (2010) analyze a similar setting, the key difference being the level of product differentiation. They restrict their attention to situations in which products are relatively close substitutes, namely $v/t > 2$. Below I argue that this simplifying assumption has a surprisingly large impact on the nature of equilibria, hence extending the analysis to the case of intermediate capacity levels provides new insights into the mechanisms of capacity-constrained oligopolies.

Finally, I complete the analysis with the cases of very high level of product differentiation, $v/t \leq 1$; and low product differentiation $1.5 < v/t \leq 2$. The following list describes the structure of this article:

- $v/t \leq 1$: discussed in Section 4.1,
- $1 < v/t \leq 1.5$: discussed in Sections 2 and 3,
- $1.5 < v/t \leq 2$: discussed in Section 4.2,
- $2 < v/t$: discussed in Boccard and Wauthy (2010).

2.2 The profit function

Assuming rational consumers the following two constraints are straightforward. The participation constraint (PC) ensures that a consumer located at point x buys from Firm 1 only if her net surplus derived from this purchase is non-negative:

$$v \geq p_1 + t \cdot x \tag{PC}$$

The market splitting condition (MS) ensures that a consumer located at point x buys from Firm 1 only if this provides her a net surplus higher than buying from the competitor:

$$v - p_1 - t \cdot x \geq v - p_2 - t \cdot (1 - x) \tag{MS}$$

Let T_1 be the marginal consumer who is indifferent whether to buy from Firm 1 or not. In the absence of capacity constraints it is easy to see that T_1 is the minimum of the solutions of the binding constraints (PC) and (MS).

Let $\overline{T}_1(p_2)$ be the location of the consumer who is indifferent between buying from Firm 2 and not buying at all. The location of this consumer will be crucial in the analysis to determine whether Firm 1 is better-off competing against its rival or being a local monopolist. Formally,

$$v - p_2 - t(1 - \overline{T}_1(p_2)) = 0 \quad \Rightarrow \quad \overline{T}_1(p_2) = \frac{p_2 - v + t}{t}.$$

Importantly, $\overline{T}_1$ plays the role of partitioning the price space according to market coverage.[3] The net surplus being decreasing in the distance from Firm 1 implies that (PC) is binding for $T_1 \leq \overline{T}_1$ and (MS) is binding if $T_1 \geq \overline{T}_1$. Symmetric formulas apply to Firm 2. Therefore, in case capacities are abundant, inverse demand for Firm 1 is given by

$$p_1 = \begin{cases} v - t \cdot T_1 & \text{if } T_1 \leq \overline{T}_1, \\ p_2 + t - 2 \cdot t \cdot T_1 & \text{if } T_1 \geq \overline{T}_1. \end{cases} \tag{1}$$

Naturally, the existence of capacity constraints means for Firm 1 that it cannot serve more than k_1 consumers. Assume that after each consumer chooses the firm to buy from (or to abstain from buying), firms have the possibility to select which consumers to serve and they serve those who are the closest to them. In this setting this corresponds to the assumption of efficient rationing rule, which is extensively used in the literature. Therefore the additional constraints caused by the fixed capacity levels can be written as:

$$T_1 \leq k_1 \quad \text{and} \quad 1 - T_2 \leq k_2 \tag{CC}$$

It is important to notice that in some cases, when Firm 2 is capacity-constrained, Firm 1 can extract a higher surplus from some consumers by knowing that they cannot purchase from the rival even if they wanted to since Firm 2 does not serve them. Practically, this means that the participation constraint (PC) will always be binding on $\left[\overline{T}_1, 1 - k_2\right]$ whenever this interval is not empty, i.e. whenever the rival's capacity is sufficiently small: $k_2 \leq 1 - \overline{T}_1$. Using this observation, one can reformulate the inverse demand in (1) for any capacity level:

[3]For notational simplicity, subsequently I will not indicate the argument of $\overline{T}_1(p_2)$.

$$p_1 = \begin{cases} v - t \cdot T_1 & \text{if } T_1 \leq \max\{\overline{T}_1, 1 - k_2\}\ , \\ p_2 + t - 2 \cdot t \cdot T_1 & \text{if } T_1 > \max\{\overline{T}_1, 1 - k_2\} \end{cases} \tag{2}$$

Firm 1's profit can be simply written as $\pi_1 = p_1 T_1$. Given the competitor's capacity and its price choice, determining the unit price p_1 is equivalent to determining the marginal consumer T_1. The observation that prices and quantities can be used interchangeably will simplify the solution of the model.[4] Importantly, the firms decide about prices, however, the quantities those prices imply are more directly comparable with the size of capacities.

The profit can thus be rewritten as

$$\pi_1(T_1, p_2) = \begin{cases} (v - t \cdot T_1) \cdot T_1 & \text{if } T_1 \leq \max\{\overline{T}_1, 1 - k_2\}, \\ (p_2 + t - 2 \cdot t \cdot T_1) \cdot T_1 & \text{if } T_1 > \max\{\overline{T}_1, 1 - k_2\} \end{cases} \tag{3}$$

The optimization problem of the firm consists of finding the value T_1 which maximizes the above expression satisfying the capacity constraint (CC). The main complexity of finding the equilibria in this pricing game comes from the shape of the profit functions. As illustrated below in Figure 2, the profit $\pi_1(T_1, p_2)$ is discontinuous at point $1 - k_2$ if $\overline{T}_1 < 1 - k_2$. Otherwise, it is continuous but non-differentiable at $\overline{T}_1$ as illustrated in Figure 1.

For notation simplicity, let the two branches of the profit function be denoted by

$$\pi_1^{LM} \equiv (v - t \cdot T_1) \cdot T_1 \quad \text{and} \quad \pi_1^C \equiv (p_2 + t - 2 \cdot t \cdot T_1) \cdot T_1.$$

The superscript LM stands for Local Monopoly because the firm extracts all the consumer surplus from the marginal consumer when (PC) binds. Similarly, the superscript C stands for Competition since the marginal consumer is indifferent between the offer of the two firms whenever (MS) binds.

Note that the profit function reveals another interpretation of $\overline{T}_1$: it is the point where π_1^{LM} and π_1^C cross (other than their crossing at 0).

2.3 Potential best reply strategies

Define $T_1^{LM} = \arg\max_{T_1} \pi_1^{LM}$ and $T_1^C = \arg\max_{T_1} \pi_1^C$, the values at which the two quadratic curves attain their maxima, hence they are local maxima of the profit

[4] The technique of arguing in terms of quantities instead of prices is also used by Yin (2004).

function $\pi_1(T_1)$.

The relative order of the five variables

$$T_1^{LM}, \quad T_1^C, \quad \overline{T}_1, \quad 1-k_2 \quad \text{and} \quad k_1$$

is crucial in solving the maximization problem. The main difficulty in the solution of the firms' maximization program is twofold. On the one hand, the profit function is discontinuous at $1-k_2$ whenever $k_2 < 1-\overline{T}_1$ and non-differentiable at $\overline{T}_1$ otherwise. On the other hand, the values

$$\overline{T}_1 = \frac{p_2 - v + t}{t} \quad \text{and} \quad T_1^C = \frac{p_2 + t}{4t}$$

depend on the choice of the other firm, p_2. The following lemma simplifies the solution considerably.

Lemma 1.
$T_1^{LM} \leq \overline{T}_1$ implies $T_1^C \leq \overline{T}_1$ and $T_1^C \geq \overline{T}_1$ implies $T_1^{LM} \geq T_1^C \geq \overline{T}_1$.

Proof: It is straightforward to derive

$$T_1^{LM} = \frac{v}{2t}, \quad \overline{T}_1 = \frac{p_2 - v + t}{t} \quad \text{and} \quad T_1^C = \frac{p_2 + t}{4t}.$$

Then for any $t > 0$

$$T_1^{LM} \leq \overline{T}_1 \iff \frac{v}{2t} \leq \frac{p_2 - v + t}{t} \iff p_2 \geq \frac{3}{2}v - t$$

and similarly

$$T_1^C \leq \overline{T}_1 \iff \frac{p_2 + t}{4t} \leq \frac{p_2 - v + t}{t} \iff p_2 \geq \frac{4}{3}v - t$$

also

$$T_1^{LM} \leq T_1^C \iff \frac{v}{2t} \leq \frac{p_2 + t}{4t} \iff p_2 \geq 2v - t$$

This proves the two parts of the lemma for any $v > 0$. □

The form of Firm 1's profit function hinges on the relative order of $\overline{T}_1$ and $1-k_2$. Therefore in the following discussion I will distinguish two cases: In Case A, the capacity of Firm 2 is relatively large, $1-k_2 < \overline{T}_1$. In Case B, $1-k_2 \geq \overline{T}_1$, which means that Firm 1 may be able to take advantage of the fact that its adversary is relatively capacity-constrained.

