Intermittency for the stochastic heat equation with Lévy noise

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Abstract

We investigate the moment asymptotics of the solution to the stochastic heat equation driven by a $(d+1)$-dimensional Lévy space–time white noise. Unlike the case of Gaussian noise, the solution typically has no finite moments of order $1+2/d$ or higher. Intermittency of order p, that is, the exponential growth of the pth moment as time tends to infinity, is established in dimension $d=1$ for all values $p \in (1,3)$, and in higher dimensions for some $p \in (1, 1+2/d)$. The proof relies on a new moment lower bound for stochastic integrals against compensated Poisson measures. The behavior of the intermittency exponents when $p \to 1+2/d$ further indicates that intermittency in the presence of jumps is much stronger than in equations with Gaussian noise. The effect of other parameters like the diffusion constant or the noise intensity on intermittency will also be analyzed in detail.

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1 Introduction

We consider the stochastic heat equation on $\mathbb{R}^d$ given by

$$\begin{aligned} \partial_t Y(t,x) &= \frac{\kappa}{2}\Delta Y(t,x) + \sigma(Y(t,x))\dot{\Lambda}(t,x), \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ Y(0,\cdot) &= f, \end{aligned} \tag{1.1}$$

where $\kappa \in (0,\infty)$ is the diffusion constant, σ a globally Lipschitz function and f a bounded measurable function on $\mathbb{R}^d$. The forcing term $\dot{\Lambda}$ that acts in a multiplicative way on the right-hand side of (1.1) is a *Lévy space–time white noise*, which is the distributional derivative of a *Lévy sheet* in $d+1$ parameters. More precisely, we assume that Λ takes the form

$$\Lambda(\mathrm{d}t,\mathrm{d}x) = b\,\mathrm{d}t\,\mathrm{d}x + \rho\, W(\mathrm{d}t,\mathrm{d}x) + \int_{\mathbb{R}} z\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z), \tag{1.2}$$

where $b\in\mathbb{R}$ is the mean of Λ, $\rho\in\mathbb{R}$ is the Gaussian part of Λ, W is a Gaussian space–time white noise (see [26]), μ is a Poisson measure on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}$ with intensity measure $\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) = \mathrm{d}t\,\mathrm{d}x\,\lambda(\mathrm{d}z)$, and λ is a Lévy measure satisfying

$$\lambda(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (1\wedge |z|^2)\,\lambda(\mathrm{d}z) < \infty.$$

Under the assumption that there exists $p\in[1,1+2/d)$ with

$$m_\lambda(p) := \left(\int_{\mathbb{R}} |z|^p\,\lambda(\mathrm{d}z)\right)^{\frac{1}{p}} < \infty, \tag{1.3}$$

it is shown in [23] that (1.1) admits a unique *mild solution* Y satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|Y(t,x)\|_p = \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^p]^{\frac{1}{p}} < \infty \tag{1.4}$$

for all $T\geq 0$. A mild solution to (1.1) is a predictable process Y satisfying the stochastic Volterra equation

$$Y(t,x) = Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)\sigma(Y(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \tag{1.5}$$

where

$$Y_0(t,x) := \int_{\mathbb{R}^d} g(t,x-y)f(y)\,\mathrm{d}y, \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \tag{1.6}$$

and

$$g(t,x) := g(\kappa;t,x) := \frac{1}{(2\pi\kappa t)^{d/2}} e^{-\frac{|x|^2}{2\kappa t}}, \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \tag{1.7}$$

is the heat kernel in dimension d. As proved in [11], condition (1.3) can be relaxed to include Lévy noises with bad moment properties such as α-stable noises, but in this paper, we will work with (1.3) as a standing assumption.

Our goal is to investigate the behavior of the moments of the solution Y as time tends to infinity. In particular, we are interested in conditions under which the solution Y to (1.5) exhibits the phenomenon of intermittency. The following definition follows [7], Definition III.1.1, [12], Equations (1.6) and (1.7), and [20], Definition 7.5.

Definition 1.1 Let Y be the mild solution to (1.5) and $p \in (0,\infty)$.

(1) Y is said to be *weakly intermittent of order* p if

$$0 < \underline{\gamma}(p) \leq \overline{\gamma}(p) < \infty, \tag{1.8}$$

where the *lower* and *upper moment Lyapunov exponents* $\underline{\gamma}(p)$ and $\overline{\gamma}(p)$ are defined as

$$\underline{\gamma}(p) := \liminf_{t\to\infty} \frac{1}{t} \inf_{x\in\mathbb{R}^d} \log \mathbb{E}[|Y(t,x)|^p] \quad \text{and} \quad \overline{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}^d} \log \mathbb{E}[|Y(t,x)|^p]. \tag{1.9}$$

(2) Y is said to have a *linear intermittency front of order* p if

$$0 < \underline{\lambda}(p) \leq \overline{\lambda}(p) < \infty, \tag{1.10}$$

where the *lower* and *upper intermittency fronts* $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ are defined as

$$\begin{aligned} \underline{\lambda}(p) &:= \sup\left\{\alpha > 0\colon \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log \mathbb{E}[|Y(t,x)|^p] > 0\right\}, \\ \overline{\lambda}(p) &:= \inf\left\{\alpha > 0\colon \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log \mathbb{E}[|Y(t,x)|^p] < 0\right\}, \end{aligned} \tag{1.11}$$

with the convention that $\sup\emptyset := 0$ and $\inf\emptyset := +\infty$.

For important classes of random fields, the purely moment based notion of weak intermittency in (1.8) translates into an interesting path property called *physical intermittency*: With high probability, the random field exhibits an extreme mass concentration at large times, in the sense that it almost vanishes on $\mathbb{R}^d$ except for exponentially small areas where it develops a whole cascade of exponentially sized peaks. We refer to [4, Section 2.4] for a precise statement.

Similarly, if the initial condition f decays at infinity (in this case we cannot expect to have (1.8) because of lacking uniformity in the spatial variable), the property (1.10) would indicate that intermittency peaks, originating from the initial mass around the origin, spread in space at a (quasi-)linear speed.

Review of literature

The intermittency problem has been investigated by many authors in various situations. For example, [7] is a classical reference for intermittency in the *parabolic Anderson model (PAM)* on $\mathbb{Z}^d$, which is the discrete-space analogue of (1.1) with

$$\sigma(x) = \sigma_0 x, \quad x \in \mathbb{R}, \tag{1.12}$$

for some $\sigma_0 > 0$. For the stochastic heat equation, and in particular the continuous PAM driven by a Gaussian space–time white noise, this is analyzed in all its facets in [4, 9, 12, 17, 18], just to name a few. We also refer to [20] for a good overview of the subject.

When it comes to stochastic PDEs with non-Gaussian noise, there is much less literature on this topic. Apart from work on the discrete PAM (see [1, 13] and the references therein), we are only aware of [3] that considers the intermittency problem in continuous space and time. This article investigates the Lévy-driven *stochastic wave equation* in one spatial dimension, and shows that the solution is weakly intermittent of any order $p \geq 2$ under natural assumptions. For the proof of the intermittency upper bounds, the authors employ predictable moment estimates for Poisson stochastic integrals, which are surveyed in [21] in detail. The proof of the lower bound, by contrast, relies on L^2-techniques, which are the same as in the Gaussian case treated in [14] or [20].

Summary of results

For the stochastic heat equation (1.1), however, there is an important difference that necessitates the development of new techniques for the intermittency analysis. Namely, as soon as Λ contains a non-Gaussian part, the solution to (1.1) will typically have finite moments only up to the order $(1+2/d)-\epsilon$, even if Λ itself has moments of all orders or has bounded jump sizes like in the case of a standard Poisson noise, see Theorem 3.1. In particular, as soon as we are in dimension $d \geq 2$, the solution has *no* finite second moment. This is in sharp contrast to the Gaussian case where it is well known that the solution to the stochastic heat equation, if it exists, has finite moments of all orders. And because, as a consequence of the *comparison principle* in Theorem 3.3, we cannot expect in general that the solution is weakly intermittent of order 1, we are forced to consider moments of *non-integer* orders in the range $(1, 1+2/d) \subseteq (1,2)$. Therefore, well-established techniques for estimating *integer* moments of the solution (see [4, 9]) do not apply in this setting.

This problem can be remedied by an appropriate use of the Burkholder–Davis–Gundy (BDG) inequalities for verifying the intermittency *upper* bounds, see Theorem 2.4. However, for the corresponding *lower* bounds, the moment estimates that are available in the literature (including again the BDG inequalities, but also "predictable" versions thereof, see e.g. [21]) do not combine well with the recursive Volterra structure of (1.5). So although these estimates are sharp, we cannot apply them to produce the desired intermittency lower bounds. In order to circumvent this, we use decoupling techniques to establish an – up to our knowledge – new moment lower bound for Poisson stochastic integrals in Lemma 3.4, which we think is of independent interest. With this inequality we then prove the weak intermittency of (1.1) under quite general assumptions. More precisely, if Λ has mean zero, we show in Theorem 3.5 and Theorem 3.6 that we have pth order intermittency for all $p \in (1,3)$ in dimension 1, and for some $p \in (1, 1+2/d)$ in dimensions $d \geq 2$. In the latter case, a small diffusion constant κ, or a high noise intensity also leads to intermittency of any desired order. Noises with positive or negative mean are treated in Theorem 3.10 or Theorem 3.12, respectively. Moreover, the moment estimates in Lemma 3.4 also permit us to determine the asymptotics of the intermittency exponents as $p \to 1+2/d$ or $\kappa \to 0$, see Theorem 4.1. The results suggest that intermittency in the Lévy case is much more pronounced than with Gaussian noise.

Our proofs further indicate that the principal source of intermittency is different between the jump and the Gaussian case. In fact, intermittency in the Gaussian case is caused by the slow decrease in time of the heat kernel, so peaks in the past are remembered for a long time and accumulate to new peaks in the future. By contrast, in the Lévy-driven equation, it is the singularity of the heat kernel at the origin that causes the high-order intermittent behavior of the solution. So here, for p close to $1+2/d$, peaks of order p amplify over short time and hence generate even higher peaks. We refer to Remark 3.9 for details.

In the sequel, we will use the letter C to denote a constant whose value may change from line to line and does not depend on anything important in the given context. Sometimes, if we want to stress the dependence of the constant on an important parameter, say p, we will write C_p. Furthermore, for reasons of brevity, we write $\iint_a^b$ and $\iiint_a^b$ for $\int_a^b \int_{\mathbb{R}^d}$ and $\int_a^b \int_{\mathbb{R}^d} \int_{\mathbb{R}}$, respectively.

2 Intermittency upper bounds

We first investigate the upper indices $\overline{\gamma}(p)$ and $\overline{\lambda}(p)$, respectively. For a random field $\Phi(t,x)$, indexed by $(t,x) \in (0,\infty) \times \mathbb{R}^d$, and exponents $\beta \in \mathbb{R}$, $c \in [0,\infty)$ and $p \in [1,\infty)$, we use the notation

$$\|\Phi\|_{p,\beta,c} := \sup_{t\in(0,\infty)} \sup_{x\in\mathbb{R}^d} e^{-\beta t + c|x|} \|\Phi(t,x)\|_p, \tag{2.1}$$

and

$$(g \circledast \Phi)(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Phi(s,y)\,\Lambda(\mathrm{d}s,\mathrm{d}y) \tag{2.2}$$

if Φ is predictable and the stochastic integral (2.2) exists for all $(t,x) \in (0,\infty)\times\mathbb{R}^d$. The key ingredient for the intermittency upper bounds is the following L^p-estimate for stochastic convolutions. The Gaussian case has been obtained in [12, Proposition 2.5] and [20, Proposition 5.2].

Proposition 2.1 (Weighted stochastic Young inequality) *Let $d \in \mathbb{N}$, $1 \leq p < 1 + 2/d$ and assume that $\rho = 0$ if $p < 2$. For any $c \geq 0$ and $\beta > \kappa c^2 d/2$, we have*

$$\|g \circledast \Phi\|_{p,\beta,c} \leq C_{\beta,c}(\kappa,p)\, \|\Phi\|_{p,\beta,c} \tag{2.3}$$

with

$$\begin{aligned} C_{\beta,c}(\kappa,p) = C_p \Bigg(&\frac{2^d|b|}{\beta - \frac{1}{2}\kappa c^2 d} + \frac{2^{\frac{d(3-p)}{2p}} \Gamma(1-\frac{d}{2}(p-1))^{\frac{1}{p}} m_\lambda(p)}{p^{\frac{2+(2-p)d}{2p}} (\pi\kappa)^{\frac{d(p-1)}{2p}} (\beta - \frac{1}{2}\kappa c^2 d)^{\frac{2-d(p-1)}{2p}}} \\ &+ \frac{m_\lambda(2) + |\rho|}{(2\kappa(\beta - \frac{1}{2}\kappa c^2))^{\frac{1}{4}}} \mathbb{1}_{\{d=1,\ p\geq 2\}} \Bigg), \end{aligned} \tag{2.4}$$

where $C_p > 0$ does not depend on Λ, κ, β, c or d, and it is bounded on $[1+\epsilon, 1+2/d)$ for any $\epsilon > 0$.

The assumption in Proposition 2.1 that $\rho = 0$ if $p < 2$ means that if $d \geq 2$, then necessarily the Gaussian part vanishes because $p < 1 + 2/d \leq 2$. This is reasonable since the stochastic heat equation (1.1) has no function-valued solution in general if $d \geq 2$ and $\rho > 0$, see e.g. [20, Section 3.5]. Moreover, in dimension $d = 1$, we shall only consider the case $p \geq 2$ if $\rho > 0$. The reason behind is that in the case of Gaussian noise, intermittency of order less than 2 is open, see the remark after Theorem 3.5.

Remark 2.2 The three terms in (2.4) illustrate in a nice way the different contributions of the noise to the size of $g \circledast \Phi$. The first part comes from the deterministic drift of the noise, the second summand is the L^p-contribution originating from the jumps, and the third term is the L^2-contribution of the jumps and the Gaussian part (if $p \geq 2$). It is important to notice that a Gaussian noise alone has no extra L^p-contribution to $C_{\beta,c}(\kappa, p)$ for $p > 2$, which reflects the equivalence of moments of the normal distribution. Furthermore, as $p \to 1+2/d$, the second term explodes for all non-trivial Lévy measures λ, no matter how good their integrability properties are. This is a first indication that the solution to a Lévy-driven stochastic heat equation (1.1) usually has no finite moments of order $1+2/d$ or higher. We confirm this rigorously in Theorem 3.1 below.