Case A: $1 - k_2 < \overline{T}_1$**.** When the capacity of Firm 2 is relatively large, (1) shows the relation between the price p_1 charged by Firm 1 and its demand (captured by the marginal consumer T_1). Using Lemma 1 three different subcases can be identified depending on the parameter values of the model and the competitor's choice.

Lemma 2. *Assume* $1 - k_2 < \overline{T}_1$.

(A1) if $T_1^{LM} \leq \overline{T}_1$ *then the optimal choice of Firm 1 is* $\min(T_1^{LM}, k_1)$,

(A2) if $T_1^C \geq \overline{T}_1$ *then the optimal choice of Firm 1 is* $\min(T_1^C, k_1)$,

(A3) if $T_1^C \leq \overline{T}_1 \leq T_1^{LM}$ *then the optimal choice of Firm 1 is* $\min(\overline{T}_1, k_1)$.

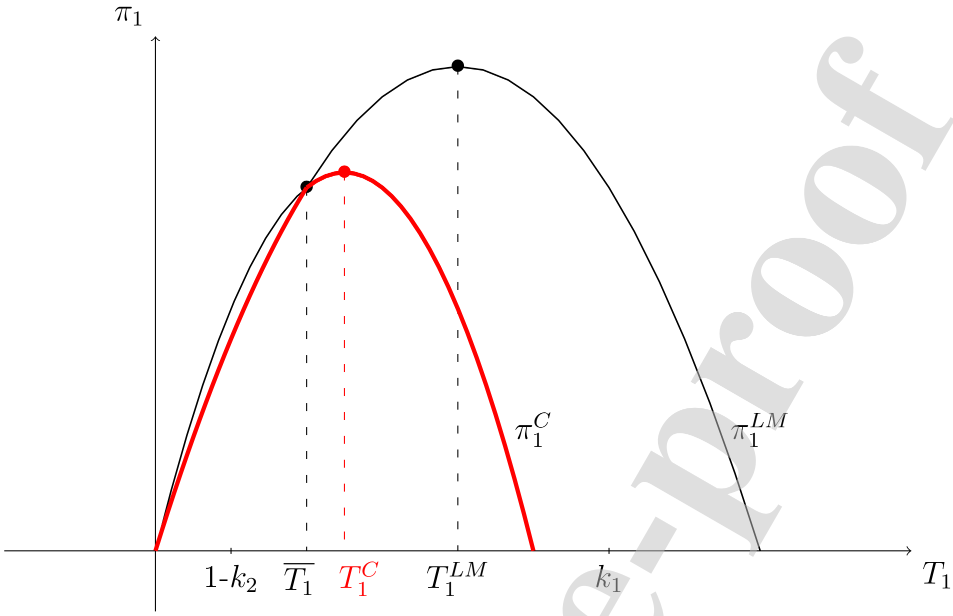
Considering Lemma 1 it is easy to see that cases A1, A2 and A3 provide a complete partitioning of Case A. Hence for any parameter values in Case 1 and for every possible behavior of the competitor, the lemma identifies the best response strategy of Firm 1. Symmetric formulas apply for Firm 2. The complete proof of this lemma is relegated to the Appendix.

However, for an intuition, first notice that the two branches of the profit function, π_1^{LM} and π_1^C are both quadratic functions of T_1 that by definition cross each other at 0 and at $\overline{T}_1$. Then depending on the values t, v and T_2 one of the three possibilities above will hold. As an illustration of Case A2 when $T_1^C < k_1$ see Figure 1. Using Lemma 1 the condition of the case $T_1^C \geq \overline{T}_1$ immediately implies $T_1^{LM} \geq \overline{T}_1$. We know that the profit function is composed of the function π_1^{LM} on the interval $[0, \overline{T}_1]$ then it switches to function π_1^C . The actual profit function is thus the thick (red) curve in the figure. Then using the figure it is straightforward to find the optimal choice of Firm 1. Since the two quadratic and concave functions cross each other before either of them reaches its maximum, the maximal profit will be attained on the second segment where $\pi_1 = \pi_1^C$. By definition, $\arg\max_{T_1} \pi_1^C = T_1^C$ is the optimal choice, and the assumption $T_1^C < k_1$ makes this feasible.

Case B: $\overline{T}_1 \leq 1 - k_2$**.** In Case B, the rival of Firm 1 disposes of relatively low capacity. Therefore Firm 1 might be inclined to take advantage of the fact that Firm 2 is not capable of serving consumers located on the interval $[0, 1 - k_2]$. On this segment Firm 1 does not have to care about its competitor's price and the market splitting condition (MS), it is only threatened by some consumers choosing the outside option of not buying the product (PC) and eventually by its own capacity constraint.

Lemma 3. *Assume* $\overline{T}_1 \leq 1 - k_2$*. Then*

Figure 1: Illustration of Case A2 ($T_1^C < k_1$)



(B1) if $T_1^{LM} \leq \overline{T}_1$ *then the optimal choice of Firm 1 is* $\min(T_1^{LM}, k_1)$*,*

(B2) if $\overline{T}_1 \leq T_1^C \leq 1 - k_2$ *then the optimal choice of Firm 1 is* $\min(1 - k_2, T_1^{LM}, k_1)$*,*

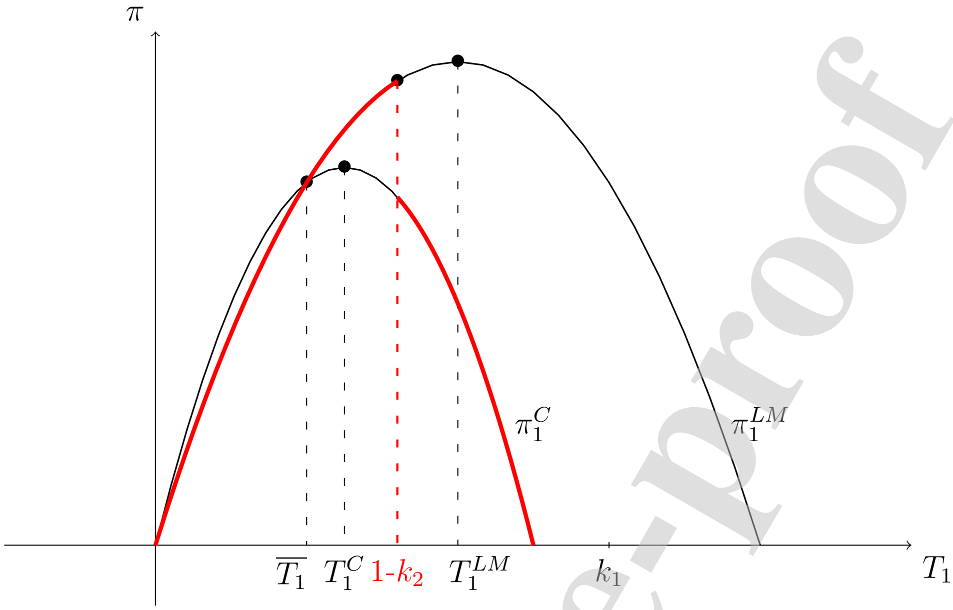
(B3) if $\overline{T}_1 \leq 1 - k_2 \leq T_1^C$ *then the optimal choice of Firm 1 is either* $\min(1 - k_2, k_1)$ *or* $\min(T_1^C, k_1)$*,*

(B4) if $T_1^C \leq \overline{T}_1 \leq 1 - k_2 \leq T_1^{LM}$ *then the optimal choice of Firm 1 is* $\min(1 - k_2, k_1)$*.*

(B5) if $T_1^C \leq \overline{T}_1 \leq T_1^{LM} \leq 1 - k_2$ *then the optimal choice of Firm 1 is* $\min(T_1^{LM}, k_1)$*.*

Notice that case B1 corresponds exactly to case A1 of Lemma 2 and B5 also describes a very similar situation. However, the other cases are affected by the limited capacity of the rival firm. The case closest to case A2 pictured above is case B2. The only difference is in the size of the rival firm's capacity, here it is assumed to be much smaller. As an illustration of this situation, see Figure 2 (where k_1 is assumed to be large in order to draw a clearer picture). As is clear from the figure and true in general, $\pi_1^{LM}(x) > \pi_1^C(x)$ whenever $x > \overline{T}_1$ i.e. to the right of the crossing point of the two curves. Hence the profit function is not only non-differentiable as in the above case, it is also discontinuous at $1 - k_2$. Therefore the assumption $T_1^C \leq 1 - k_2 \leq T_1^{LM}$ immediately implies that $1 - k_2$ is the optimal choice of Firm 1, i.e. it produces up to the capacity of the other firm. The profit curve and the optimal solution are shown in thick (red) on Figure 2.

Figure 2: Illustration of Case B2 ($1 - k_2 < T_1^{LM} < k_1$)



The most interesting case is arguably B3 where 3 different best replies may arise depending on the exact parameters of the model and the competitor's choice. This is also the most problematic case in Boccard and Wauthy (2010) in the sense that this discontinuity inhibits the possible existence of pure-strategy equilibrium. As I will show below, case B3 never arises in equilibrium when assuming intermediate levels of product differentiation. However, in Section 4.2 when discussing the case of $1.5 < v/t \leq 2$ I show that it does arise and the discontinuity is exactly the reason for the non-existence of equilibria in pure strategies for low levels of product differentiation.

The next section describes the numerous equilibria of the game using the conditional best replies of firms described above.

3 Equilibria for intermediate levels of product differentiation

In this section I will determine which kinds of equilibria may arise in the intermediate product differentiation case as a function of firms' capacities and the other parameters of the model (v and t). The calculations will be based on the results of Lemmas 2 and 3 that describe the firms' conditional best responses.

As is clear from those lemmas, there are 5 potential equilibrium strategies for

Firm 1:

$$T_1^{LM}, \quad T_1^C, \quad \overline{T}_1, \quad 1-k_2 \quad \text{and} \quad k_1.$$

The exercise of finding all equilibria consists of comparing the conditions for potential equilibrium strategies (described in cases A1-A3 and B1-B5) of Firm 1 to those of Firm 2 one-by-one and determining whether the conditions are compatible. In case they are, one also has to formulate the conditions in terms of the parameters of the model. Since the cases described in the two lemmas are exhaustive, this method finds all the existing equilibria of the game. An advantage of such a complete characterization is that for parameter regions where I find one equilibrium only, that one is clearly the unique equilibrium.

These case-by-case calculations are by nature tedious so they are relegated to the Appendix. The following proposition summarizes the main result of the paper, i.e. the findings for intermediate levels of product differentiation.

Proposition 1. *For $1 < v/t \leq 1.5$ there exists at least one equilibrium in pure strategies for any capacity pair (k_1, k_2). The nature of the equilibria depends on the relative size of the capacity levels, and the relative value of consumers' willingness-to-pay v and their transportation cost t.*

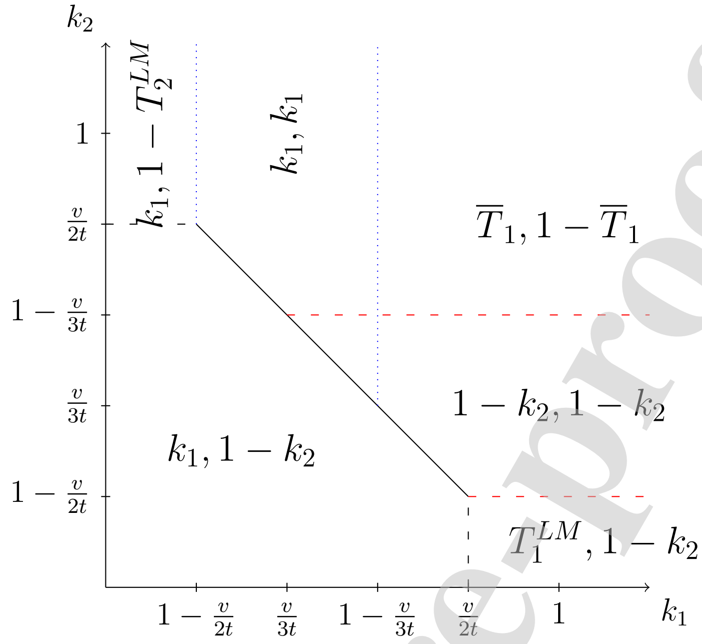
Proposition 1 is in contrast to most of the existing results about Bertrand-Edgeworth oligopolies. The usual finding in the existing literature is that there is at least one region of capacity levels for which there does not exist a pure-strategy equilibrium. This clearly shows that the presence of substantial local monopoly power changes Bertrand-Edgeworth competition drastically. Even Boccard and Wauthy (2010) who investigate the case of slightly differentiated products face the problem of non-existence of pure-strategy equilibrium, indeed, their main contribution is a partial characterization of the mixed-strategy equilibrium.

By restricting attention to intermediate levels of product differentiation, one can provide a complete characterization of the equilibria of the model. Figure 3 illustrates the different types of equilibria that arise as a function of the parameters. For simplicity, the figure depicts the case of $1.2 < v/t \leq 1.5$.[5]

The capacities of Firm 1 and Firm 2 are shown on the horizontal and the vertical axis, respectively. The values in the parentheses in every parameter region show the addresses of the farthest consumers Firm 1 and Firm 2 serve, respectively.

[5] The complement case of $1 < v/t \leq 1.2$ is qualitatively equivalent and the same type of equilibria arise. The only difference is in the ordering of the different values on the axes, in particular, the ordering of $\frac{v}{2t}$ and $1 - \frac{v}{3t}$ reverses at 1.2.

Figure 3: Equilibria with substantial product differentiation ($1.2 < v/t \leq 1.5$)



Note that the figure is symmetric to the diagonal, which is a direct consequence of the firms being identical apart from their capacities.[6]

Capacity-constrained equilibria The simplest case is the one where k_1 and k_2 are both very low ($k_1 + k_2 < 1$) which inhibits the interaction between the two firms. Consequently they maximize their profits independently by producing up to their capacity. Therefore $(k_1, 1-k_2)$ is the unique equilibrium in this region.

Assuming a similarly low capacity for Firm 2 ($k_2 < 1 - \frac{v}{2t}$) but a larger one for Firm 1 ($k_1 \geq \frac{v}{2t}$), one gets to the region where Firm 1 cannot profitably increase its production and implements its unconstrained local monopoly profit $T_1^{LM} = \frac{v}{2t}$. Hence $(T_1^{LM}, 1 - k_2)$ is the unique equilibrium here.

Capacity-constrained secret handshake equilibria The most interesting region is arguably the one where the capacity of one firm is not very low but not

[6]The notation may be misleading in this aspect, for example seeing the symmetry between (k_1, k_1) and $(1 - k_2, 1 - k_2)$ is non-trivial. The values in the parentheses denote the address of the farthest consumer firms are willing to serve. For Firm 2, this is generally not equal to its demand, i.e. its strategy, instead it is given by one minus its demand. Thus the strategies used in (k_1, k_1) and $(1 - k_2, 1 - k_2)$ are in fact symmetric: in both cases one firm serves up to its capacity and the rival serves all the residual demand.

very high either ($1-\frac{v}{2t} < k_2 < \min(1-\frac{v}{3t}, \frac{v}{2t})$) and the industry capacity is sufficient to cover the market ($k_1 + k_2 \geq 1$). Firm 2 producing up to its capacity and Firm 1 deciding to serve the remaining $1 - k_2$ consumers is a pure-strategy equilibrium of this region. Notice that the size of their capacity would allow firms to enter into direct competition, however, it would not be profitable for Firm 1. Instead it prefers to match the residual demand of the market. Essegaier et al. (2002) find similar equilibrium behavior in their model with heterogeneous demand and call it a "secret handshake" equilibrium.

Notice that in the triangle-shaped region $k_1, k_2 < \min(1-\frac{v}{3t}, \frac{v}{2t})$ and $k_1 + k_2 \geq 1$ (i.e. the one delimited by the dotted blue line, the dashed red line and the black line in Figure 3) either firm producing up to its capacity with the other one engaging in the secret handshake constitutes an equilibrium. Thus in this region the two pure-strategy equilibria (k_1, k_1) and $(1 - k_2, 1 - k_2)$ co-exist. Clearly, both firms would prefer serving up to their own capacity, therefore none of the equilibria Pareto dominates the other one. To illustrate how these two equilibria are sustained, consider the incentives of the firms at $(1-k_2, 1-k_2)$. It is the best outcome for Firm 2, so the more interesting question is why Firm 1 will not deviate from it. First, $1 - k_2 < \frac{v}{2t}$ ensures that $1 - k_2$ is on the increasing part of π_1^{LM}, therefore all strategies below $1 - k_2$ are dominated. Second, as Firm 2's strategy implies a price of $p_2 = v - k_2 t$, the condition $k_2 < 1 - \frac{v}{3t}$ is equivalent to $T_1^C(v - k_2 t) < 1 - k_2$. Thus $1 - k_2$ is on the decreasing part of the π_1^C curve, which means that all strategies in the interval $(1 - k_2, k_1]$ are also dominated.