With the help of Proposition 2.1, we can extend the local moment bound (1.4) obtained in [23] to a global bound.

Proposition 2.3 *Assume that f satisfies $|f(x)| = O(e^{-c|x|})$ as $|x| \to \infty$ for some $c \geq 0$ and that σ in* (1.5) *is Lipschitz continuous with*

$$|\sigma(x) - \sigma(y)| \leq L|x - y|, \quad x, y \in \mathbb{R},$$

for some $L > 0$, and also $\sigma(0) = 0$ if $c > 0$. Further suppose that Λ takes the form (1.2) *and satisfies* (1.3) *for some $1 \leq p < 1+2/d$ as well as $\rho = 0$ if $p < 2$. Then there exists a number $\beta_0 > 0$ such that the stochastic heat equation* (1.5) *has a unique mild solution Y (up to modifications) with $\|Y\|_{p,\beta,c} < \infty$ for all $\beta \geq \beta_0$.*

We obtain as an immediate consequence upper bounds for the moments of the solution Y to the stochastic heat equation (1.5).

Theorem 2.4 (Intermittency upper bounds) *Grant the assumptions and notations of Proposition 2.3.*

(1) We have $\overline{\gamma}(p) < \infty$.

(2) If $c > 0$ and $\sigma(0) = 0$, then $\overline{\lambda}(p) < \infty$.

3 Intermittency lower bounds

3.1 High moments

One important difference between the stochastic heat equation with jump noise and with Gaussian noise is that the solution Y to (1.5) has no large moments, even in dimension $d = 1$ and no matter how good the integrability properties of the jumps are. In order to understand this, let us consider the situation where $\sigma \equiv 1$, $f \equiv 0$, and Λ is a standard Poisson random measure, that is, $\lambda = \delta_1$, $b = 1$, and $\rho = 0$. Denoting by (S_i, Y_i) the space–time locations of the jumps of Λ, we have for $(t, x) \in (0, \infty) \times \mathbb{R}^d$,

$$Y(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) \, \Lambda(\mathrm{d}s, \mathrm{d}y) = \sum_{i=1}^{\infty} g(t - S_i, x - Y_i) \mathbb{1}_{\{S_i < t\}}.$$

If $t > 1$, conditionally on the event that at least one point falls into $(t-1,t) \times \prod_{i=1}^d (x_i - 1, x_i)$, we have

$$Y(t,x) \geq g(U,V) = \frac{1}{(2\pi\kappa U)^{\frac{d}{2}}} e^{-\frac{|V|^2}{2\kappa U}},$$

where $U, V_1, \ldots, V_d$ are independent and uniformly distributed on $(0,1)$, and $V = (V_1, \ldots, V_d)$. Now

$$\begin{aligned}\mathbb{E}[g(U,V)^p] &= \frac{1}{(2\pi\kappa)^{\frac{pd}{2}}} \int_0^1 u^{-\frac{pd}{2}} \left(\int_0^1 e^{-\frac{pv^2}{2\kappa u}} \,\mathrm{d}v \right)^d \mathrm{d}u = \frac{1}{(2\pi\kappa)^{\frac{pd}{2}}} \int_0^1 u^{\frac{d(1-p)}{2}} \left(\int_0^{\frac{1}{\sqrt{u}}} e^{-\frac{py^2}{2\kappa}} \,\mathrm{d}y \right)^d \mathrm{d}u \\ &\geq \frac{1}{(2\kappa\pi)^{\frac{pd}{2}}} \left(\int_0^1 e^{-\frac{py^2}{2\kappa}} \,\mathrm{d}y \right)^d \int_0^1 u^{\frac{d(1-p)}{2}} \,\mathrm{d}u,\end{aligned}$$

which is finite if and only if $p < 1 + 2/d$. So we conclude that

$$\mathbb{E}[|Y(t,x)|^{1+\frac{2}{d}}] = \infty$$

for all $(t,x) \in (1,\infty) \times \mathbb{R}^d$, and, in fact for all $t > 0$. It is not surprising that this holds in a much more general setting. The following results also answers an open problem posed in [3, Remark 1.5]. Its proof will be given after the proof of Theorem 3.6.

Theorem 3.1 (Non-existence of high moments) *Consider the situation described in Proposition 2.3 and assume that $\lambda \not\equiv 0$. Furthermore, suppose that there exists $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^d$ such that*

$$\sigma(Y_0(t_0, x_0)) \neq 0, \tag{3.1}$$

where Y_0 is defined in (1.6)*. If Y denotes the unique mild solution to* (1.1)*, then*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^{1+\frac{2}{d}}] = +\infty \tag{3.2}$$

for all $T > t_0$.

Remark 3.2 The arguments presented in [4] linking the notion of weak intermittency as defined in Definition 1.1 with physical intermittency remain valid even if $\underline{\gamma}(p) = \infty$ for large values of p, provided we have $\underline{\gamma}(p) \uparrow \infty$ for $p \uparrow p_{\max} = \inf\{p > 0 \colon \underline{\gamma}(p) = \infty\} \leq 1 + 2/d$. Under mild assumptions, this is indeed the case as we will see in Theorem 4.1.

3.2 The martingale case

In this subsection, we assume that Λ has mean zero, that is, $b = 0$. As in the Gaussian case, we cannot hope for weak intermittency of order 1 in general. This is a consequence of the following comparison principle for the stochastic heat equation driven by a nonnegative pure-jump Lévy noise, whose proof we postpone to the end of Section 5.2. The Gaussian analogue was established in [22, Theorem 3.1].

Theorem 3.3 (Comparison principle) *Let σ be a non-decreasing Lipschitz function and Λ be a Lévy noise as in* (1.2) *with $b \in \mathbb{R}$, $\rho = 0$ and λ satisfying $\lambda((-\infty, 0]) = 0$ and $m_\lambda(p) < \infty$ for some $p \in [1, 1+2/d)$. Assume that $f_1 \geq f_2 \geq 0$ are two bounded measurable initial conditions, and Y_1 and Y_2 the corresponding mild solutions to* (1.1)*. There exist modifications of Y_1 and Y_2 such that, with probability* 1*, we have $Y_1(t,x) \geq Y_2(t,x)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$.*

In particular, if we have in addition that f is a bounded nonnegative function and $0 \leq \sigma(x) \leq Lx$ for some $L > 0$, then the mild solution Y to (1.1) *has a nonnegative modification with*

$$e^{(b\wedge 0)Lt} \int_{\mathbb{R}^d} g(t,x-y)f(y)\,\mathrm{d}y \leq \mathbb{E}[|Y(t,x)|] = \mathbb{E}[Y(t,x)] \leq e^{(b\vee 0)Lt} \int_{\mathbb{R}^d} g(t,x-y)f(y)\,\mathrm{d}y \quad (3.3)$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$. So if $b = 0$, we have $\overline{\gamma}(1) = 0$ if f is strictly positive on a set of positive Lebesgue measure; $\underline{\gamma}(1) = 0$ if $\inf_{x\in\mathbb{R}^d} f(x) > 0$; $\overline{\lambda}(1) = 0$ if $f(x) = O(e^{-c|x|})$ for some $c > 0$; and $\underline{\lambda}(1) = 0$ by definition.

Thus, we are left to consider exponents in the region $p \in (1, 1+2/d)$. In dimension 1, we can use Itô's isometry to calculate second moments, and there are essentially no differences to the estimates (or exact formulae) obtained in the Gaussian case ([9, 12, 17]). However, for $d \geq 2$, we cannot use Itô's isometry because p is strictly between 1 and 2. Instead, our main tool for proving intermittency in the regime $p < 2$ are the following moment lower bounds for stochastic integrals with respect to compensated Poisson random measures, which are of independent interest and complement existing sharp (but for our purposes not feasible) estimates in the literature (see [21]).

Lemma 3.4 *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on the underlying probability space and N be an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure on $[0,\infty) \times E$, where E is a Polish space. Further suppose that m denotes the intensity measure of N, and $H\colon \Omega \times [0,\infty) \times E \to \mathbb{R}$ is an $(\mathcal{F}_t)_{t\geq 0}$-predictable process such that the process*

$$t \mapsto \int_0^t \int_E H(s,x)\,\tilde{N}(\mathrm{d}s,\mathrm{d}x)$$

is a well-defined $(\mathcal{F}_t)_{t\geq 0}$-local martingale, where $\tilde{N}(\mathrm{d}t,\mathrm{d}x) := N(\mathrm{d}t,\mathrm{d}x) - m(\mathrm{d}t,\mathrm{d}x)$ is the compensation of N.

Then there exists for every $p \in (1,2]$ a constant $C_p > 0$ that is independent of H and m such that

$$\mathbb{E}\left[\left|\iint_{[0,\infty)\times E} H(t,x)\,\tilde{N}(\mathrm{d}t,\mathrm{d}x)\right|^p\right] \geq C_p \frac{\iint_{[0,\infty)\times E} \mathbb{E}[|H(t,x)|^p]\,m(\mathrm{d}t,\mathrm{d}x)}{(1 \vee m([0,\infty)\times E))^{1-\frac{p}{2}}}, \quad (3.4)$$

where $\infty/\infty := 0$. In particular, if the right-hand side of (3.4) *is infinite, then also the left-hand side of* (3.4) *is infinite. Furthermore, for every $p' \in (1,2]$, the constants C_p can be chosen to be bounded away from* 0 *for $p \in [p', 2]$.*

We are now ready to state the intermittency lower bounds for (1.1) that complement the corresponding upper bounds in Theorem 2.4. We start with non-vanishing initial data.

Theorem 3.5 (Intermittency lower bounds – I) *Let Y be the solution to* (1.5) *constructed under the assumptions of Proposition 2.3. Additionally assume that*

$$L_f := \inf_{x\in\mathbb{R}^d} f(x) > 0 \quad \text{and} \quad L_\sigma := \inf_{x\in\mathbb{R}\setminus\{0\}} \frac{|\sigma(x)|}{|x|} > 0, \quad (3.5)$$

and that Λ *has the properties*

$$b = 0, \quad \lambda \not\equiv 0 \quad \text{and} \quad \int_{\mathbb{R}} |z|^{1+\frac{2}{d}} \mathbb{1}_{\{|z|>1\}} \, \lambda(\mathrm{d}z) < \infty. \tag{3.6}$$

Then the following statements are valid.

(1) There exists a value $p_0 = p_0(\Lambda, \kappa, \sigma) \in [1, 1 + 2/d)$ *such that we have* $\underline{\gamma}(p) > 0$ *for all exponents* $p_0 < p < 1 + 2/d$.

(2) For given $p \in (1, 1 + 2/d)$*, there exists* $\kappa_0 = \kappa_0(\Lambda, p, \sigma) \in (0, \infty]$ *such that* $\underline{\gamma}(p) > 0$ *for all diffusion constants* $0 < \kappa < \kappa_0$.

(3) Given $p \in (1, 1 + 2/d)$ *and* $\kappa > 0$*, there exists* $L_0 = L_0(\Lambda, p, \kappa) \in [0, \infty)$ *such that* $\underline{\gamma}(p) > 0$ *if* σ *has the property* $L_\sigma > L_0$.

(4) In dimension $d = 1$*, we can take* $p_0 = 1$*,* $\kappa_0 = \infty$ *and* $L_0 = 0$.

To paraphrase, under the assumptions of Theorem 3.5, we have weak intermittency of order p for every $p \in (1, 3)$ in dimension 1, while for higher dimensions we have this if p is close enough to $1 + 2/d$, or κ is small enough, or the size of σ (or equivalently, the noise intensity) is large enough. It remains an open question whether in dimension $d \geq 2$, we always have intermittency of all orders $p \in (1, 1 + 2/d)$. Also, in contrast to the jump case where we have an affirmative answer, it seems to be open whether the solution to (1.1) in $d = 1$ with Gaussian noise is weakly intermittent of order $p \in (1, 2)$.

For decaying initial condition, we have the following counterpart for the indices $\underline{\lambda}(p)$.

Theorem 3.6 (Intermittency lower bounds – II) *Let* Y *be the solution to* (1.5) *constructed in Proposition 2.3. Further assume that* $c > 0$*,* $L_\sigma > 0$ *(as defined in* (3.5)*),* $\sigma(0) = 0$*, that* f *is nonnegative and strictly positive on a set of positive Lebesgue measure,* $f(x) = O(e^{-c|x|})$ *as* $|x| \to \infty$*, and that* Λ *satisfies* (3.6)*.*

(1) There exists a value $p_1 = p_1(\Lambda, \kappa, \sigma) \in [1, 1+2/d)$ *such that* $\underline{\lambda}(p) > 0$ *for all* $p \in (p_1, 1+2/d)$.

(2) Given $p \in (1, 1 + 2/d)$*, there exists* $\kappa_1 = \kappa_1(\Lambda, p, \sigma) \in (0, \infty]$ *such that* $\underline{\lambda}(p) > 0$ *for all* $0 < \kappa < \kappa_1$.

(3) Given $p \in (1, 1 + 2/d)$ *and* $\kappa > 0$*, there exists* $L_1 = L_1(\Lambda, p, \kappa) \in [0, \infty)$ *such that* $\underline{\lambda}(p) > 0$ *for all* σ *satisfying* $L_\sigma > L_1$.

(4) In $d = 1$*, we can take* $p_1 = 1$*,* $\kappa_1 = \infty$ *and* $L_1 = 0$.

Remark 3.7 If $d = 1$, $m_\lambda(2) < \infty$ and we consider the indices $\overline{\gamma}(2)$, $\underline{\gamma}(2)$, $\underline{\lambda}(2)$ and $\overline{\lambda}(2)$, there is – thanks to Itô's isometry – absolutely no difference between a Lévy and a Gaussian noise if we replace σ by $\sqrt{v}\sigma$ where $v = \rho^2 + m_\lambda(2)^2$ is the variance of Λ. For example, the explicit formulae derived in [9] immediately extend to the Lévy case.

Remark 3.8 In [9], the authors consider the stochastic heat equation with a measure-valued (e.g., a Dirac delta) initial condition. Their proof for the existence and uniqueness of solutions can be adapted to the Lévy setting by replacing L^2-estimates with L^p-type estimates from the BDG inequalities. Furthermore, since the heat operator smooths out a rough initial condition immediately, the intermittency properties of the solution will only depend on its decay and support properties. For example, Theorem 3.6 as well as the Theorems 3.10(2), 3.12 and 4.1(2) continue to hold for the solution with a Dirac delta initial condition.