Unconstrained secret handshake equilibria Lastly, when both capacities are large ($k_1, k_2 > \min(1 - \frac{v}{3t}, \frac{v}{2t})$) there is a continuum of equilibria in pure strategies. As $\overline{T}_1$ depends on p_2 and thus on T_2 and vice versa, the location of the indifferent consumer ($\overline{T}_1 = \overline{T}_2$) may take any values in between $1 - \frac{v}{2t}$ and $\min(1 - \frac{v}{3t}, \frac{v}{2t})$. Furthermore, these equilibria could also be described as a type of secret handshake since here $\overline{T}_1 = \overline{T}_2$ holds so the market is exactly covered by the two firms. In the proof of Proposition 1, I show that $v/t \leq 1.5$ is a necessary condition for the existence of unconstrained secret handshake equilibria. Note that the multiplicity of equilibria is a standard result for Hotelling models with substantial product differentiation without capacity constraints (Economides [1984]), so its presence is natural for the case of abundant capacities.

Notice that the unconstrained secret handshake equilibria are similar to the equilibria in Hunold and Muthers (2019) where firms only serve their "home market", i.e. the consumers closest to them. The main difference is that firms charge location-specific prices as opposed to uniform prices as in the present model.

4 Other levels of product differentiation

In order to obtain a more complete picture of how Hotelling-type product differentiation and capacities interact, in this Section I investigate the cases of very high product differentiation ($v/t < 1$) and the case of low product differentiation ($1.5 < v/t \leq 2$) which have been missing from the literature. The analysis of the remaining case ($v/t > 2$) can be found in Boccard and Wauthy (2010). To see how the results of the baseline model are related to the existing literature, I then compare the type of equilibria that can arise for intermediate capacity levels for all possible levels of product differentiation.

4.1 Case of very high product differentiation: $v/t \leq 1$

When products are very differentiated, the analysis is fairly straightforward. First, notice that choosing $T_1^{LM} = \frac{v}{2t}$ is always optimal whenever available as it maximizes π_1^{LM}, the local monopoly branch of the profit function, and $\pi_1^{LM}(T_1^{LM}) > \pi_1^C(T_1^C)$. Therefore, any price leading to $T_1 > \frac{v}{2t}$ would be suboptimal. In other words, neither firm wants to choose low prices that would attract more than $\frac{v}{2t}$ consumers. However, this means that the firms want to serve less than $1/2$ of the market in the case of $v/t \leq 1$. Therefore, firms never enter into direct competition, they both act as local monopolists, serving $\frac{v}{2t}$ consumers if their capacity permits it. The following Lemma summarizes these findings.

Lemma 4. *In case of very high product differentiation, i.e. for $v/t \leq 1$, there is a unique pure-strategy equilibrium for any capacity pair. Both firms act as local monopolists: firm i serves $\min\{k_i, v/2t\}$ consumers at price $v - t\min\{k_i, v/2t\}$.*

4.2 Case of low product differentiation: $1.5 < v/t \leq 2$

In this Subsection I complete the analysis with the case of low product differentiation: $1.5 < v/t \leq 2$. I show that there always exist a non-empty range of parameters for which a pure-strategy equilibrium fails to exist. This highlights the importance of the intermediate level of product differentiation in the main model. Moreover, I show that the semi-mixed equilibrium identified in Boccard and Wauthy (2010, Lemma 4) also exists for $1.5 < v/t \leq 2$. This slightly unusual type of equilibrium consists of one firm playing a pure strategy (a single price) and its rival randomizing between two strategies.

To see that there are capacity pairs for which no pure-strategy equilibrium exists for $1.5 < v/t \leq 2$, I highlight the three main differences with respect to the main model. First, $v/t > 3/2$ excludes the existence of unconstrained secret handshake equilibria, i.e. firms playing $(\overline{T}_1, \overline{T}_2)$. Intuitively, for $v/t > 3/2$ the price firms would charge in such an equilibrium ($2v + t$) is so high compared to the competitive price (which is independent of v) that the best reply is undercutting it by playing the competitive strategy and gaining a large market share.

The second main difference is that for $v/t > 3/2$ a classic Hotelling-type equilibrium of both firms choosing the competitive strategy and prices equal to t (leading to market shares $T_1^C = 1 - T_1^C = T_2^C = \frac{1}{2}$) exists for sufficiently high capacity levels.

Third, $v/t > 3/2$ means that $1 - \frac{v}{3t} < \frac{1}{2} < \frac{v}{3t}$, i.e. the ordering of some key cut-off values is reversed with respect to the main model. This has surprisingly important consequences. On the one hand, $1 - \frac{v}{3t} < \frac{1}{2}$ is the upper bound for capacity-constrained secret handshake equilibria $(1-k_2, 1-k_2)$. On the other hand, I show in the Appendix that

$$\overline{k} \equiv 1 - \frac{v/t - \sqrt{(v/t)^2 - 2}}{2} > \frac{1}{2}$$

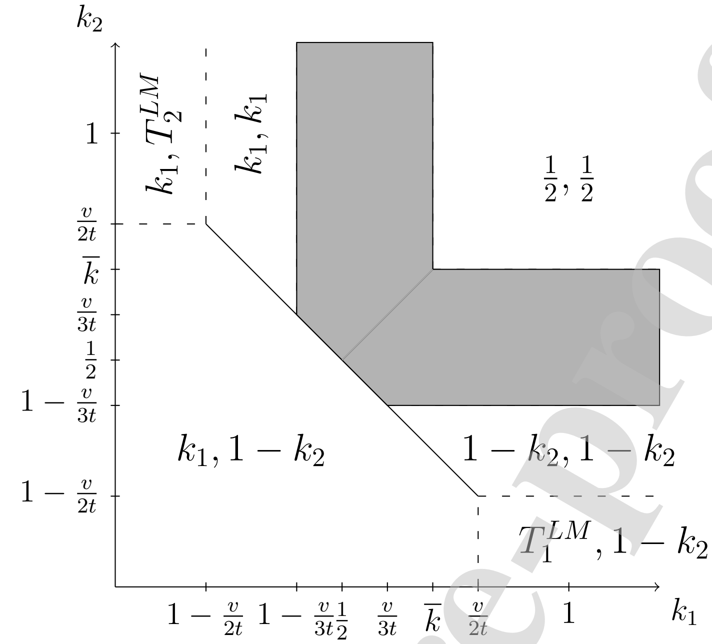
is the lower bound for the existence of a Hotelling-type equilibrium. The next Proposition states that there is no pure-strategy equilibrium in the parameter range (partly) delimited by $1 - \frac{v}{3t} < k_2 < \overline{k}$.

Proposition 2. *For low levels of product differentiation, i.e. for* $1.5 < v/t \leq 2$ *there exists a non-empty range of capacity pairs for which no pure-strategy equilibrium exists. This range is defined by capacity pairs* (k_1, k_2) *that jointly satisfy*
(i) $k_1 + k_2 > 1$ *and (ii)* $\min\{k_1, k_2\} \geq 1 - \frac{v}{3t}$ *and (iii)* $\max\{k_1, k_2\} \leq \overline{k}$.

Figure 3 depicts the different equilibria for $1.5 < v/t \leq 2$. The shaded area is the range of capacity pairs with no pure-strategy equilibrium. Clearly, $\overline{k} > 1/2$ ensures that this area is non-empty. Moreover, the lower the level of product differentiation (the higher v/t is), the larger this area becomes since $\overline{k}$ is increasing in v/t and $1 - \frac{v}{3t}$ is decreasing in v/t.

The proof of Proposition 2, relegated to the Appendix, consists of finding all pure-strategy equilibria, revealing the lack of such equilibria for the parameter range defined in the Proposition. Intuitively, for capacity pairs in this range there is an “Edgeworth-cycle” that can be described as follows. Assume Firm 1 playing $1 - k_2$ and the corresponding relatively high price. It is then in Firm 2’s interest to undercut that price by choosing its competitive best reply and gaining a large market share. However, facing a low price, it is in Firm 1’s interest to regain market share by also lowering its price. Both firms playing low competitive prices cannot be an equilibrium, either: it is then in Firm 2’s interest to raise its

Figure 4: Equilibria with low product differentiation ($1.5 < v/t \leq 2$)



price by playing $1-k_2$. In turn, Firm 1 also raises its price and the cycle starts again.

Finally, this intuition may also help explain the existence of a semi-mixed equilibrium in which one firm plays a pure strategy while the other randomizes between two strategies. I show in the Appendix that this type of equilibrium, identified in Boccard and Wauthy (2010, Lemma 4) for the case of $v/t > 2$, also exists for $1.5 < v/t \leq 2$. In particular, Firm 2 choosing its price equal to

$$\overline{p}_2 \equiv \sqrt{8t(v-t(1-k_2))(1-k_2)} - t$$

makes Firm 1 indifferent between playing T_1^C or $1-k_2$. In turn, Firm 1 playing $1-k_2$ with probability w and T_1^C with probability $1-w$ with

$$0 < w = \frac{3\sqrt{2(v-t(1-k_2))(1-k_2)} - 3\sqrt{t}}{3\sqrt{2(v-t(1-k_2))(1-k_2)} + (2k_2-3)\sqrt{t}} < 1$$

makes $\overline{p}_2$ the optimal choice for Firm 2. This type of equilibrium is arguably interesting when comparing the different types of equilibria that arise for different levels of product differentiation, an exercise I do in the next subsection.