Remark 3.9 The intermittency of (1.1) with Gaussian noise is analytically due to the non-integrable tails of g^2 at $t = +\infty$ (see [12, 17]). Translated into the picture of physical intermittency, this suggests that peaks in the past remain "visible" for a long time, and finally add up to new peaks. In the Lévy case, our proofs hint at the same phenomenon in dimension 1 for the intermittency islands of low order (i.e., p close to 1). However, regardless of dimension, peaks of orders close to $1 + 2/d$, which are the dominating ones from a macroscopic level, arise from the singularity of the heat kernel at small times (this is further confirmed in the asymptotics we derive in Theorem 4.1). It seems that high-order intermittency islands immediately trigger the formation of similar (or even larger) islands, leading to "clusterings" of peaks. It would be interesting for future research to specify and prove these heuristics.

3.3 Noise with positive or negative drift

In this section we consider the intermittency problem for (1.1) when the noise Λ has a non-zero mean. If Λ has a positive mean, that is, if $b > 0$, then under natural assumptions, the solution to (1.1) is even weakly intermittent of order 1 (and hence also of all orders $p \in [1, 1 + 2/d)$).

Theorem 3.10 (Intermittency for noises with positive drift) *Suppose that Y is the solution to* (1.1) *constructed in Proposition 2.3 and assume that σ is a nonnegative Lipschitz continuous function with $L_\sigma > 0$ (as defined in* (3.5)*). Furthermore, if $c = 0$, suppose that L_f, as defined in* (3.5)*, is strictly positive, while for $c > 0$, suppose that f is nonnegative and strictly positive on a set of positive Lebesgue measure. If $b > 0$, the following statements are valid.*

(1) If $c = 0$, then $\underline{\gamma}(1) > 0$.

(2) If $c > 0$, then $\underline{\lambda}(1) > 0$.

If Λ has a negative drift, we restrict ourselves to the *parabolic Anderson model* where σ is given by (1.12). In this case, we can reformulate (1.1) as an equation driven by the martingale part of Λ only. In fact, decomposing $\Lambda(\mathrm{d}t, \mathrm{d}x) = b\,\mathrm{d}t\,\mathrm{d}x + M(\mathrm{d}t, \mathrm{d}x)$, equation (1.1) can be written in the form

$$\begin{aligned} \partial_t Y(t,x) &= \frac{\kappa}{2}\Delta Y(t,x) + b\sigma_0 Y(t,x) + \sigma_0 Y(t,x)\dot{M}(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ Y(0,\cdot) &= f. \end{aligned} \tag{3.7}$$

This is the d-dimensional *stochastic cable equation* driven by the zero-mean Lévy space–time white noise $\dot{M}$. In a similar form, it has been studied in [26] for Gaussian driving noise in dimension

$d = 1$. Its mild form is the same as in (1.5) but with g replaced by

$$g'(t,x) = g(t,x)e^{b\sigma_0 t}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d.$$

Proposition 3.11 *Under the assumptions of Proposition 2.3, there exists $\beta_1 > 0$ such that* (3.7) *has a unique mild solution Y satisfying $\|Y\|_{p,\beta,c} < \infty$ for all $\beta \geq \beta_1$. Furthermore, it is a modification of the unique mild solution to* (1.1) *constructed in Proposition 2.3.*

We omit the proof since the existence and uniqueness result follows exactly as in the proof for Proposition 2.3. Moreover, the second statement holds because weak and mild solutions are equivalent in our present setting: The proof is the same as in [26, Theorem 3.2] for Gaussian M and $d = 1$.

Theorem 3.12 (Intermittency for noises with negative drift) *Let Y be the mild solution to* (1.1) *as in Proposition 2.3. Suppose that $b < 0$, $m_\lambda(1+2/d) < \infty$ and that σ is given by* (1.12) *with $\sigma_0 > 0$. If $c = 0$, also assume that $L_f > 0$, and if $c > 0$, that f is nonnegative and positive on a set of positive Lebesgue measure.*

(1) If $\lambda \not\equiv 0$, Theorem 3.5(1)–(3) and Theorem 3.6(1)–(3) continue to hold.

(2) Let a value $p \in (1, 1+2/d)$ be given, with the restriction $p \geq 2$ if $\rho \neq 0$. Whenever κ or $|b|$ is large enough, or σ_0 is small enough (each time keeping the other two variables fixed), we have $\underline{\gamma}(p) \leq \overline{\gamma}(p) < 0$ and $\underline{\lambda}(p) = \overline{\lambda}(p) = 0$.

4 Asymptotics of intermittency exponents

As seen in the previous sections, the intermittency of the mild solution to (1.1) is stronger for higher values of p or smaller values of κ. In this section, we investigate the limiting behavior of $\overline{\gamma}(p)$, $\underline{\gamma}(p)$, $\overline{\lambda}(p)$ and $\underline{\lambda}(p)$ as

$$p \to 1 + \frac{2}{d} \quad \text{and} \quad \kappa \to 0.$$

In (4.2) and (4.4) below, one should keep in mind that, although not explicitly indicated in the notation, the indices $\overline{\gamma}(p)$ etc. also depend on κ.

Theorem 4.1 (Asymptotics of intermittency exponents) *Consider a noise Λ with non-zero Lévy measure λ.*

(1) Let $c = 0$ and grant the assumptions of Theorem 3.5, Theorem 3.10 or Theorem 3.12 depending on whether Λ has mean $b = 0$, $b > 0$ or $b < 0$. If $b > 0$ or $b < 0$, we also impose that σ is of the form (1.12)*. Then we have*

$$\lim_{p \to 1+\frac{2}{d}} \frac{1 + \frac{2}{d} - p}{\left|\log\left(1 + \frac{2}{d} - p\right)\right|} \log \underline{\gamma}(p) = \lim_{p \to 1+\frac{2}{d}} \frac{1 + \frac{2}{d} - p}{\left|\log\left(1 + \frac{2}{d} - p\right)\right|} \log \overline{\gamma}(p) = \frac{2}{d}, \tag{4.1}$$

$$0 < \liminf_{\kappa \to 0} \kappa^{\frac{p-1}{1+2/d-p}} \underline{\gamma}(p) \leq \limsup_{\kappa \to 0} \kappa^{\frac{p-1}{1+2/d-p}} \overline{\gamma}(p) < \infty. \tag{4.2}$$

(2) *Let $c > 0$ and grant the assumptions of Theorem 3.6, Theorem 3.10 or Theorem 3.12 depending on whether Λ has mean $b = 0$, $b > 0$ or $b < 0$. If $b > 0$ or $b < 0$, we also impose that σ is of the form* (1.12). *Then we have*

$$\frac{1}{d} \leq \liminf_{p\to 1+\frac{2}{d}} \frac{1+\frac{2}{d}-p}{\left|\log\left(1+\frac{2}{d}-p\right)\right|} \log \underline{\lambda}(p) \leq \limsup_{p\to 1+\frac{2}{d}} \frac{1+\frac{2}{d}-p}{\left|\log\left(1+\frac{2}{d}-p\right)\right|} \log \overline{\lambda}(p) \leq \frac{2}{d}. \tag{4.3}$$

If in addition the initial condition decays superexponentially in the sense that $|f(x)| = O(e^{-c|x|})$ as $|x| \to \infty$ for every $c \geq 0$, then

$$0 < \liminf_{\kappa\to 0} \kappa^{-\frac{1+1/d-p}{1+2/d-p}} \underline{\lambda}(p) \leq \limsup_{\kappa\to 0} \kappa^{-\frac{1+1/d-p}{1+2/d-p}} \overline{\lambda}(p) < \infty. \tag{4.4}$$

Remark 4.2 (1) Equation (4.1) asserts that the moment Lyapunov exponents $\underline{\gamma}(p)$ and $\overline{\gamma}(p)$, which determine the exponential rates at which $\mathbb{E}[|Y(t,x)|^p]$ grows for $t \to \infty$, themselves increase at a superexponential speed as p approaches $1 + 2/d$. This is much faster than in the Gaussian case, where for the PAM (1.12) in $d = 1$ with constant f [4, Theorem 2.6] and [20, Theorem 6.4] showed that the Lyapunov exponents have a cubic growth as $n \to \infty$:

$$\underline{\gamma}(n) = \overline{\gamma}(n) = \frac{\sigma_0^4}{4!\kappa} n(n^2 - 1), \quad n \in \mathbb{N}. \tag{4.5}$$

We conclude that the intermittent behavior of the stochastic heat equation with jumps is much stronger than with Gaussian noise.

(2) Similarly, (4.3) states that the velocity at which pth order intermittency peaks propagate in space grows superexponentially when $p \to 1 + 2/d$. Again, this is on a much faster scale than in the Gaussian case, where the indices $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ typically only increase linearly in p: see [19, Proposition 3.11] where for the PAM (1.12) in $d = 1$ with compactly supported initial data f, the authors showed that

$$0 < \liminf_{n\to\infty} \frac{\underline{\lambda}(n)}{n} \leq \limsup_{n\to\infty} \frac{\overline{\lambda}(n)}{n} < \infty. \tag{4.6}$$

We also remark that in the jump case, the asymptotics of the exponents $\underline{\gamma}(p)$ and $\overline{\gamma}(p)$ as $p \to 1 + 2/d$ are similar to the exponents $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$, in contrast to the Gaussian case, cf. (4.5) and (4.6).

(3) Regarding the asymptotics for $\kappa > 0$, a notable difference between jump and Gaussian noise is that in the former case, the rate at which $\underline{\gamma}(p)$ and $\overline{\gamma}(p)$ increases as $\kappa \to 0$ explicitly depends on p, whereas in the latter case, at least for $p \in \mathbb{N}$, it typically does not, see (4.5).

(4) Another interesting observation is that for jump noises, the asymptotics of $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ for $\kappa \to 0$ exhibit a phase transition at $p = 1 + 1/d$. If $p \in (1, 1 + 1/d)$, they decrease like $\kappa^{(1+1/d-p)/(1+2/d-p)}$, if $p = 1 + 1/d$, they are bounded away from zero and infinity in κ, and for $p \in (1 + 1/d, 1 + 2/d)$, they increase like $\kappa^{-(p-1+1/d)/(1+2/d-p)}$. Intuitively speaking, this is because for small κ there are two effects that counteract each other: On the one hand, a

small diffusion constant reduces the speed at which the initial mass at the origin can spread. On the other hand, if κ is small, once an intermittency peak is built up, it takes longer for the Laplace operator to smooth it out, which facilitates the development and transmission of further peaks. Thus, for small values of p, the first effect is dominant, while for large values of p, it is the second effect that wins. In the Gaussian case, the behavior is again different. Here for any $p \in [2, \infty)$, we have

$$0 < \liminf_{\kappa \to 0} \underline{\lambda}(p) \leq \limsup_{\kappa \to 0} \overline{\lambda}(p) < \infty. \tag{4.7}$$

The lower bound follows from [12, Theorem 1.3] together with the fact that $\underline{\lambda}(2) \leq \underline{\lambda}(p)$ for all $p \geq 2$, while the upper bound follows as in the proof of Theorem 4.1 from the formula (2.4).

5 Proofs

5.1 Proofs for Section 2

Lemma 5.1 *Define* $g_{\beta,c}(t,x) := g(t,x)e^{-\beta t + c|x|}$ *for* $(t,x) \in (0,\infty) \times \mathbb{R}^d$. *If* $0 < p < 1 + 2/d$, $c \geq 0$ *and* $\beta > \kappa c^2 d/2$, *then*

$$\int_0^\infty \int_{\mathbb{R}^d} g^p_{\beta,c}(t,x)\,\mathrm{d}t\,\mathrm{d}x \leq \frac{2^{\frac{d}{2}(3-p)}\Gamma(1-\frac{d}{2}(p-1))}{p^{1+d(1-\frac{p}{2})}(\pi\kappa)^{\frac{d}{2}(p-1)}(\beta - \frac{1}{2}\kappa c^2 d)^{1-\frac{d}{2}(p-1)}},$$

where Γ *denotes the gamma function* $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$.

Proof. If $\beta > \kappa c^2 d/2$, then

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d} g^p_{\beta,c}(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \int_0^\infty \frac{e^{-p\beta t}}{p^{\frac{d}{2}}(2\pi\kappa t)^{\frac{d}{2}(p-1)}} \int_{\mathbb{R}^d} \frac{e^{-\frac{p}{2\kappa t}|x|^2}}{(2\pi\kappa t/p)^{\frac{d}{2}}} e^{pc|x|}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^\infty \frac{e^{-p\beta t}}{p^{\frac{d}{2}}(2\pi\kappa t)^{\frac{d}{2}(p-1)}} \left(\int_{\mathbb{R}} \frac{e^{-\frac{p}{2\kappa t}|x|^2}}{(2\pi\kappa t/p)^{\frac{1}{2}}} e^{pc|x|}\,\mathrm{d}x\right)^d \mathrm{d}t \\
&\leq \int_0^\infty \frac{2^d e^{-p\beta t}}{p^{\frac{d}{2}}(2\pi\kappa t)^{\frac{d}{2}(p-1)}} \left(\int_{\mathbb{R}} \frac{e^{-\frac{p}{2\kappa t}|x|^2}}{(2\pi\kappa t/p)^{\frac{1}{2}}} e^{pcx}\,\mathrm{d}x\right)^d \mathrm{d}t \\
&= \int_0^\infty \frac{2^d e^{-p\beta t}}{p^{\frac{d}{2}}(2\pi\kappa t)^{\frac{d}{2}(p-1)}} e^{\frac{1}{2}d\kappa p c^2 t}\,\mathrm{d}t \\
&= \frac{2^{\frac{d}{2}(3-p)}\Gamma(1-\frac{d}{2}(p-1))}{p^{1+d(1-\frac{p}{2})}(\pi\kappa)^{\frac{d}{2}(p-1)}(\beta - \frac{1}{2}\kappa c^2 d)^{1-\frac{d}{2}(p-1)}}.
\end{aligned}$$