4.3 Comparison of product differentiation levels

In this Subsection, I compare the case of intermediate capacity levels with varying degrees of product differentiation.

(i) $v/t = \infty$: mixed-strategy equilibria with continuous support

(ii) $2 < v/t < \infty$: mixed-strategy equilibria with finite support

(iii) $1.5 < v/t \leq 2$: semi-mixed equilibria

(iv) $1 < v/t \leq 1.5$: nontrivial pure-strategy equilibria

(v) $v/t \leq 1$: trivial pure-strategy equilibrium

(i) is the case of homogeneous goods which is the seminal result of Kreps and Scheinkman (1983). (ii) is the main result of Boccard and Wauthy (2010). Furthermore, they prove that the number of atoms used in equilibrium is decreasing in v/t. (iii) is the case investigated in Subsection 4.2. (iv) is the main result of this article, described in Section 3. (v) is the local monopoly case discussed in Section 4.1.

In light of this comparison, the main result of the paper, the existence of pure-strategy equilibrium for all capacity pairs, can be seen as the number of atoms used in equilibrium being reduced all the way to one for intermediate levels of product differentiation. This is also why the semi-mixed equilibrium arising for $1.5 < v/t \leq 2$ is of interest: one can view it as a smooth transition between completely mixed equilibria with finite support and pure-strategy equilibria.

5 Vertical product differentiation extension

In the next section, I will analyze a model where firms are asymmetric in the following sense: Firm 2 will be located at point $1 + a$, with $0 < a < 1$, while Firm 1 remains at 0 and consumers are located uniformly on $[0, 1]$.[7] Thus Firm 2 is disadvantaged: It is located on average a units farther from the consumers than its rival. This setup can also be thought of as a particular form of vertical product differentiation. Therefore this asymmetric model will serve as a robustness check for the baseline model of pure horizontal product differentiation.

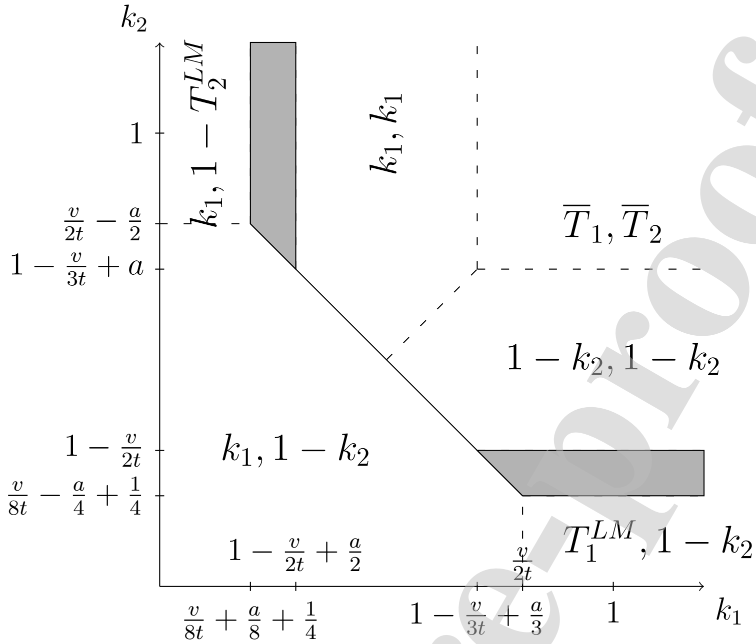
The main difference with the baseline model is in the net surplus consumers derive from purchasing from Firm 2. For a consumer located at $0 \leq x \leq 1$, it is given by

$$v - p_2 - (1 + a - x)t$$

thus each consumer buying from Firm 2 incurs an additional transportation cost of at compared to the baseline model. Consequently, both the participation constraint and the market splitting constraint of consumers of Firm 2 are changed:

[7] I would like to thank to Xavier Wauthy for the idea of this model variant.

Figure 5: Asymmetric equilibria $(1.2 + 0.6a < v/t \leq 1.5)$



$$v - p_2 - (1 + a - x)t \geq 0 \qquad \text{(PC')}$$

and

$$v - p_1 - tx \geq v - p_2 - (1 + a - x)t \qquad \text{(MS')}$$

It is also important to note that the capacity constraints are unchanged. Despite being farther from the consumers, Firm 2 still has the possibility of serving k_2 consumers if it can attract them with a low price.

Naturally, the 5 potential equilibrium strategies of both firms are also affected by the asymmetry. Somewhat surprisingly, I can show that despite these changes, both Lemma 1 and Lemma 2 hold for $0 < a < 1$. The proofs are relegated to the Appendix. The next proposition summarizes the main result of the asymmetric model.

Proposition 3. *For $1 < v/t \leq 1.2 + 0.4a$ there exists at least one equilibrium in pure strategies for any capacity pair (k_1, k_2). For $1.2 + 0.4a < v/t \leq 1.5$ there is no pure-strategy equilibrium for capacity levels satisfying $k_1 + k_2 > 1$ and $\frac{v}{8t} + \frac{1-a}{4} < k_2 < 1 - \frac{v}{2t}$. Moreover, for $1.2 + 0.6a < v/t \leq 1.5$ there is no pure-strategy equilibrium for capacity-pairs satisfying $k_1 + k_2 > 1$ and $\frac{v}{8t} + \frac{a}{8} + \frac{1}{4} < k_1 < 1 - \frac{v}{2t} + \frac{a}{2}$ either.*

Proposition 3 states that the main result of the baseline model holds in the asymmetric model as well for relatively high levels of product differentiation.

However, for lower levels, it identifies two parameter regions without pure-strategy equilibrium. Hence the existence result of pure-strategy equilibria for any capacity level is only partially robust to the the introduction of vertical product differentiation.

Intuitively, if the level of product differentiation is high ($1 < v/t \leq 1.2 + 0.4a$), then the local monopoly power of firms is sufficiently strong to impede direct competition. Similarly to the baseline model, firms act as local monopolies if their capacity is small and engage in secret handshake equilibria for higher capacity levels. However, for lower levels of product differentiation (but still assuming $v/t \leq 1.5$) the asymmetry in firms' location results in the lack of equilibria for intermediate capacity levels.

As illustrated in Figure 5, the size of the two areas without pure-strategy equilibrium (shaded in the figure) depend on a in an intuitive way: the larger the asymmetry, the larger the regions without pure-strategy equilibria. Conversely, these two areas disappear as the two firms become symmetric (i.e. $a \to 0$), in other words this model variant converges to the baseline model. To see this, it is sufficient to check that all cut-offs converge to their equivalent in the baseline model, moreover, the width of the two areas, i.e.

$$\left(1 - \frac{v}{2t} + \frac{a}{2}\right) - \left(\frac{v}{8t} + \frac{a}{8} + \frac{1}{4}\right) \quad \text{and} \quad \left(1 - \frac{v}{2t}\right) - \left(\frac{v}{8t} + \frac{1-a}{4}\right)$$

go to zero as a goes to zero for $v/t = 1.2$ and they become negative for product differentiation levels that correspond to $v/t > 1.2$.

6 Conclusion

This paper analyzes a Bertrand-Edgeworth duopoly with exogenous capacity constraints and a non-negligible degree of product differentiation. The complete characterization of the model's equilibria was feasible and showed that there exists at least one pure-strategy equilibrium for any capacity level. This contrasts with the usual result of existing Bertrand-Edgeworth models that find nonexistence of such equilibria for some capacity levels. Moreover, the analysis of an asymmetric model revealed that the existence result is partially robust to the introduction of vertical product differentiation.

The main finding of the paper illuminates the importance of local monopoly power in the price setting of capacity-constrained industries, especially in the short run where exogenous capacities are realistic. As Hunold and Muthers (2019) describe more in detail, such findings can be interesting for competition policy cases, for example in the cement industry. In particular, if the competitive equilibrium is secret handshake then competition authorities may mistake equilibrium behavior for collusion.

A clear limitation of the present model is that capacities are exogenously given. However, endogenizing the choice of capacity investment seems to be intractable due to the multiplicity of equilibria in certain parameter regions.