∎

Proof of Proposition 2.1. We use the triangle inequality to split

$$\begin{aligned}\|(g\circledast\Phi)(t,x)\|_p &\leq |\rho|\left\|\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)\Phi(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right\|_p\\ &\quad+\left\|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}} g(t-s,x-y)\Phi(s,y)z\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right\|_p\\ &\quad+|b|\left\|\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)\Phi(s,y)\,\mathrm{d}s\,\mathrm{d}y\right\|_p\\ &=: I_1(t,x)+I_2(t,x)+I_3(t,x)\end{aligned}$$

into a Gaussian, a pure-jump and a drift part. Recall that I_1 vanishes for $d\geq 2$. For $d=1$ and $p\in[2,3)$, we have from the BDG inequalities (see [15, Theorem VII.92]) together with Minkowski's integral inequality that

$$\begin{aligned}e^{-\beta t+c|x|}I_1(t,x) &\leq |\rho|C_p e^{-\beta t+c|x|}\left(\int_0^t\int_{\mathbb{R}} g^2(t-s,x-y)\|\Phi(s,y)\|_p^2\,\mathrm{d}s\,\mathrm{d}y\right)^{\frac12}\\ &\leq |\rho|C_p\|\Phi\|_{p,\beta,c}\left(\int_0^t\int_{\mathbb{R}} g^2(t-s,x-y)e^{-2\beta(t-s)+2c(|x|-|y|)}\,\mathrm{d}s\,\mathrm{d}y\right)^{\frac12}\\ &\leq |\rho|C_p\|\Phi\|_{p,\beta,c}\left(\int_0^\infty\int_{\mathbb{R}} g^2_{\beta,c}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right)^{\frac12}.\end{aligned}$$

So we deduce from Lemma 5.1 that

$$\sup_{(t,x)\in(0,\infty)\times\mathbb{R}} e^{-\beta t+c|x|}I_1(t,x)\leq C_p|\rho|\frac{1}{(2\kappa(\beta-\frac12\kappa c^2))^{\frac14}}\|\Phi\|_{p,\beta,c}. \tag{5.1}$$

In order to estimate I_3 we only need Minkowski's integral inequality and Lemma 5.1 to obtain

$$\begin{aligned}I_3(t,x) &\leq |b|\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)\|\Phi(s,y)\|_p\,\mathrm{d}s\,\mathrm{d}y\\ &\leq |b|e^{\beta t-c|x|}\|\Phi\|_{p,\beta,c}\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)e^{-\beta(t-s)+c(|x|-|y|)}\,\mathrm{d}s\,\mathrm{d}y\\ &\leq |b|e^{\beta t-c|x|}\|\Phi\|_{p,\beta,c}\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)e^{-\beta(t-s)+c(|x-y|)}\,\mathrm{d}s\,\mathrm{d}y\\ &\leq \frac{2^d|b|}{\beta-\frac12\kappa c^2 d}e^{\beta t-c|x|}\|\Phi\|_{p,\beta,c}.\end{aligned} \tag{5.2}$$

We turn to the estimation of I_2. If $p\leq 2$, we use the BDG inequality to deduce

$$\begin{aligned}I_2(t,x)^p &\leq C_p^p\,\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}}|g(t-s,x-y)\Phi(s,y)z|^2\,\mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^{\frac p2}\right]\\ &\leq C_p^p\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}} g^p(t-s,x-y)\|\Phi(s,y)\|_p^p|z|^p\,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\\ &\leq C_p^p(m_\lambda(p))^p e^{p\beta t-pc|x|}\frac{2^{\frac d2(3-p)}\Gamma(1-\frac d2(p-1))}{p^{1+d(1-\frac p2)}(\pi\kappa)^{\frac d2(p-1)}(\beta-\frac12\kappa c^2 d)^{1-\frac d2(p-1)}}\|\Phi\|^p_{p,\beta,c}.\end{aligned}$$

At the second inequality we used that $(\sum_{i=1}^{\infty} a_i)^r \leq \sum_{i=1}^{\infty} a_i^r$ for any $r \in [0,1]$ and nonnegative numbers $(a_i)_{i\in\mathbb{N}}$. If $d = 1$ and $2 < p < 3$, we use [21, Theorem 1] with $\alpha = 2$ to obtain

$$\begin{aligned} I_2(t,x)^p \leq C_p^p \Bigg(\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |g(t-s,x-y)\Phi(s,y)z|^2 \, \nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \right)^{\frac{p}{2}} \right] \\ + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g^p(t-s,x-y) \|\Phi(s,y)\|_p^p |z|^p \, \nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \Bigg). \end{aligned} \tag{5.3}$$

For the first term, again by Minkowski's integral inequality and Lemma 5.1, we have

$$\left(\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |g(t-s,x-y)\Phi(s,y)z|^2 \, \nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} \leq \frac{m_\lambda(2) e^{\beta t - c|x|}}{(2\kappa(\beta - \frac{1}{2}\kappa c^2))^{\frac{1}{4}}} \|\Phi\|_{p,\beta,c}, \tag{5.4}$$

while for the second term,

$$\left(\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g^p(t-s,x-y) \|\Phi(s,y)\|_p^p |z|^p \, \nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \right)^{\frac{1}{p}} \leq \frac{2^{\frac{3-p}{2p}} \Gamma(\frac{3-p}{2})^{\frac{1}{p}} m_\lambda(p) e^{\beta t - c|x|}}{p^{\frac{4-p}{2p}} (\pi\kappa)^{\frac{p-1}{2p}} (\beta - \frac{1}{2}\kappa c^2)^{\frac{3-p}{2p}}} \|\Phi\|_{p,\beta,c}. \tag{5.5}$$

Substituting (5.4) and (5.5) back into (5.3), we obtain

$$e^{-\beta t + c|x|} I_2(t,x) \leq C_p \left(\frac{m_\lambda(2)}{(2\kappa(\beta - \frac{1}{2}\kappa c^2))^{\frac{1}{4}}} + \frac{2^{\frac{3-p}{2p}} \Gamma(\frac{3-p}{2})^{\frac{1}{p}} m_\lambda(p)}{p^{\frac{4-p}{2p}} (\pi\kappa)^{\frac{p-1}{2p}} (\beta - \frac{1}{2}\kappa c^2)^{\frac{3-p}{2p}}} \right) \|\Phi\|_{p,\beta,c}. \tag{5.6}$$

The statement now follows from inequalities (5.1), (5.2) and (5.6). Finally, since C_p comes from BDG inequalities, it remains bounded on $[1+\epsilon, 1+2/p)$. □

Proof of Proposition 2.3. The proof combines Proposition 2.1 with arguments in [12, Theorem 1.1] (see also [20, Theorem 8.1]).

As usual, we consider the Picard iteration sequence $Y^{(0)} = Y_0$ and

$$Y^{(n)}(t,x) = Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)\sigma(Y^{(n-1)}(s,y)) \, \Lambda(\mathrm{d}s,\mathrm{d}y)$$

for $n \in \mathbb{N}$, and define $u^{(n)} = Y^{(n)} - Y^{(n-1)}$. After possibly enlarging the value of L, we can assume that $|\sigma(x)| \leq L(1+|x|)$ for all $x \in \mathbb{R}$. Now let us choose $\beta_0 > \frac{1}{2}\kappa c^2 d$ large enough such that the factor $C_{\beta,c}(\kappa,p)$ in front of $\|\Phi\|_{p,\beta,c}$ on the right-hand side of (2.3) satisfies

$$C_{\beta,c}(\kappa,p) < \frac{1}{L} \quad \text{for all } \beta \geq \beta_0. \tag{5.7}$$

Using the Lipschitz property of σ, we obtain for all $\beta \geq \beta_0$ and $n \in \mathbb{N}$ as a consequence of Proposition 2.1,

$$\begin{aligned} \|u^{(n)}\|_{p,\beta,c} &= \|g \circledast (\sigma(Y^{(n-1)}) - \sigma(Y^{(n-2)}))\|_{p,\beta,c} \leq C_{\beta,c}(\kappa,p) \|\sigma(Y^{(n-1)}) - \sigma(Y^{(n-2)})\|_{p,\beta,c} \\ &\leq q\|u^{(n-1)}\|_{p,\beta,c} \leq \ldots \leq q^{n-1}\|u^{(1)}\|_{p,\beta,c} \end{aligned}$$

for some $q = q_c(\kappa, p) < 1$. If $c = 0$, the last term is less than or equal to $Cq^n(1 + \|Y_0\|_{p,\beta,c})$, while it is bounded by $Cq^n\|Y_0\|_{p,\beta,c}$ if $c > 0$ (and therefore $\sigma(0) = 0$). Since $\beta \geq \beta_0 > \frac{1}{2}\kappa c^2 d$,

$$\begin{aligned}
\|Y_0\|_{p,\beta,c} &= \sup_{t\in(0,\infty)} \sup_{x\in\mathbb{R}^d} e^{-\beta t + c|x|} \left| \int_{\mathbb{R}^d} g(t, x-y) f(y) \, \mathrm{d}y \right| \\
&\leq \sup_{x\in\mathbb{R}^d} e^{c|x|}|f(x)| \sup_{t\in(0,\infty)} e^{-\beta t} \int_{\mathbb{R}^d} g(t,x) e^{c|x|} \, \mathrm{d}x \\
&\leq C \sup_{t\in(0,\infty)} e^{-\beta t} \left(\int_{\mathbb{R}} g(t,x) e^{cx} \, \mathrm{d}x \right)^d \\
&= C \sup_{t\in(0,\infty)} e^{-(\beta - \frac{1}{2}\kappa c^2 d)t} < \infty,
\end{aligned}$$

it follows that $(Y^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{p,\beta,c}$, converging in $\|\cdot\|_{p,\beta,c}$ to some limit Y. That Y satisfies (1.5) and is unique up to modifications, follows as in [10, Theorem 3.1]. □

Proof of Theorem 2.4. The first part follows immediately from $\|Y\|_{p,\beta,0} < \infty$ for $\beta \geq \beta_0$ with β_0 as in the proof of Proposition 2.3. Concerning the second part of the theorem, observe that $\|Y\|_{p,\beta,c} < \infty$ for $\beta \geq \beta_0$ implies $\mathbb{E}[|Y(t,x)|^p] \leq Ce^{\beta pt - cp|x|}$ for all $t > 0$, $x \in \mathbb{R}^d$ and some finite constant $C > 0$. Hence,

$$\sup_{|x|\geq\alpha t} \mathbb{E}[|Y(t,x)|^p] \leq Ce^{\beta pt - cp\alpha t},$$

and therefore,

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log \mathbb{E}[|Y(t,x)|^p] < 0 \tag{5.8}$$

for all $\alpha > \beta_0/c$. □

5.2 Proofs for Section 3

Lemma 5.2 *If X_λ has a Poisson distribution with parameter λ, then there exists for every $r > 0$ a constant $C_r > 0$ such that*

$$\mathbb{E}[X_\lambda^r] \geq C_r \begin{cases} \lambda^r & \text{for } \lambda > 1, \\ \lambda & \text{for } \lambda \leq 1. \end{cases}$$

Proof. Suppose that $(X_\lambda)_{\lambda\geq 0}$ forms a standard Poisson process. The law of large numbers implies that $X_\lambda/\lambda \to 1$ a.s. as $\lambda \to \infty$. The convergence also takes place in L^p for every $p \geq 1$ because $\mathbb{E}[X_\lambda^n]$ is a polynomial in λ of degree n for every $n \in \mathbb{N}$ so that $\sup_{\lambda\geq 1} \mathbb{E}[X_\lambda^n]/\lambda^n < \infty$. In particular, we obtain for every $r > 0$ that $\mathbb{E}[X_\lambda^r]/\lambda^r \to 1$ as $\lambda \to \infty$, which implies the claim for $\lambda > 1$. The bound for $\lambda \leq 1$ follows from the definition of the expectation and $\mathbb{P}[X_\lambda^r = 1] = \mathbb{P}[X_\lambda = 1] = \lambda e^{-\lambda} \geq \lambda e^{-1}$. □

The following *decoupling inequalities* can be found in [25, Theorem 2.4.1]. Because of its importance for proving Lemma 3.4, and because the proof in the reference is given for processes with values in Banach spaces, we reproduce the proof in the real-valued setting for the reader's convenience. In the following lemma, for notational ease, a random variable $\xi\colon \Omega \to \mathbb{R}$ is identified with its natural extension to the product space $\Omega \times \overline{\Omega}$, i.e., $\xi(\omega, \overline{\omega}) = \xi(\omega)$.

Lemma 5.3 *Consider two probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, each of them equipped with a discrete-time filtration $(\mathcal{F}_i)_{i\geq 0}$ and $(\overline{\mathcal{F}}_i)_{i\geq 0}$, respectively. Furthermore, let $(\xi_i)_{i\geq 1}$ be a zero-mean $(\mathcal{F}_i)_{i\geq 1}$-adapted sequence such that ξ_i is independent of $\mathcal{F}_{i-1}$ under $\mathbb{P}$ for all $i \geq 1$, and let $(\overline{\xi}_i)_{i\geq 1}$ be a sequence with analogous properties on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ and the same distribution as $(\xi_i)_{i\geq 1}$. Finally, assume that $(H_i)_{i\geq 1}$ is a sequence of random variables on $(\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \otimes \overline{\mathbb{P}})$ such that H_i is $\mathcal{F}_{i-1} \otimes \overline{\mathcal{F}}_{i-1}$-measurable for all $i \geq 1$. Then for every $p \in (1, \infty)$ there exist constants $C_p, C'_p > 0$ that are independent of $(\xi_i)_{i\geq 1}$ and $(H_i)_{i\geq 1}$ such that for every $N \in \mathbb{N}$,*

$$(C'_p)^{-1}\mathbb{E}\left[\overline{\mathbb{E}}\left[\left|\sum_{i=1}^N H_i\overline{\xi}_i\right|^p\right]\right] \leq \mathbb{E}\left[\overline{\mathbb{E}}\left[\left|\sum_{i=1}^N H_i\xi_i\right|^p\right]\right] \leq C_p\mathbb{E}\left[\overline{\mathbb{E}}\left[\left|\sum_{i=1}^N H_i\overline{\xi}_i\right|^p\right]\right].$$

Proof. Define the random variables

$$D_{2i-1} := \frac{1}{2}(H_i\xi_i + H_i\overline{\xi}_i), \quad D_{2i} := \frac{1}{2}(H_i\xi_i - H_i\overline{\xi}_i), \quad i = 1, \ldots, N,$$

and a filtration $(\mathcal{G}_i)_{i=0,\ldots,2N}$ by

$$\mathcal{G}_0 := \{\emptyset, \Omega\}, \quad \mathcal{G}_{2i-1} := \sigma(\mathcal{F}_{i-1} \otimes \overline{\mathcal{F}}_{i-1}, \xi_i + \overline{\xi}_i), \quad \mathcal{G}_{2i} := \mathcal{F}_i \otimes \overline{\mathcal{F}}_i, \quad i = 1, \ldots, N.$$

Obviously, $(D_i)_{i=1,\ldots,2N}$ is adapted to $(\mathcal{G}_i)_{i=1,\ldots,2N}$. In addition, denoting by $\mathbb{E}\otimes\overline{\mathbb{E}}$ the expectation with respect to $\mathbb{P} \otimes \overline{\mathbb{P}}$, we have for all $i = 1, \ldots, N$,

$$\begin{aligned}\mathbb{E} \otimes \overline{\mathbb{E}}[D_{2i+1} \mid \mathcal{G}_{2i}] &= \frac{1}{2}H_{i+1}\mathbb{E} \otimes \overline{\mathbb{E}}[\xi_{i+1} + \overline{\xi}_{i+1}] = 0,\\ \mathbb{E} \otimes \overline{\mathbb{E}}[D_{2i} \mid \mathcal{G}_{2i-1}] &= \frac{1}{2}H_i\mathbb{E} \otimes \overline{\mathbb{E}}[\xi_i - \overline{\xi}_i \mid \xi_i + \overline{\xi}_i] = 0,\end{aligned}$$

where the last identity holds because ξ_i and $\overline{\xi}_i$ are independent with the same distribution. It follows from [24, Theorem VII.1.1] that the processes $(\sum_{i=1}^n D_i)_{n=0,\ldots,2N}$ and $(\sum_{i=1}^n (-1)^{i+1} D_i)_{n=0,\ldots,2N}$ are discrete-time local martingales with respect to $(\mathcal{G}_i)_{i=0,\ldots,2N}$.