Another limitation is that the location of firms on the Hotelling line is fixed. On the one hand, from d'Aspremont, Gabszewicz, and Thisse (1979) we know that there is no pure-strategy equilibrium in a game of endogenous location choice followed by pricing if the capacity constraints are very large, i.e. if $k_1, k_2 > 1$ (and transportation costs are linear as in the present paper). On the other hand, there is clearly a continuum of pure-strategy equilibria if capacities are very small, i.e. $k_1 + k_2 < 1$, in which Firm i serves the closest $k_i/2$ consumers on each side, for the local monopoly price of $v - tk_i/2$, and firms locate at a distance of at least $(k_1+k_2)/2$ from each other so that there is no overlap in their customers. However, for intermediate levels of capacities the model in Section 5 suggests[8] a potential lack of pure-strategy equilibria in the pricing stage when location choices are asymmetric. For this reason, the analysis of endogenous location choice for intermediate capacity levels is beyond the scope of this paper.

7 Appendix

Proof of Lemma 2

(A1) First assume $T_1^{LM} < k_1$. By Lemma 2 the condition $T_1^{LM} < \overline{T}_1$ implies $T_1^C < \overline{T}_1$. By definition T_1^{LM} is the profit maximizing quantity on the π_1^{LM} curve. Hence

$$\pi_1^{LM}(T_1^{LM}) \geq \pi_1^{LM}(\overline{T}_1) = \pi_1^C(\overline{T}_1) \geq \pi_1^C(x) \quad \text{for all } x > \overline{T}_1$$

where the last inequality holds because $T_1^C < \overline{T}_1$ means that π_1^C is decreasing on the interval in question.

[8] Section 5 discusses a model where the asymmetry comes from one of the firms sitting *outside* the Hotelling line and finds that pure-strategy equilibria fail to exist for some intermediate capacity pairs and product differentiation levels. However, I believe this result is a strong indication for the case relevant for the analysis of endogenous location choice where the asymmetry comes from the firms being located in the *interior* of the Hotelling line.

k_1 is clearly the optimal choice when $T_1^{LM} \geq k_1$ as π_1^{LM} is increasing up to T_1^{LM}.

(A2) is proved in the main text.

(A3) Assume $\overline{T}_1 < k_1$. Firstly, $T_1^C \leq \overline{T}_1$ implies that

$$\pi_1^{LM}(\overline{T}_1) = \pi_1^C(\overline{T}_1) \geq \pi_1^C(x) \quad \text{for all } x > \overline{T}_1$$

Secondly, $\overline{T}_1 \leq T_1^{LM}$ implies that

$$\pi_1^{LM}(x) \leq \pi_1^{LM}(\overline{T}_1) = \pi_1^C(\overline{T}_1) \quad \text{for all } x < \overline{T}_1$$

This means that the profit function is increasing up to $\overline{T}_1$ and then it is decreasing. Again, k_1 is clearly the optimal choice when $\overline{T}_1 \geq k_1$ as π_1^{LM} is increasing up to $\overline{T}_1$.

□

Proof of Lemma 3

(B1) The proof of case (B1) is identical to the proof of case (A1) above.

(B2) is proved in the main text.

(B3) $\overline{T}_1 \leq 1 - k_2 \leq T_1^C$ implies that Firm 1 must compare $\pi_1^{LM}(1-k_2)$ to $\pi_1^C(T_1^C)$ which are the two local maxima of the profit function, except if k_1 is low, then the capacity might be the optimal choice.

(B4) Given the condition $\overline{T}_1 < 1 - k_2$, the constraint (PC) binds on $[0, 1-k_2]$. The profit function π_1^{LM} is increasing up to $1 - k_2$ since $T_1^{LM} > 1 - k_2$. Moreover, $\pi_1^{LM}(1-k_2) > \pi_1^C(1-k_2)$ and also π_1^C is decreasing above $1 - k_2$.

(B5) Given the condition $\overline{T}_1 < 1 - k_2$, the constraint (PC) binds on $[0, 1-k_2]$. The unconstrained optimum at $T_1^{LM}(< 1 - k_2)$ is feasible for Firm 1 whenever its capacity is sufficiently large.

□

Proof of Proposition 1 The proof builds heavily on the results of Lemmata 2 and 3 that identify parameter regions in which one of the 5 potential equilibrium strategies dominate any other strategy for a given firm. In the following I check the conditions of the 15 possible combinations of the potentially dominating strategies of the two firms and determine whether they are compatible or not.

Firstly, notice that any case where $k_1 + k_2 \leq 1$ is trivial: the firms do not have sufficient capacity to cover the market, they can never enter into competition. Hence $\pi_i = \pi_i^{LM}$ and the only possible equilibrium is Firm 1 playing $\min(T_1^{LM}, k_1)$ and similarly, Firm 2 playing $\max(T_2^{LM}, 1 - k_2)$. In the following, I consider the case of $k_1 + k_2 > 1$.

Consider the 5 cases in which Firm 1 plays T_1^{LM}:

T_2^{LM}: When Firm 2 plays T_2^{LM} both firms serve $v/2t$ consumers and their price is equal to $p_1 = p_2 = v/2$. This may only happen if condition (A1) or (B1) or (B5) is satisfied for both firms. (A1) and (B1) imply $p_i > \frac{3}{2}v - t$ which in turn implies $v/t < 1$ which contradicts Assumption 1. The only remaining possibility is that (B5) holds for both firms. However, that cannot be, as it necessitates $k_1 < T_2^{LM} < \overline{T}_2 = \overline{T}_1 < T_1^{LM} < 1 - k_2$, which in turn implies $k_1 + k_2 \leq 1$. Therefore this case will never arise in equilibrium if $k_1 + k_2 > 1$.

T_2^C: Firm 1 playing T_1^{LM} while Firm 2 plays T_2^C can never happen since by definition this would entail (MS) binding for Firm 2 and slack for Firm 1 which is a contradiction.

$\overline{T}_2$: Firm 2 cannot play $\overline{T}_2$ for the same reason it cannot play T_2^C.

k_1: Firm 2 playing k_1 is incompatible with Firm 1 playing T_1^{LM}. To see this, notice that playing k_1 can only be optimal for Firm 2 if the condition of Case B is satisfied, namely $k_1 < \overline{T}_2$, thus

$$\frac{v}{2t} < k_1 < \overline{T}_2 = \frac{v - p_1}{t} = \frac{v}{2t},$$

which is a contradiction. The last inequality is a result of $p_1 = v/2$.

$1 - k_2$: Next I show that Firm 2 playing $1 - k_2$ and Firm 1 playing T_1^{LM} is an equilibrium if $k_1 > v/2t$ and $k_2 < 1 - v/2t$. Notice that $p_1 = v/2$ and $p_2 = v - t \cdot k_2$. By replacing these values into the formulas, it is easy to see that

$$1 - T_2^C < 1 - \overline{T}_2 < 1 - T_2^{LM}$$

which means by Lemma 3 that Firm 2 should play $\max(\overline{T}_2, 1 - k_2)$. Since $\overline{T}_2 = v/2t < 1 - k_2$ it is optimal for Firm 2 to play $1 - k_2$. Finally, notice that according to case (B5), $k_1 > v/2t$ implies that playing T_1^{LM} is a best reply for Firm 1 as well.

Now consider the 4 cases where Firm 1 plays $1 - k_2$. (The remaining fifth such case is symmetric to one case analyzed above.) This may only be optimal for the firm if one of the conditions (B2), (B3) or (B4) holds. Notice that it is common among these conditions that $\overline{T}_1 \leq 1 - k_2$, moreover, $1 - k_2$ is only played when (PC) binds so $p_1 = v - t \cdot (1 - k_2)$.

$1 - k_2$: If Firm 2 plays $1 - k_2$, $p_2 = v - t \cdot k_2$ always holds. Conditions for (B2) imply $p_2 < \frac{4}{3}v - t$ and $T_1^C < 1 - k_2$ which imply $1 - v/3t < k_2 < 1 - v/3t$ so (B2) is not compatible with k_2.

Conditions for (B3) require that $\pi_1^{LM}(1 - k_2) > \pi_1^C(T_1^C)$ which is equivalent to

$$0 > \frac{(v + t(1 - k_2))^2}{8t} - (v - (1 - k_2)(1 - k_2)) \iff 0 > [v - 3t(1 - k_2)]^2$$

which is impossible, so (B3) is also incompatible with k_2.

Conditions for (B4) are in turn compatible with Firm 2 playing $1 - k_2$. The conditions for a $(1 - k_2, 1 - k_2)$-type equilibrium are the following:

$$1 - \frac{v}{2t} < k_2 < \min(1 - \frac{v}{3t}, \frac{v}{2t}) \quad \text{and} \quad k_1 + k_2 > 1.$$

Firstly, it is optimal for Firm 1 to play $1 - k_2$ to Firm 2 playing $1 - k_2$ if and only if $1 - \frac{v}{2t} < k_2 < 1 - \frac{v}{3t}$. Secondly, $1 - k_2$ is a best reply for Firm 2 to Firm 1 playing $1 - k_2$ if and only if $\frac{v}{3t} < k_2 < \frac{v}{2t}$ or $\frac{v}{3t} \geq k_2$ which reduces to the additional constraint of $k_2 < \frac{v}{2t}$.