Observing that

$$\sum_{i=1}^N H_i\xi_i = \sum_{i=1}^{2N} D_i, \quad \sum_{i=1}^N H_i\overline{\xi}_i = \sum_{i=1}^{2N}(-1)^{i+1}D_i$$

by construction, the claim is a consequence of the classical BDG inequalities because the two discrete-time local martingales above can be canonically embedded into continuous-time local martingales with the same quadratic variation process. □

Proof of Lemma 3.4. We first prove (3.4) for simple integrands of the form

$$Z(\omega, t, x) = \sum_{i,j=1}^K X_{ij}(\omega)\mathbb{1}_{(t_{i-1},t_i]\times B_j}(t, x), \quad (\omega, t, x) \in \Omega \times [0, \infty) \times E, \tag{5.9}$$

where $0 \leq t_0 \leq \ldots \leq t_K < \infty$, $(B_j)_{j=1,\ldots,K}$ are pairwise disjoint Borel subsets of E, and X_{ij} are $\mathcal{F}_{t_{i-1}}$-measurable random variables for all $i, j = 1, \ldots, K$.

Using Lemma 5.3, we can assume without loss of generality that Z is deterministic, that is, the variables $X_{ij}(\omega)$ do not depend on ω. To see this, define

$$\begin{aligned}
\xi_{ij}(\omega) &= N((t_{i-1}, t_i] \times B_j)(\omega) - m((t_{i-1}, t_i] \times B_j), \\
\overline{\xi}_{ij}(\overline{\omega}) &= \overline{N}((t_{i-1}, t_i] \times B_j)(\overline{\omega}) - m((t_{i-1}, t_i] \times B_j), \\
H_{ij}(\omega, \overline{\omega}) &= X_{ij}(\omega),
\end{aligned}$$

where $\overline{N}$ lives on a copy $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original probabilty space, with the same distribution as N. Since ξ_{ij} is $\mathcal{F}_{t_i}$-measurable and H_{ij} is $\mathcal{F}_{t_{i-1}} \otimes \overline{\mathcal{F}}_{t_{i-1}}$-measurable, Lemma 5.3 applies and yields

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{i,j=1}^{K} X_{ij}(\omega)\,(N((t_{i-1}, t_i] \times B_j)(\omega) - m((t_{i-1}, t_i] \times B_j))\right|^p\right] \\
&\geq (C_p')^{-1}\mathbb{E}\left[\overline{\mathbb{E}}\left[\left|\sum_{i,j=1}^{K} X_{ij}(\omega)\left(\overline{N}((t_{i-1}, t_i] \times B_j)(\overline{\omega}) - m((t_{i-1}, t_i] \times B_j)\right)\right|^p\right]\right].
\end{aligned}$$

As $X_{ij}(\omega)$ does not depend on $\overline{\omega}$, it is indeed enough to prove (3.4) for deterministic integrands.

By the BDG inequalities, there exists $C_p > 0$ (which is bounded away from 0 for $p > p'$) such that

$$\begin{aligned}
\mathbb{E}\left[\left|\iint_{[0,\infty)\times E} Z(t,x)\,\tilde{N}(\mathrm{d}t, \mathrm{d}x)\right|^p\right] &\geq C_p\mathbb{E}\left[\left|\iint_{[0,\infty)\times E} Z^2(t,x)\,N(\mathrm{d}t, \mathrm{d}x)\right|^{\frac{p}{2}}\right] \\
&= C_p\mathbb{E}\left[\left|\sum_{i,j=1}^{K} X_{ij}^2 N((t_{i-1}, t_i] \times B_j)\right|^{\frac{p}{2}}\right].
\end{aligned}$$

Inequality (3.4) is shown for integrands of the form (5.9) once we can show that

$$\mathbb{E}\left[\left|\sum_{i=1}^{K} a_i N(A_i)\right|^r\right] \geq C\frac{\sum_{i=1}^{K} a_i^r m(A_i)}{(1 \vee m([0,\infty) \times E))^{1-r}} \tag{5.10}$$

for all $a_i \in [0, \infty)$, pairwise disjoint $A_i \in \mathcal{B}([0, \infty) \times E)$ and $r \in (1/2, 1]$. By the tower property of conditional expectations,

$$\mathbb{E}\left[\left(\sum_{i=1}^{K} a_i N(A_i)\right)^r\right] = \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\sum_{i=1}^{K} a_i N(A_i)\right)^r \,\middle|\, \sum_{i=1}^{K} N(A_i) = n\right] \mathbb{P}\left[\sum_{i=1}^{K} N(A_i) = n\right].$$

On the event $\sum_{i=1}^{K} N(A_i) = n$, at most n summands in $\sum_{i=1}^{K} a_i N(A_i)$ are different from zero. Therefore, by rewriting $a_i N(A_i)$ as a sum $a_i + \ldots + a_i$ of $N(A_i)$ terms, $\sum_{i=1}^{K} a_i N(A_i)$ becomes a sum of $\sum_{i=1}^K N(A_i) = n$ (possibly repeated) terms. Thus, using the estimate $(\sum_{i=1}^n c_i)^r \geq n^{r-1} \sum_{i=1}^n c_i^r$ for nonnegative $c_1, \dots, c_n$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^K a_i N(A_i)\right)^r\right] &\geq \sum_{n=1}^\infty n^{r-1} \mathbb{E}\left[\sum_{i=1}^K a_i^r N(A_i) \,\middle|\, \sum_{i=1}^K N(A_i) = n\right] \mathbb{P}\left[\sum_{i=1}^K N(A_i) = n\right] \\
&= \sum_{i=1}^K a_i^r \sum_{n=1}^\infty n^{r-1} \mathbb{E}\left[N(A_i) \,\middle|\, \sum_{i=1}^K N(A_i) = n\right] \mathbb{P}\left[\sum_{i=1}^K N(A_i) = n\right] \\
&= \sum_{i=1}^K a_i^r \sum_{n=1}^\infty n^r \frac{m(A_i)}{\sum_{j=1}^K m(A_j)} \mathbb{P}\left[\sum_{i=1}^K N(A_i) = n\right] \\
&= \frac{\sum_{i=1}^K a_i^r m(A_i)}{\sum_{j=1}^K m(A_j)} \sum_{n=1}^\infty n^r \mathbb{P}\left[\sum_{i=1}^K N(A_i) = n\right] \\
&= \frac{\sum_{i=1}^K a_i^r m(A_i)}{\sum_{j=1}^K m(A_j)} \mathbb{E}\left[\left(\sum_{i=1}^K N(A_i)\right)^r\right].
\end{aligned}$$

Since the constant C_r in Lemma 5.2 can be taken independently of r when $r \in (1/2, 1]$, we derive

$$\mathbb{E}\left[\left(\sum_{i=1}^K a_i N(A_i)\right)^r\right] \geq C \frac{\sum_{i=1}^K a_i^r m(A_i)}{\left(1 \vee \sum_{j=1}^K m(A_j)\right)^{1-r}} \geq C \frac{\sum_{i=1}^K a_i^r m(A_i)}{(1 \vee m([0,\infty) \times E))^{1-r}},$$

which is (5.10).

For a general $(\mathcal{F}_t)_{t\geq 0}$-predictable process H, one can choose a sequence H_n of processes of the form (5.9) such that $|H_n| \leq |H|$ for all $n \in \mathbb{N}$ and $H_n \to H$ as $n \to \infty$, pointwise in (ω, t, x). If the right-hand side of (3.4) is finite, then inequality (3.4) follows from the dominated convergence theorem for stochastic integrals (see [6, Equation (2.6)]) on the left-hand side and for Lebesgue integrals on the right-hand side. If the right-hand side of (3.4) is infinite, then the estimates we have established for simple integrands, together with the BDG inequalities, imply that also the left-hand side of (3.4) is infinite. □

Lemma 5.4 *Suppose that $a \in \mathbb{R}$ and X is a random variable with zero mean. Then for every $p \in (1, 3]$, we have*

$$\mathbb{E}[|a + X|^p] \geq C_p(|a|^p + \mathbb{E}[|X|^p])$$

where $C_p = 1/4$ for $p \in (1, 2]$ and $C_p = 1/6$ for $p \in (2, 3]$.

Proof. First we prove the statement for $a = 1$ and $p \in (1, 2]$. The proof follows from the following simple inequalities:

$$\begin{aligned}
(y-1)^p &\geq \frac{1}{3}\left(y^p - 2y + 1\right), \quad y \geq 1, \\
(1-y)^p &\geq \frac{1}{3}\left(y^p - 2y + \frac{3}{4}\right), \quad y \in [0, 1], \qquad (5.11) \\
(y+1)^p &\geq \frac{1}{3}\left(y^p + 2y + 1\right), \quad y \geq 0.
\end{aligned}$$

Indeed, denoting the distribution function of X by F, (5.11) and $\mathbb{E}[X]=0$ imply

$$\begin{aligned}\mathbb{E}[|1+X|^p] &= \int_{-\infty}^{\infty} |1+y|^p\, F(\mathrm{d}y)\\ &\geq \int_{-\infty}^{-1} \frac{1}{3}\left((-y)^p-2(-y)+1\right)\, F(\mathrm{d}y) + \int_{-1}^{0} \frac{1}{3}\left((-y)^p-2(-y)+\frac{3}{4}\right)\, F(\mathrm{d}y)\\ &\quad + \int_0^\infty \frac{1}{3}\left(y^p+2y+1\right)\, F(\mathrm{d}y)\\ &\geq \frac{1}{3}\mathbb{E}[|X|^p] + \frac{2}{3}\mathbb{E}[X] + \frac{1}{4} \geq \frac{1}{4}\left(\mathbb{E}[|X|^p]+1\right).\end{aligned}$$

For general $a\in\mathbb{R}$, the statement follows from

$$\mathbb{E}[|a+X|^p] = |a|^p\, \mathbb{E}\left[\left|1+\frac{X}{a}\right|^p\right] \geq \frac{|a|^p}{4}\left(\frac{\mathbb{E}[|X|^p]}{|a|^p}+1\right) = \frac{1}{4}\mathbb{E}[|X|^p] + \frac{1}{4}|a|^p.$$

Here is the proof of (5.11). The first inequality holds for $y=1$, and

$$p(y-1)^{p-1} \geq p(y^{p-1}-1) = \frac{1}{3}(py^{p-1}-2) + \frac{2p}{3}y^{p-1} + \frac{2}{3} - p \geq \frac{1}{3}(py^{p-1}-2),$$

that is the derivative of the left-hand side is greater than that of the right-hand side for all $y\geq 1$. Thus the first inequality follows. For the second, using $y^p+(1-y)^p\leq 1$, $y\in[0,1]$, we have

$$\begin{aligned}3(1-y)^p - y^p + 2y - \frac{3}{4} &\geq 3(1-y)^p - (1-(1-y)^p) + 2y - \frac{3}{4} \geq 4(1-y)^2 + 2y - \frac{7}{4}\\ &= \left(2y-\frac{3}{2}\right)^2,\end{aligned}$$

which is nonnegative, so the second inequality is proved. Finally, for $y\geq 0$

$$(y+1)^p \geq y^p + 1 = \frac{1}{3}(y^p+2y+1) + \frac{2}{3}(y^p-y+1) \geq \frac{1}{3}(y^p+2y+1).$$

The proof is similar for $p\in(2,3]$, once the inequalities

$$\begin{aligned}(y-1)^p &\geq \frac{1}{6}\left(y^p-6y+1\right), \quad y\geq 1,\\ (1-y)^p &\geq \frac{1}{3}\left(y^p-3y+1\right) \geq \frac{1}{6}\left(y^p-6y+1\right), \quad y\in[0,1],\\ (y+1)^p &\geq \frac{1}{3}\left(y^p+3y+1\right) \geq \frac{1}{6}\left(y^p+6y+1\right), \quad y\geq 0,\end{aligned} \tag{5.12}$$

are established. We leave the proof of (5.12) to the interested reader. □

Proof of Theorem 3.5. (1) We assume $d\geq 2$ here as the case $d=1$ will be treated in part (4). In particular, p is always less than 2 and Λ contains no Gaussian part. By Lemma 5.4 and the BDG inequalities, we have for all $p\in(1,1+2/d)$

$$\mathbb{E}[|Y(t,x)|^p] \geq C_p\left(L_f^p + \mathbb{E}\left[\left(\iiint_0^t g^2(t-s,x-y)\sigma^2(Y(s,y))z^2\,\mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^{\frac{p}{2}}\right]\right). \tag{5.13}$$