$\overline{T}_2$: Notice that when Firm 2 plays $\overline{T}_2$ and Firm 1 plays $1 - k_2$, $\overline{T}_2 = 1 - k_2$ so the cut-off value for Firm 2 exactly coincides with it serving consumers up to capacity. This means that this case is identical to the one above.

T_2^C: Notice that T_2^C is only played by Firm 2 if $T_2^C > \overline{T}_2$ which implies $p_1 < \frac{4}{3}v - t$ which is equivalent to $k_2 < v/3t$. However, $T_2^C < k_2$ which entails $k_2 > v/3t$ is also necessary. This shows that T_2^C is incompatible with Firm 1 playing $1 - k_2$.

k_1: Firm 2 playing k_1 is incompatible withFirm 1 playing $1 - k_2$. These quantities entail prices $p_1 = v - t \cdot (1 - k_2)$ and $p_2 = v - t(1 - k_1)$ which imply $\overline{T}_2 = 1 - k_2$ and $\overline{T}_1 = k_1$. However, conditions for Firm 1 playing $1 - k_2$ (case B) require $k_1 = \overline{T}_1 \leq 1 - k_2$ which is ruled out by $k_1 + k_2 > 1$.

Now consider the 3 cases when Firm 1 plays $\overline{T}_1$.

$\overline{T}_2$: There is an equilibrium where Firm 2 plays $\overline{T}_2$ and Firm 1 plays $\overline{T}_1$. The conditions of optimality translate to $p_1 + p_2 = 2v - t$ and also $\frac{4}{3}v - t < p_2 < \frac{3}{2}v - t$. Furthermore, conditions concerning the capacities require $k_1, k_2 \geq \min(1 - \frac{v}{3t}, \frac{v}{2t})$. Thus there is a continuum of of equilibria in this capacity range.

$1-k_2$: Firm 2 playing $1-k_2$ and Firm 1 playing $\overline{T}_1$ is possible only if $1-k_2=\overline{T}_2$ otherwise the (MS) constraint would bind for the one firm but not for the other. If this is true, the case is naturally identical to the case above.

T_2^C: Firm 2 playing T_2^C is impossible when firm plays $\overline{T}_1$ because then the constraint (MS) would be binding for Firm 1 and slack for Firm 2 which is a contradiction.

Now consider the 2 cases when Firm 1 plays T_1^C.

T_2^C: Both firms playing the competitive strategy leads to $p_1=p_2=t$ and both firms serving exactly 1/2 of the market. However, this requires $\overline{T}_1<T_1^C$ (in both A2 and B3) which from Lemma 2 implies $p_2=t\leq\frac{4}{3}v-t$. This in turn implies that product differentiation is low, $v/t>1.5$ which case is excluded in this Proposition.

$1-k_2$: Firm 2 playing $1-k_2$ and Firm 1 playing T_2^C is possible only if $k_2=T_2^C$ otherwise the (MS) constraint would bind for the one firm but not for the other. If this is true, the case is naturally identical to the case above.

The remaining case is when both firms serve consumers up to their capacity. However, Lemma 3 ensures it can only be optimal for firms to do so if (PC) is binding for their marginal consumers. This is clearly impossible when $k_1+k_2>1$.

□

Proof of Proposition 2 The proof consists of two steps.

First, it is straightforward to verify that the results in Lemmata 2 and 3 also hold under $1.5<v/t\leq 2$.

Second, analogously to the proof of Proposition 1, one can analyze all potential equilibrium strategies to find all pure-strategy equilibria. Clearly, most of the 15 cases above are unchanged by the level of product differentiation. Therefore, in the following I investigate only potential equilibrium strategy pairs where the assumption $1.5<v/t\leq 2$ induces a change with respect to the case of intermediate product differentiation.

$(T_1^C,1-T_1^C)$: Both firms playing the competitive strategy leads to $p_1=p_2=t$ and both firms serving exactly 1/2 of the market. For $v/t>1.5$, this is possible under the conditions of (A2) or (B3). The former implies $k_2\geq 1-\overline{T}_1=v/t-1$ using that $p_2=t$. We have a symmetric condition for Firm 1's capacity: $k_1\geq v/t-1$. Next, (B3) requires $\pi_1^{LM}(T_1^{LM})\leq\pi_1^C(T_1^C)$. Straightforward calculations show that this condition is equivalent to

$$k_2 \geq \overline{k} \equiv 1 - \frac{v/t - \sqrt{(v/t)^2 - 2}}{2}.$$

Clearly, we also need a symmetric condition for Firm 1's capacity for $(T_1^C, 1 - T_1^C)$ to be an equilibrium: $k_1 \geq \overline{k}$. Furthermore, $\overline{k}$ is strictly increasing in v/t and its value is exactly 0.5 at $v/t = 1.5$, thus $\overline{k} > 0.5$ for $v/t > 1.5$. Finally, note that $\overline{k} < v/t - 1$ holds for all $v/t > 1.5$, therefore the second condition is weaker. Consequently, $k_1, k_2 > \overline{k}$ is necessary and sufficient for the existence of a $(T_1^C, 1 - T_1^C)$-type equilibrium.

$(\overline{T}_1, 1 - \overline{T}_1)$: As opposed to the case of $v/t \leq 1.5$, this type of equilibrium cannot arise for $1.5 < v/t \leq 2$. To see this, note that in such an equilibrium $p_1 + p_2 = 2v - t$ must hold and also $\frac{4}{3}v - t < p_1,\ p_2 < \frac{3}{2}v - t$. These three conditions imply $\frac{v}{2} < p_1,\ p_2 < \frac{2}{3}v$. For such prices to exist on must have

$$\frac{4}{3}v - t < \frac{2}{3}v \quad \Leftrightarrow \quad \frac{2}{3}v < t$$

which does not hold for $1.5 < v/t$.

□

Proof of existence of semi-mixed equilibria Following Boccard and Wauthy (2010, Lemma 4), the proof is by construction. Let $\overline{p}_2$ denote the price that makes Firm 1 indifferent between playing $1 - k_2$ and T_1^C. Then

$$\frac{(\overline{p}_2 + t)^2}{8t} = (v - t(1 - k_2))(1 - k_2) \quad \Leftrightarrow \quad \overline{p}_2 = \sqrt{8t(v - t(1 - k_2))(1 - k_2)} - t.$$

It remains to show that there exist weights (with values between 0 and 1) for Firm 1's strategies that indeed make $\overline{p}_2$ the optimal choice for Firm 2. The profit of Firm 2 can be written as

$$\pi_2(p_2) = p_2 \left(w k_2 + (1 - w) \frac{t - p_2 + p_1}{2t} \right)$$

as the demand of Firm 2 exceeds k_2 when Firm 1 plays $1 - k_2$ by $k_2 < \overline{k}$ and the fraction provides the location of the indifferent consumer when Firm 1 plays the competitive price response $p_1 = \frac{\overline{p}_2 + t}{2}$ while Firm 2 plays $\overline{p}_2$. The profit is clearly concave in p_2, so the optimal price for Firm 2 is given by the first order condition $\pi_2'(\tilde{p}_2) = 0$. Tedious but straightforward calculations show that the weight w that equate this optimal price $\tilde{p}_2$ with $\overline{p}_2$ that makes Firm 1 indifferent is given by

$$w = \frac{3\sqrt{2(v-t(1-k_2))(1-k_2)} - 3\sqrt{t}}{3\sqrt{2(v-t(1-k_2))(1-k_2)} + (2k_2-3)\sqrt{t}}.$$

To see that $0 < w < 1$, note that the numerator of this fraction is strictly smaller than its denominator, so it suffices to show that the numerator is strictly positive. This in turn is equivalent to

$$\sqrt{2(v-t(1-k_2))(1-k_2)} - \sqrt{t} > 0 \quad \Leftrightarrow \quad 2(v-t(1-k_2))(1-k_2) > t$$

which in turn is ensured by the condition $\overline{p}_2 > t$ which always hold in the region without pure-strategy equilibria. Finally, a last necessary condition for existence of such a semi-mixed equilibrium is

$$k_1 \geq T_1^C(\overline{p}_2) \quad \Leftrightarrow \quad k_1^2 \geq \frac{1}{2}\left(\frac{v}{2t}\right)^2 \text{ or } k_2 \geq 1 - \frac{v}{2t} + \sqrt{\left(\frac{v}{2t}\right)^2 - 2k_1^2},$$

where the last cut-off is clearly below 1 for any k_1. Therefore there always exist capacity pairs without pure-strategy equilibria for which the semi-mixed equilibrium derived here exists.

Proof of Proposition 3 Firstly, I show that Lemma 1 holds in the asymmetric model as well. The logic of the proof is similar to the original one. The new values of potential equilibrium strategies of Firm 1 are the following:

$$T_1^{LM} = \frac{v}{2t}, \quad T_1^C = \frac{p_2 + (1+a)t}{4t}, \quad \overline{T}_1 = 1 + a - \frac{v - p_2}{t},$$

therefore

$$T_1^{LM} \leq \overline{T}_1 \iff p_2 \geq \frac{3}{2}v - (1+a)t$$

and

$$T_1^C \leq \overline{T}_1 \iff p_2 \geq \frac{4}{3}v - (1+a)t$$

and

$$T_1^{LM} \leq T_1^C \iff p_2 \geq 2v - (1+a)t.$$

The above inequalities prove Lemma 1 holds for Firm 1. Similarly, the new potential equilibrium strategies of Firm 2 are:

$$T_2^{LM} = 1 - \frac{v}{2t} + \frac{a}{2}, \quad T_2^C = 1 - \frac{p_1 + t}{4t} + \frac{a}{4}, \quad \overline{T}_2 = \frac{v - p_1}{t},$$

therefore

$$T_2^{LM} \geq \overline{T}_2 \iff p_1 \geq \frac{3}{2}v - t - \frac{at}{2}$$

and

$$T_2^{C} \geq \overline{T}_2 \iff p_1 \geq \frac{4}{3}v - t - \frac{at}{3}$$

and

$$T_2^{LM} \geq T_2^{C} \iff p_1 \geq 2v - at.$$

It is easy to see that $a < v/t$ implies both

$$\frac{4}{3}v - t - \frac{at}{3} \leq \frac{3}{2}v - t - \frac{at}{2} \leq 2v - at.$$

Thus the assumptions of $a < 1$ and $1 \leq v/t$ together imply that Lemma 1 holds for Firm 2 as well.

Secondly, Lemma 1 being satisfied in the asymmetric model directly imply that Lemmas 2 and 3 will also hold.

Thirdly, one must repeat the steps of the proof of Proposition 1 to find the pure-strategy equilibria of the asymmetric model. Below I will only show calculations for strategy-pairs forming an equilibrium or where the reasoning is different than the one in Proposition 1. For all other strategy-pairs the logic of the proof of Proposition 1 remains the same with obvious modifications. Let (T_1, T_2) denote a strategy-pair with Firm 1 choosing T_1 and Firm 2 choosing T_2 as its marginal consumer.

$(k_1, 1-k_2)$: Clearly, both firms serving up to capacity is still an equilibrium if $k_1 + k_2 \leq 1$, $k_1 \leq T_1^{LM}$ and $k_2 \leq 1 - T_2^{LM}$.

(T_1^{LM}, T_2^{LM}): Importantly, (T_1^{LM}, T_2^{LM}) can be an equilibrium of the asymmetric model even if

$$k_1 + k_2 > 1 \quad \text{and} \quad k_1 \geq \frac{v}{2t} \quad \text{and} \quad k_2 \geq 1 - \frac{v}{2t}$$

given that $v/t \leq 1 + a/2$. Indeed, if (A1) or (B1) holds for both firms, ensuring that the strategies are mutual best replies, then $p_1 = v/2$ and $p_2 = v/2 - a/2t$ imply $v/t \leq 1 + a/2$ and vica versa.

$(1-k_2, 1-k_2)$: Using the same arguments as in the proof of Proposition 1, one obtains that both firms choosing $1 - k_2$ is an equilibrium if and only if

$$1 - \frac{v}{2t} < k_2 \leq \min(1 - \frac{v}{3t} + a,\ \frac{v}{2t} - \frac{a}{2}) \quad \text{and} \quad k_1 + k_2 > 1.$$

(k_1, k_1): Similarly to the equilibrium above, one can show that both firms choosing k_1 as their marginal consumer is an equilibrium if and only if

$$1 - \frac{v}{2t} + \frac{a}{2} < k_1 \leq \min(1 - \frac{v}{3t} + \frac{a}{3}, \frac{v}{2t}) \quad \text{and} \quad k_1 + k_2 > 1.$$

$(\overline{T}_1, \overline{T}_2)$: The conditions of optimality translate to $p_1 + p_2 = 2v - t - at$ and also $\frac{v}{2} < p_1 < \frac{2}{3}v$. Consequently conditions concerning the capacities require

$$k_1, k_2 \geq \min(1 - \frac{v}{3t}, \frac{v}{2t} - \frac{a}{2})$$

for this strategy-pair to constitute an equilibrium.

$(T_1^{LM}, 1 - k_2)$: The capacity-pairs for which these strategies form an equilibrium are crucially different for $a > 0$ than for $a = 0$. Firstly, the condition for Firm 1 playing T_1^{LM} being a best reply to Firm 2 playing $1 - k_2$ is simply

$$k_2 < 1 - \frac{v}{2t} \quad \text{and} \quad k_1 \geq \frac{v}{2t}.$$

However, $1 - k_2$ being a best reply for Firm 2 to T_1^{LM} depends on the degree of product differentiation and a. There are 3 cases:

(i) $v/t \leq 1 + 0.5a$: Then $1 - k_2$ is optimal to play if and only if $k_2 \leq \frac{v}{2t} - \frac{a}{2}$. Thus $(T_1^{LM}, 1 - k_2)$ is an equilibrium if and only if

$$k_1 \geq \frac{v}{2t} \quad \text{and} \quad k_2 \leq \frac{v}{2t} - \frac{a}{2} < 1 - \frac{v}{2t},$$

where the last inequality stems from the assumption of case (i).

(ii) $1 + 0.5a < v/t \leq 1.2 + 0.4a$: Then $1 - k_2$ is optimal to play if and only if $k_2 \leq 1 - \frac{v}{2t}$ which is exactly the condition for optimality for Firm 1, thus in this case $(T_1^{LM}, 1 - k_2)$ is an equilibrium if and only if

$$k_2 < 1 - \frac{v}{2t} \quad \text{and} \quad k_1 \geq \frac{v}{2t}.$$

(iii) $v/t > 1.2+0.4a$: Then $1-k_2$ is optimal to play if and only if $k_2 \leq \frac{v}{8t} + \frac{1-a}{4}$. Thus $(T_1^{LM}, 1 - k_2)$ is an equilibrium if and only if

$$k_1 \geq \frac{v}{2t} \quad \text{and} \quad k_2 \leq \frac{v}{8t} + \frac{1 - a}{4} < 1 - \frac{v}{2t},$$

where the last inequality stems from the assumption of case (iii).

(k_1, T_2^{LM}): Similarly to the equilibrium above, three cases can be distinguished, and similar reasoning reveals that

(i) If $v/t \leq 1 + 0.5a$ then (k_1, T_2^{LM}) is an equilibrium if and only if

$$k_2 \geq \frac{v}{2t} - \frac{a}{2} \quad \text{and} \quad k_1 \leq \frac{v}{2t} \leq 1 - \frac{v}{2t} + \frac{a}{2}.$$

(ii) If $1 + 0.5a < v/t \leq 1.2 + 0.6a$ then (k_1, T_2^{LM}) is an equilibrium if and only if

$$k_2 \geq \frac{v}{2t} - \frac{a}{2} \quad \text{and} \quad k_1 \leq 1 - \frac{v}{2t} + \frac{a}{2}.$$

(iii) If $v/t > 1.2 + 0.6a$ then (k_1, T_2^{LM}) is an equilibrium if and only if

$$k_2 \geq \frac{v}{2t} - \frac{a}{2} \quad \text{and} \quad k_1 \leq \frac{v}{8t} + \frac{a}{8} + \frac{1}{4} < 1 - \frac{v}{2t} + \frac{a}{2}.$$

A comparison of the capacity thresholds delimiting the different kinds of equilibria reveals the first part of Proposition 3. Indeed, for $v/t \leq 1.2 + 0.4a$ the capacity-pairs for which there exist at least one equilibrium cover the whole positive quadrant. However, for $v/t > 1.2 + 0.4a$ there is no pure-strategy equilibrium for capacity-pairs satisfying

$$k_1 + k_2 > 1 \quad \text{and} \quad \frac{v}{8t} + \frac{1 - a}{4} < k_2 < 1 - \frac{v}{2t}.$$

Moreover, for $v/t > 1.2 + 0.6a$ there is no pure-strategy equilibrium for capacity-pairs satisfying

$$k_1 + k_2 > 1 \quad \text{and} \quad \frac{v}{8t} + \frac{a}{8} + \frac{1}{4} < k_1 < 1 - \frac{v}{2t} + \frac{a}{2}$$

either.

$\square$

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