This estimate remains valid if we replace μ on the right-hand side by the measure

$$\mu_{\epsilon,\delta}^{(t,x)}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) := \mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{g(t-s,x-y)>\epsilon\}}\mathbb{1}_{[-\delta,\delta]^{\mathrm{c}}}(z)\,\mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \tag{5.14}$$

where $\epsilon > 0$ is arbitrary and $\delta > 0$ is chosen small enough such that $\lambda([-\delta,\delta]^{\mathrm{c}}) > 0$. The corresponding intensity measure is given by

$$\nu_{\epsilon,\delta}^{(t,x)}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) := \mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{g(t-s,x-y)>\epsilon\}}\mathbb{1}_{[-\delta,\delta]^{\mathrm{c}}}(z)\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z),$$

and satisfies

$$\begin{aligned}\nu_{\epsilon,\delta}^{(t,x)}([0,\infty)\times\mathbb{R}^d\times\mathbb{R}) &= \lambda([-\delta,\delta]^{\mathrm{c}})\iint_0^t \mathbb{1}_{\{g(s,y)>\epsilon\}}\,\mathrm{d}s\,\mathrm{d}y \\ &\leq \lambda([-\delta,\delta]^{\mathrm{c}})\iint_0^\infty \mathbb{1}_{\{g(s,y)>\epsilon\}}\,\mathrm{d}s\,\mathrm{d}y < \infty,\end{aligned}$$

with an upper bound independent of (t,x). By Lemma 3.4, we obtain (keeping in mind that $L_f > 0$ and $L_\sigma > 0$, and using the BDG inequality from the first to the second line)

$$\begin{aligned}\mathbb{E}[|Y(t,x)|^p] &\geq C_p\left(1+\mathbb{E}\left[\left(\iiint_0^t g^2(t-s,x-y)\sigma^2(Y(s,y))z^2\,\mu_{\epsilon,\delta}^{(t,x)}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^{\frac{p}{2}}\right]\right) \\ &\geq C_p\left(1+\mathbb{E}\left[\left|\iiint_0^t g(t-s,x-y)\sigma(Y(s,y))z\,(\mu_{\epsilon,\delta}^{(t,x)}-\nu_{\epsilon,\delta}^{(t,x)})(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^p\right]\right) \\ &\geq C_p\Bigg(1+\frac{\int_{\mathbb{R}}|z|^p\mathbb{1}_{\{|z|>\delta\}}\,\lambda(\mathrm{d}z)}{(1\vee\lambda([-\delta,\delta]^{\mathrm{c}})\iint_0^\infty\mathbb{1}_{\{g(s,y)>\epsilon\}}\,\mathrm{d}s\,\mathrm{d}y)^{1-\frac{p}{2}}} \\ &\qquad\times\iint_0^t g^p(t-s,x-y)\mathbb{1}_{\{g(t-s,x-y)>\epsilon\}}\mathbb{E}[|Y(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y\Bigg)\end{aligned} \tag{5.15}$$

with a constant C_p independent of (t,x). As a consequence, the function

$$I_p(t) := \inf_{x\in\mathbb{R}^d}\mathbb{E}[|Y(t,x)|^p]$$

satisfies

$$I_p(t) \geq a_p + \int_0^t w_p(t-s)I_p(s)\,\mathrm{d}s \tag{5.16}$$

for some $a_p > 0$ where

$$w_p(t) = C_p\frac{\int_{\mathbb{R}}|z|^p\mathbb{1}_{\{|z|>\delta\}}\,\lambda(\mathrm{d}z)}{(1\vee\lambda([-\delta,\delta]^{\mathrm{c}})\iint_0^\infty\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x)^{1-\frac{p}{2}}}\int_{\mathbb{R}^d}g^p(t,x)\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}x. \tag{5.17}$$

Recall from Lemmata 3.4 and 5.4 that both C_p and a_p can be assumed to be bounded away from 0 if p is bounded away from 1. Since $g \notin L^{1+2/d}([0,T]\times\mathbb{R}^d)$ for any $T > 0$ (cf. the calculations before Theorem 3.1), and the heat kernel decays exponentially in space, we have $\iint_0^\infty g^p(t,x)\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x \to \infty$ as $p \to 1+2/d$ and consequently,

$$\lim_{p\to 1+\frac{2}{d}} \int_0^\infty w_p(t)\,\mathrm{d}t = \infty. \tag{5.18}$$

Hence, there exists $p_0 \in (1, 1+2/d)$ such that $\int_0^\infty w_{p_0}(t)\,\mathrm{d}t > 1$. By classical renewal theory, see e.g. [2, Theorem V.7.1], it follows that the solution to the equation

$$i(t) = a_p + \int_0^t w_{p_0}(t-s)i(s)\,\mathrm{d}s$$

satisfies $i(t) \geq e^{\gamma t}$ for all $t \geq t_0$ and some $t_0 > 0$ and $\gamma > 0$. Since $I_{p_0}(t) \geq i(t)$ by [20, Theorem 7.11], we conclude that $\underline{\gamma}(p_0) > 0$, and by Jensen's inequality, also $\underline{\gamma}(p) > 0$ for all $p_0 \leq p < 1+2/d$.

(2) Again, we only consider the case $d \geq 2$. A direct computation shows that

$$\begin{aligned}
\iint_0^\infty g^p(t,x)\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x &= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^{\frac{1}{2\pi\kappa\epsilon^{2/d}}}\int_0^{\sqrt{-d\kappa t\log(2\pi\kappa\epsilon^{2/d}t)}} \frac{e^{-\frac{pr^2}{2\kappa t}}}{(2\pi\kappa t)^{\frac{pd}{2}}} r^{d-1}\,\mathrm{d}r\,\mathrm{d}t \\
&= \frac{\pi^{\frac{d}{2}}}{\pi\Gamma(\frac{d}{2})\kappa\epsilon^{\frac{2}{d}-p}}\int_0^1\int_0^{\sqrt{\frac{-ds\log(s)}{2\pi\epsilon^{2/d}}}} s^{-\frac{pd}{2}} e^{-\frac{pr^2\pi\epsilon^{2/d}}{s}} r^{d-1}\,\mathrm{d}r\,\mathrm{d}s \\
&= \frac{(\frac{d}{2})^{\frac{d}{2}}}{\pi\Gamma(\frac{d}{2})\kappa\epsilon^{1+\frac{2}{d}-p}}\int_0^1\int_0^1 s^{\frac{pd(z^2-1)}{2}}(-s\log(s))^{\frac{d}{2}} z^{d-1}\,\mathrm{d}z\,\mathrm{d}s \\
&= \frac{C_p}{\kappa\epsilon^{1+\frac{2}{d}-p}}
\end{aligned} \tag{5.19}$$

for all $p \in (0, 1+2/d)$. This formula is still valid for $p = 0$. Thus, for the function in (5.17), which we denote by $w_\kappa(t)$ now since κ is the parameter that interests us, there exists $C > 0$ that is independent of κ such that

$$\lim_{\kappa\to 0}\int_0^\infty w_\kappa(t)\,\mathrm{d}t = C\lim_{\kappa\to 0}\frac{\kappa^{1-\frac{p}{2}}}{\kappa} = \lim_{\kappa\to 0} C\kappa^{-\frac{p}{2}} = \infty. \tag{5.20}$$

The proof can now be completed as in the first part of the theorem.

(3) This part follows as before because L_σ enters C_p in (5.17) in a multiplicative way.

(4) In dimension 1 it suffices by Jensen's inequality to consider $p \in (1,2)$. Furthermore, by Lemma 5.4 and the BDG inequalities, we may assume $\rho = 0$ without loss of generality. Then the proof of part (1) remains valid up to equation (5.17). Instead of varying the value of p, we now let $\epsilon \to 0$, keeping $p \in (1,2)$, $\kappa > 0$ as well as $\delta > 0$ fixed. Writing $w_\epsilon(t)$ in the following instead of $w_p(t)$ for the function in (5.17), it follows from (5.19) that

$$\lim_{\epsilon\to 0}\int_0^\infty w_\epsilon(t)\,\mathrm{d}t = C\lim_{\epsilon\to 0}\frac{\epsilon^{3(1-\frac{p}{2})}}{\epsilon^{3-p}} = C\lim_{\epsilon\to 0}\epsilon^{-\frac{p}{2}} = \infty,$$

and the assertion follows.

□

Proof of Theorem 3.6. Let $\alpha > 0$, $p \in (1, 2 \wedge (1 + 2/d))$ and write $x = (x_1, \dots, x_d)$. By Proposition 2.3, $x \mapsto \mathbb{E}[|Y(t,x)|^p]$ is integrable, so we deduce from (5.15) (with $\epsilon, \delta > 0$ sufficiently small) and the hypothesis $L_\sigma > 0$,

$$\int_{x_1 \geq \alpha t} \mathbb{E}[|Y(t,x)|^p] \,\mathrm{d}x \geq \frac{1}{4}\Bigg(\int_{x_1 \geq \alpha t} |Y_0(t,x)|^p \,\mathrm{d}x \\ + C \int_{x_1 \geq \alpha t} \left(\iint_0^t g^p(t-s, x-y) \mathbb{1}_{\{g(t-s,x-y) > \epsilon\}} \mathbb{E}[|Y(s,y)|^p] \,\mathrm{d}s \,\mathrm{d}y \right) \mathrm{d}x \Bigg),$$

where the constant C is given by

$$C = \int_{\mathbb{R}} |z|^p \mathbb{1}_{\{|z|>\delta\}} \,\lambda(\mathrm{d}z) \left(1 \vee \lambda([-\delta, \delta]^c) \iint_0^\infty \mathbb{1}_{\{g(t,x) > \epsilon\}} \,\mathrm{d}t \,\mathrm{d}x \right)^{\frac{p}{2}-1}. \tag{5.21}$$

Let us write $v(t) := \int_{x_1 \geq \alpha t} \mathbb{E}[|Y(t,x)|^p] \,\mathrm{d}x$, $v_0(t) := \int_{x_1 \geq \alpha t} |Y_0(t,x)|^p \,\mathrm{d}x$ and

$$h(t) := \int_{x_1 \geq \alpha t} g^p(t,x) \mathbb{1}_{\{g(t,x) > \epsilon\}} \,\mathrm{d}x.$$

Using that $x_1 - y_1 \geq \alpha(t-s)$ and $y_1 \geq \alpha s$ imply $x_1 \geq \alpha t$, we obtain

$$v(t) \geq \frac{1}{4} \left(v_0(t) + C \int_0^t h(t-s) v(s) \,\mathrm{d}s \right)$$

for all $t \geq 0$. A straightforward extension of [9, Lemma 4.2] to the d-dimensional setting shows that $v_0(t) > 0$ for all $t > 0$. So on the one hand, if

$$C \int_0^\infty h(t) \,\mathrm{d}t > 4, \tag{5.22}$$

it follows from renewal theory (see the proof of Theorem 3.5) that

$$\limsup_{t \to \infty} e^{-\beta t} v(t) = \limsup_{t \to \infty} e^{-\beta t} \int_{x_1 \geq \alpha t} \mathbb{E}[|Y(t,x)|^p] \,\mathrm{d}x = \infty \tag{5.23}$$

whenever $\beta > 0$ is sufficiently small. On the other hand, from Proposition 2.3, we know that

$$\int_{x_1 \geq \alpha' t} \mathbb{E}[|Y(t,x)|^p] \,\mathrm{d}x \leq C e^{\beta' t} \int_{x_1 \geq \alpha' t} e^{-c|x|} \,\mathrm{d}x \leq C e^{\beta' t} \int_{x_1 \geq \alpha' t} e^{-\frac{c}{\sqrt{d}}(|x_1| + \ldots + |x_d|)} \,\mathrm{d}x \\ \leq C e^{\beta' t} \int_{\alpha' t}^\infty e^{-\frac{c}{\sqrt{d}} x_1} \,\mathrm{d}x_1 \leq C e^{(\beta' - \alpha' \frac{c}{\sqrt{d}})t}$$

for all $\alpha' > 0$, some $\beta' > 0$ and some $C > 0$ that is independent of t. Thus, the last expression decays exponentially whenever α' is large enough, so in this case, (5.23) implies

$$\limsup_{t \to \infty} e^{-\beta t} \int_{\alpha t \leq x_1 < \alpha' t} \mathbb{E}[|Y(t,x)|^p] \,\mathrm{d}x = \infty,$$

from which the assertion follows because

$$\sup_{|x|\geq \alpha t} \mathbb{E}[|Y(t,x)|^p] \geq \sup_{x_1\geq \alpha t} \mathbb{E}[|Y(t,x)|^p] \geq ((\alpha'-\alpha)t)^{-1} \int_{\alpha t\leq x_1<\alpha' t} \mathbb{E}[|Y(t,x)|^p]\,\mathrm{d}x.$$

So the only thing left to show is that we can achieve (5.22) by proper choices of the parameters involved. Since the heat kernel is radially symmetric, we have $\int_0^\infty h(t)\,\mathrm{d}t \geq \frac{1}{2d}\int_0^\infty \tilde{h}(t)\,\mathrm{d}t$ where

$$\tilde{h}(t) = \int_{|x|\geq \tilde{\alpha} t} g^p(t,x)\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}x \tag{5.24}$$

and $\tilde{\alpha} = \sqrt{d}\alpha$. Using polar coordinates and changing variables $s = 2\pi\kappa\epsilon^{2/d}t$ and $u = r^2 p\pi\epsilon^{2/d}/s$, we obtain

$$\begin{aligned}
\int_0^\infty \tilde{h}(t)\,\mathrm{d}t &= \int_0^{\frac{1}{2\pi\kappa\epsilon^{2/d}}} \int_{\mathbb{R}^d} g^p(t,x)\mathbb{1}_{\left\{\tilde{\alpha}t\leq|x|<\sqrt{-2\kappa t\log(\epsilon(2\pi\kappa t)^{d/2})}\right\}}\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^{\frac{1}{2\pi\kappa\epsilon^{2/d}}}\int_0^\infty \frac{e^{-\frac{pr^2}{2\kappa t}}}{(2\pi\kappa t)^{\frac{pd}{2}}}\mathbb{1}_{\left\{\tilde{\alpha}t\leq r<\sqrt{-2\kappa t\log(\epsilon(2\pi\kappa t)^{d/2})}\right\}} r^{d-1}\,\mathrm{d}r\,\mathrm{d}t \\
&= \frac{\epsilon^{-(1+\frac{2}{d}-p)}}{\Gamma(\frac{d}{2})p^{\frac{d}{2}}(2\pi\kappa)}\int_0^1\int_0^\infty s^{-\frac{d}{2}(p-1)}e^{-u}u^{\frac{d}{2}-1}\mathbb{1}_{\left\{\frac{\tilde{\alpha}^2}{\kappa^2\epsilon^{2/d}}\frac{sp}{4\pi}\leq u\leq \frac{pd}{2}\log s^{-1}\right\}}\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned} \tag{5.25}$$

Note that the latter integral depends on the parameters $\tilde{\alpha}$, κ, and ϵ only through the ratio

$$R = R(\tilde{\alpha},\kappa,\epsilon) = \frac{\tilde{\alpha}^2}{\kappa^2\varepsilon^{2/d}}. \tag{5.26}$$

Since s is increasing and $\log s^{-1}$ is decreasing,

$$R\frac{p}{4\pi}s \leq \frac{pd}{2}\log s^{-1}, \quad s\in(0,s_0),$$

with $s_0 = \min\{2\pi d/R, e^{-1}\}$. Therefore, the integral in (5.25) is not less than

$$\begin{aligned}
&\int_0^{s_0}\int_{\frac{Rp}{4\pi}s}^{\frac{pd}{2}\log s^{-1}} s^{-\frac{d}{2}(p-1)}e^{-u}u^{\frac{d}{2}-1}\mathrm{d}u\,\mathrm{d}s \geq \int_0^{\frac{Rp}{4\pi}s_0} e^{-u}u^{\frac{d}{2}-1}\int_0^{\frac{4\pi}{Rp}u} s^{-\frac{d}{2}(p-1)}\mathrm{d}s\,\mathrm{d}u \\
&= \left(\frac{4\pi}{Rp}\right)^{1-\frac{d}{2}(p-1)}\frac{2}{d(1+\frac{2}{d}-p)}\gamma\left(1+d\left(1-\frac{p}{2}\right),\frac{Rps_0}{4\pi}\right),
\end{aligned}$$

where the second inequality follows from Fubini's theorem, and

$$\gamma(x,T) = \int_0^T t^{x-1}e^{-t}\mathrm{d}t, \quad x>0, \quad T\geq 0, \tag{5.27}$$

stands for the lower incomplete gamma function. Substituting back into (5.25), we obtain

$$\begin{aligned}
\int_0^\infty \tilde{h}(t)\,\mathrm{d}t &\geq \frac{2^{2-d(p-1)}\kappa^{-1}\epsilon^{-(1+\frac{2}{d}-p)}R^{\frac{d}{2}(p-1)-1}}{d\Gamma(\frac{d}{2})\pi^{\frac{d}{2}(p-1)}p^{1+d(1-\frac{p}{2})}(1+\frac{2}{d}-p)}\gamma\left(1+d\left(1-\frac{p}{2}\right),\frac{Rs_0}{4\pi}\right) \\
&= \frac{2(2\kappa)^{1-d(p-1)}\tilde{\alpha}^{-2(1-\frac{d}{2}(p-1))}}{d\Gamma(\frac{d}{2})\pi^{\frac{d}{2}(p-1)}p^{1+d(1-\frac{p}{2})}(1+\frac{2}{d}-p)}\gamma\left(1+d\left(1-\frac{p}{2}\right),\frac{Rs_0}{4\pi}\right),
\end{aligned} \tag{5.28}$$

where R is given in (5.26).

Consequently, when $\tilde{\alpha}^2\epsilon^{-2/d} \geq 2\pi\kappa^2$, we have

$$\int_0^\infty \tilde{h}(t)\,\mathrm{d}t \geq \frac{2(2\kappa)^{1-d(p-1)}\tilde{\alpha}^{-2(1-\frac{d}{2}(p-1))}\gamma(1+d(1-\frac{p}{2}),\frac{1}{6})}{d\Gamma(\frac{d}{2})\pi^{\frac{d}{2}(p-1)}p^{1+d(1-\frac{p}{2})}(1+\frac{2}{d}-p)}. \tag{5.29}$$

Part (1) of the theorem in dimension $d \geq 2$ now follows from the observation that the right-hand side of (5.29) tends to ∞ as $p \to 1+2/d$, for any given $\kappa, \alpha > 0$ and small values of ϵ and δ (note that the constant C in (5.22) is bounded for p in a neighborhood of $1+2/d$).

For (2) choose $\alpha = \sqrt{2\pi}\epsilon^{1/d}\kappa$, with ϵ being fixed and $\kappa \to 0$. The lower bound in (5.29) is of order κ^{-1}, while the constant C in (5.21) is of order $\kappa^{1-p/2}$ by (5.19). Thus the statement follows by choosing κ sufficiently small. Similar considerations, compare also with the proof of Theorem 3.5, also show part (3) of the theorem.

For part (4), i.e., if we are in dimension $d=1$, let us assume $\rho = 0$ without loss of generality and choose $\tilde{\alpha} = \sqrt{2\pi\kappa^2}\epsilon$, with κ and p being fixed this time. Then, for any given p and κ, the right-hand side of (5.29) is of order $\epsilon^{-2(1-(p-1)/2)} = \epsilon^{-(3-p)}$ under the hypotheses of the theorem, while the constant C in (5.22) is of order $\epsilon^{3(1-p/2)}$ by (5.21) and (5.19), so we can achieve (5.22) by taking ϵ small enough. □

Proof of Theorem 3.1. Let $T > t_0$, $p = 1+2/d$ and assume the opposite of (3.2). Then by Minkowski's integral inequality,

$$\left\| b \iint_0^t g(t-s,x-y)\sigma(Y(s,y))\,\mathrm{d}s\,\mathrm{d}y \right\|_p \leq |b| \iint_0^t g(t-s,x-y)\|\sigma(Y(s,y))\|_p\,\mathrm{d}s\,\mathrm{d}y \leq C|b|T$$

for all $(t,x) \in [0,T]\times\mathbb{R}^d$. Similarly, if $d=1$, and we have by the BDG inequality,

$$\begin{aligned}\left\| \rho \iint_0^t g(t-s,x-y)\sigma(Y(s,y))\,W(\mathrm{d}s,\mathrm{d}y) \right\|_p &\leq C_p|\rho| \left\| \iint_0^t g^2(t-s,x-y)\sigma^2(Y(s,y))\,\mathrm{d}s\,\mathrm{d}y \right\|_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq C_p|\rho| \left(\iint_0^t g^2(t-s,x-y)\|\sigma(Y(s,y))\|_p^2\,\mathrm{d}s\,\mathrm{d}y \right)^{\frac{1}{2}} \leq C_p|\rho|T^{\frac{1}{4}}.\end{aligned}$$

Therefore, we deduce, if the left-hand side of (3.2) was finite, then

$$\mathbb{E}\left[\left| \iiint_0^t g(t-s,x-y)\sigma(Y(s,y))z\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \right|^p \right] < \infty \tag{5.30}$$

as well. We now show that this cannot be true. Indeed, because $\sigma(Y_0(t_0,x_0)) \neq 0$, there exists $(t_1,x_1) \in (0,t_0]\times\mathbb{R}^d$ with $\mathbb{P}[\sigma(Y(t_1,x_1)) \neq 0] > 0$ (otherwise, we would have $\sigma(Y(s,y)) = 0$ for all $(s,y) \in (0,t_0]\times\mathbb{R}^d$ and hence $Y(t,x) = Y_0(t,x)$ for all $(t,x) \in [0,t_0] \in \mathbb{R}^d$ by (1.1), which together contradict (3.1)). Therefore, we have $\mathbb{E}[|\sigma(Y(t_1,x_1))|] > 0$, and because the solution to (1.1) is L^r-continuous on $(0,\infty)\times\mathbb{R}^d$ for all $r < 1+2/d$ (see [10, Theorem 4.7(2)]), there exist $\epsilon, \delta > 0$ such that

$$0 \leq t_1 - s < \delta,\ |x_1 - y| \leq \sqrt{\delta} \implies \mathbb{E}[|\sigma(Y(s,y))|] > \epsilon. \tag{5.31}$$

As seen in the proof of Theorem 3.5, the left-hand side of (5.30) is greater than or equal to a constant times the same expression with μ and ν replaced by $\tilde{\mu}$ and $\tilde{\nu}$, respectively, where

$$\tilde{\mu}(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \mathbb{1}_{(t_1-\delta,t_1]}(s)\mathbb{1}_{\{|x_1-y|\leq\sqrt{t_1-s}\}}(y)\mathbb{1}_{\mathbb{R}\setminus[-a,a]}(z)\,\mu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z),$$

and $\tilde{\nu}$ is its compensator. The number $a > 0$ is chosen such that $\lambda(\mathbb{R}\setminus[-a,a]) > 0$. Now if $d \geq 2$ (and consequently $p \leq 2$), Lemma 3.4 gives the estimate

$$\begin{aligned}
&\mathbb{E}\left[\left|\iiint_0^{t_1} g(t_1-s, x_1-y)\sigma(Y(s,y))z\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^p\right] \\
&\quad\geq C\frac{\iiint_0^{t_1} g^p(t_1-s,x_1-y)\mathbb{E}[|\sigma(Y(s,y))|^p]|z|^p\,\tilde{\nu}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)}{(1\vee\tilde{\nu}([0,t]\times\mathbb{R}^d\times\mathbb{R}))^{1-\frac{p}{2}}} \\
&\quad= C\left(\int_0^\delta\int_{|y|\leq\sqrt{s}}\int_{\mathbb{R}\setminus[-a,a]}\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z)\right)^{\frac{p}{2}-1} \\
&\qquad\times\int_{t_1-\delta}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}} g^p(t_1-s,x_1-y)\mathbb{1}_{\{|x_1-y|\leq\sqrt{t_1-s}\}}\mathbb{E}[|\sigma(Y(s,y))|^p]|z|^p\mathbb{1}_{\{|z|>a\}}\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) \\
&\quad\geq C\epsilon^p\delta^{(1+\frac{d}{2})(\frac{p}{2}-1)}\int_0^\delta\int_{|x|\leq\sqrt{t}} g^p(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad= C\int_0^\delta\int_{|x|\leq\sqrt{t}} g^p(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

The last line is a valid lower bound also in the case $d = 1$, possibly with another value of C, as a consequence of [21, Theorem 1]. But for $p = 1 + 2/d$, we have

$$\begin{aligned}
\int_0^\delta\int_{|x|\leq\sqrt{t}} g^p(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \int_0^\delta\int_{|x|\leq\sqrt{t}}\frac{e^{-\frac{p|x|^2}{2\kappa t}}}{(2\pi\kappa t)^{\frac{pd}{2}}}\,\mathrm{d}t\,\mathrm{d}x \\
&\geq \frac{Ce^{-\frac{p}{2\kappa}}}{(2\pi\kappa)^{\frac{pd}{2}}}\int_0^\delta t^{-\frac{pd}{2}}t^{\frac{d}{2}}\,\mathrm{d}t = C\int_0^\delta\frac{1}{t}\,\mathrm{d}t = +\infty,
\end{aligned}$$

proving that (5.30) is wrong. □

Proof of Theorem 3.10. For every $(t,x) \in (0,\infty)\times\mathbb{R}^d$, we have

$$\begin{aligned}
\mathbb{E}[|Y(t,x)|] \geq \mathbb{E}[Y(t,x)] &= Y_0(t,x) + b\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)\mathbb{E}[\sigma(Y(s,y))]\,\mathrm{d}s\,\mathrm{d}y \\
&\geq Y_0(t,x) + bL_\sigma\int_0^t\int_{\mathbb{R}^d} g(t-s,x-y)\mathbb{E}[|Y(s,y)|]\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

Since the integral of g on $(0,\infty)\times\mathbb{R}^d$ is infinite, the theorem follows from the renewal methods as used in Theorems 3.5 and 3.6. □

Proof of Theorem 3.12. By Proposition 3.11, we can equally consider the stochastic cable equation (3.7) driven by the noise $\dot{M}$.

(1) We only carry out the proof for $c = 0$; the arguments are similar for $c > 0$ and we leave the details to the reader. Starting with $d \geq 2$ and $p \in (1, 1 + 2/d)$, by virtually the same calculations as in the proof of Theorem 3.5, the function $I_p(t) = \inf_{x\in\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^p]$ satisfies (5.16) with a_p replaced by the function $a_p(t) = a_p' e^{-p|b|\sigma_0 t}$ where $a_p' > 0$ is a constant, and with g replaced by g' in the definition (5.17) of w_p. But still, we have (5.18), so the renewal methods go through and the conclusion of Theorem 3.5(1) is valid. Statement (3) of the same theorem can be derived in a similar way.

For statement (2), we observe that the truncated jump measure in (5.14) does not need to use the same kernel function as in (5.13) a priori (it only needs to have a finite intensity measure). Hence, for imitating the proof of Theorem 3.5(2), we only replace g by g' in (5.13). For the indicator function $\mathbb{1}_{\{g(t-s,x-y)>\epsilon\}}$ in (5.14), by contrast, we replace $g(t,x)$ by $g(1;t,x)$ (i.e., the heat kernel with $\kappa = 1$ and without the $e^{-|b|\sigma_0 t}$ factor). As a consequence, the function in (5.17) becomes

$$w_\kappa'(t) = C_p \frac{\int_{\mathbb{R}} |z|^p \mathbb{1}_{\{|z|>\delta\}}\, \lambda(\mathrm{d}z)}{\left(1 \vee \lambda([-\delta,\delta]^c)\right) \iint_0^\infty \mathbb{1}_{\{g(1;t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x)^{1-\frac{p}{2}}} \int_{\mathbb{R}^d} e^{-p|b|\sigma_0 t} g^p(\kappa;t,x) \mathbb{1}_{\{g(1;t,x)>\epsilon\}}\,\mathrm{d}x.$$

Only the integral term in the previous line depends on κ. Hence, we conclude from (5.39) (the calculation there is valid up to the third line for any value of β) that $\int_0^\infty w_\kappa'(t)\,\mathrm{d}t$ is of order $\kappa^{-(p-1)d/2}$.

For $d = 1$, we need to let $p \to 3$. The BDG inequalities allow us to ignore the Gaussian part, so by [21, Theorem 1] and Lemma 5.4, we have that

$$\begin{aligned}
\mathbb{E}[|Y(t,x)|^p] \geq \frac{1}{6}\Bigg(& Y_0(t,x)^p + \sigma_0^p \mathbb{E}\left[\left(\iiint_0^t |g'(t-s,x-y)Y(s,y)z|^2\,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^{\frac{p}{2}}\right] \\
& + \sigma_0^p \mathbb{E}\left[\iiint_0^t |g'(t-s,x-y)Y(s,y)z|^p\,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right]\Bigg) \\
\geq \frac{1}{6} & \left(Y_0(t,x)^p + \sigma_0^p m_\lambda^p(p) \iiint_0^t g'^p(t-s,x-y)\mathbb{E}[|Y(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y\right),
\end{aligned} \tag{5.32}$$

so we can complete the proof as in the case $d \geq 2$ above.

(2) Since $\overline{\gamma}(p) < 0$ implies $\overline{\lambda}(p) = 0$, we can assume $c = 0$ in this part of the theorem. Furthermore, by the hypotheses of Proposition 2.3, the assumption that $m_\lambda(1 + 2/d) < \infty$, and Jensen's inequality, we may assume that p is large enough such that $m_\lambda(p)$ is finite, and if $d = 1$ and $\rho \neq 0$, that $p \geq 2$. Writing $C_{\beta,c}(b,\rho,\lambda,\kappa,p)$ for the constant $C_{\beta,c}(\kappa,p)$ in Proposition 2.1 to stress the dependence of the constant on the other parameters, we obtain with identical calculations as in the proof of Proposition 2.1 that $\|g' \circledast \Phi\|_{p,\beta,0} \leq C_{\beta+|b|\sigma_0,0}(0,\rho,\lambda,\kappa,p)$. In particular, if we re-examine the proof of Proposition 2.3 and the formula (2.4), we see that whenever κ or $|b|$ is large, or σ_0 is small, there exists $\beta < 0$ such that $C_{\beta+|b|\sigma_0,0}(0,\rho,\lambda,\kappa,p) < 1/\sigma_0$ and thus $\|Y\|_{p,\beta,0} < \infty$ and $\overline{\gamma}(p) < 0$.

□

Proof of Theorem 3.3. Let us introduce the following truncations of Λ:

$$\begin{aligned}\Lambda_n(\mathrm{d}t,\mathrm{d}x) &= b\psi_n(x)\,\mathrm{d}t\,\mathrm{d}x + \psi_n(x)\int_0^\infty z\mathbb{1}_{\{z>\frac{1}{n}\}}\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)\\ &= \left(b - \int_0^\infty z\mathbb{1}_{\{z>\frac{1}{n}\}}\,\lambda(\mathrm{d}z)\right)\psi_n(x)\,\mathrm{d}t\,\mathrm{d}x + \psi_n(x)\int_0^\infty z\mathbb{1}_{\{z>\frac{1}{n}\}}\,\mu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)\\ &=: b_n\psi_n(x)\,\mathrm{d}t\,\mathrm{d}x + \Lambda_n^+(\mathrm{d}t,\mathrm{d}x), \quad n\in\mathbb{N},\end{aligned}$$

where $\psi_n(x) = \psi(|x|/n)$ and $\psi\colon [0,\infty)\to[0,1]$ is a smooth function with $\mathbb{1}_{[0,1]} \le \psi \le \mathbb{1}_{[0,2]}$. If Y_n denotes the solution to (1.1) with noise Λ_n, we have by [8, Theorem 1] that $Y_n(t,x)\to Y(t,x)$ in L^p for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$ (the cited result remains valid for the smooth truncation functions ψ_n instead of the indicator functions $\mathbb{1}_{[-n,n]^d}$). So if we can show that almost surely, with obvious notation, $Y_{n,1}(t,x)\ge Y_{n,2}(t,x)$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$, then it follows that $Y_1(t,x)\ge Y_2(t,x)$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$ upon choosing separable modifications of Y_1 and Y_2, which is always possible, see [16, Theorem II.2.4].

Now notice that for every $T>0$, the measure Λ_n^+ only has finitely many jumps on $[0,T]\times\mathbb{R}^d$ almost surely. Let $T_0=0$ and (T_i,X_i,Z_i), $i=1,\dots,N_n(T)$, be the corresponding jump times, positions and sizes. The crucial observation is now that between (T_{i-1},T_i), in absence of jumps, both $Y_{n,1}$ and $Y_{n,2}$ satisfy the deterministic PDE

$$\partial_t Y_{n,j}(t,x) = \frac{\kappa}{2}\Delta Y_{n,j}(t,x) + b_n\sigma(Y_{n,j}(t,x))\psi_n(x), \quad j=1,2,$$

respectively. Since $f_1\ge f_2$ and σ is Lipschitz continuous, the comparison principle for the deterministic heat equation (see [5, Theorem II]) implies $Y_{n,1}(t,x)\ge Y_{n,2}(t,x)$ for all $(t,x)\in[0,T_1)\times\mathbb{R}^d$. By induction, we may therefore assume that $Y_{n,1}(t,x)\ge Y_{n,2}(t,x)$ holds for all $(t,x)\in[0,T_i)\times\mathbb{R}^d$ and then prove the same relation for $(t,x)\in[T_i,T_{i+1})\times\mathbb{R}^d$. But since $Z_i\ge 0$ and hence

$$\begin{aligned}Y_{n,1}(T_i,x) &= Y_{n,1}(T_i-,x) + \sigma(Y_{n,1}(T_i-,X_i))Z_i\delta_{X_i}(x)\\ &\ge Y_{n,2}(T_i-,x) + \sigma(Y_{n,2}(T_i-,X_i))Z_i\delta_{X_i}(x) = Y_{n,2}(T_i,x)\end{aligned}$$

by the induction hypothesis and the monotonicity property of σ, this again follows from the deterministic comparison principle (by considering smooth approximations of the Dirac delta function, the result of [5] extends to the measure-valued initial conditions encountered here).

Concerning the second statement of the theorem, the nonnegativity of Y follows from the first part by comparison with the zero solution corresponding to a zero initial condition. Next, observe that the mean function $m(t,x)=\mathbb{E}[Y(t,x)]$ satisfies $m(0,x)=f(x)$ and

$$\partial_t m(t,x) = \Delta m(t,x) + b\mathbb{E}[\sigma(Y(t,x))]\begin{cases}\le \Delta m(t,x) + (b\vee 0)Lm(t,x),\\ \ge \Delta m(t,x) + (b\wedge 0)Lm(t,x).\end{cases} \tag{5.33}$$

Again by the deterministic comparison principle, $m'(t,x)\le m(t,x)\le m''(t,x)$ where m' (resp. m'') is the solution to (5.33) with equality instead of "$\ge$" (resp. "$\le$"). Since m' (resp. m'') is given by the left-hand side (resp. right-hand side) of (3.3), all assertions follow. ☐

5.3 Proofs for Section 4

Proof of Theorem 4.1. (1) If $\beta_0 > 0$ satisfies (5.7), then the proof of Proposition 2.3 reveals that $\|Y\|_{p,\beta_0,0} < \infty$, and hence $\overline{\gamma}(p) \leq p\beta_0$. When $\lambda \not\equiv 0$ and p is close enough to $1 + 2/d$, the second summand in (2.4) is always the term of leading order. Thus, (5.7) holds as soon as β_0 satisfies

$$\frac{\Gamma(1-(p-1)\frac{d}{2})^{\frac{1}{p}}}{\beta_0^{\frac{1}{p}-\frac{d}{2p}(p-1)}} < C \iff \beta_0 > C^{-\frac{2/d}{1+2/d-p}}\Gamma\left(\tfrac{d}{2}\left(1+\tfrac{2}{d}-p\right)\right)^{\frac{2/d}{1+2/d-p}}$$

for some finite constant C independent of p. Since $x\Gamma(x) = \Gamma(1+x) \to 1$ as $x \to 0$, we can choose

$$\beta_0 = C^{-\frac{2/d}{1+2/d-p}}\left(\frac{\frac{2}{d}}{1+\frac{2}{d}-p}\right)^{\frac{2/d}{1+2/d-p}}$$

when p is sufficiently close to $1 + 2/d$, which implies

$$\limsup_{p\to 1+\frac{2}{d}} \frac{1+\frac{2}{d}-p}{\left|\log\left(1+\frac{2}{d}-p\right)\right|}\log\overline{\gamma}(p) \leq \frac{2}{d}. \tag{5.34}$$

The upper bound in (4.2) follows similarly.

For the lower bounds in (4.1) and (4.2), we first consider the case $b = 0$. For $d \geq 2$ let $\beta_1 = \beta_1(p)$ be the number for which

$$\int_0^\infty w_p(t)e^{-\beta_1 t}\,\mathrm{d}t = 1$$

where w_p is given by (5.17). Recalling (5.27), and assuming that p is close to $1 + 2/d$, and $\epsilon, \delta > 0$ are small enough such that (5.15) holds, we have that

$$\begin{aligned}
&\iint_0^\infty e^{-\beta t}g^p(t,x)\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad= \int_0^{\frac{1}{2\pi\kappa\epsilon^{2/d}}}\frac{e^{-\beta t}}{(2\pi\kappa t)^{\frac{pd}{2}}}\int_{\mathbb{R}^d} e^{-\frac{p|x|^2}{2\kappa t}}\mathbb{1}_{\{|x|^2<-2\kappa t\log(\epsilon(2\pi\kappa t)^{d/2})\}}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^{\frac{1}{2\pi\kappa\epsilon^{2/d}}}\frac{e^{-\beta t}}{(2\pi\kappa t)^{\frac{pd}{2}}}\int_0^{\sqrt{-2\kappa t\log(\epsilon(2\pi\kappa t)^{d/2})}} e^{-\frac{pr^2}{2\kappa t}}r^{d-1}\,\mathrm{d}r\,\mathrm{d}t \\
&\quad= \frac{1}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}}\int_0^{\frac{1}{2\pi\kappa\epsilon^{2/d}}}\frac{e^{-\beta t}}{t^{\frac{d}{2}(p-1)}}\gamma\left(\tfrac{d}{2}, -p\log\left(\epsilon(2\pi\kappa t)^{\frac{d}{2}}\right)\right)\mathrm{d}t \qquad (5.35)\\
&\quad\geq \frac{\gamma(\frac{d}{2},1)}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}}\int_0^{\frac{e^{-2/(pd)}}{2\pi\kappa\epsilon^{2/d}}}\frac{e^{-\beta t}}{t^{\frac{d}{2}(p-1)}}\,\mathrm{d}t \\
&\quad= \frac{\gamma(\frac{d}{2},1)\gamma(1-\frac{d}{2}(p-1),(2\pi\kappa)^{-1}\epsilon^{-\frac{2}{d}}e^{-\frac{2}{pd}}\beta)}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}\beta^{1-\frac{d}{2}(p-1)}}.
\end{aligned}$$

It follows for $\beta \geq 2\pi\kappa e^{2/(pd)}\epsilon^{2/d}$ that

$$\begin{aligned}\iint_0^\infty e^{-\beta t} g^p(t,x)\mathbb{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x &\geq \frac{\gamma(\frac{d}{2},1)\gamma(1-\frac{d}{2}(p-1),1)}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}\beta^{1-\frac{d}{2}(p-1)}} \\ &\geq \frac{\gamma(\frac{d}{2},1)(1-e^{-1})}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}(1-\frac{d}{2}(p-1))\beta^{1-\frac{d}{2}(p-1)}},\end{aligned} \tag{5.36}$$

where the last step uses $\gamma(1,1) = 1-e^{-1}$ and the fact that $x\gamma(x,1)$ is a continuous decreasing function on $[0,1]$. Indeed, the latter follows from the identity $x\gamma(x,1) = \gamma(x+1,1)+e^{-1}$, which can be proved by integration by parts. Observing that the factor in front of the integral in (5.17) is bounded for p around $1+2/d$, we deduce from (5.36) that

$$\beta_1 \geq \left(\frac{C}{1-\frac{d}{2}(p-1)}\right)^{\frac{1}{1-d(p-1)/2}} = \left(C\frac{\frac{2}{d}}{1+\frac{2}{d}-p}\right)^{\frac{2/d}{1+2/d-p}} \tag{5.37}$$

for some constant C independent of p. Hence we obtain from [2, Theorem V.7.1] that

$$\underline{\gamma}(p) \geq \beta_1 \geq \left(C\frac{\frac{2}{d}}{1+\frac{2}{d}-p}\right)^{\frac{2/d}{1+2/d-p}},$$

which implies

$$\liminf_{p\to 1+\frac{2}{d}} \frac{1+\frac{2}{d}-p}{\left|\log\left(1+\frac{2}{d}-p\right)\right|}\log\underline{\gamma}(p) \geq \frac{2}{d} \tag{5.38}$$

and hence (4.1) together with (5.34). For $d=1$, if we estimate as in (5.32), the same arguments apply and only some constants would change that have no impact on the result.

For the lower bound in (4.2), the estimates (5.35) and (5.36) can be re-used in principle, but we need to make a small change in our arguments because the denominator in (5.17) involves the kernel g and therefore the parameter κ, which would lead to a suboptimal lower bound. In order to avoid this, we proceed as in the proof of Theorem 3.12(2), and construct the measure in (5.14) by using the indicator function $\mathbb{1}_{\{g(1;t-s,x-y)>\epsilon\}}$ instead of $\mathbb{1}_{\{g(t-s,x-y)>\epsilon\}}$, where $g(1;t,x)$ is the heat kernel with $\kappa=1$. Then we have for $\kappa \leq 1$ and $\beta \geq 2\pi\epsilon^{2/d}e^{2/(pd)}$,

$$\begin{aligned}&\iint_0^\infty e^{-\beta t} g^p(t,x)\mathbb{1}_{\{g(1;t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x \\ &\quad= \frac{1}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}}\int_0^{\frac{1}{2\pi\epsilon^{2/d}}} \frac{e^{-\beta t}}{t^{\frac{d}{2}(p-1)}}\gamma\left(\tfrac{d}{2}, -p\kappa^{-1}\log\left(\epsilon(2\pi t)^{\frac{d}{2}}\right)\right)\,\mathrm{d}t \\ &\quad\geq \frac{\gamma(\frac{d}{2},1)}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}}\int_0^{\frac{e^{-2/(pd)}}{2\pi\epsilon^{2/d}}} \frac{e^{-\beta t}}{t^{\frac{d}{2}(p-1)}}\,\mathrm{d}t \geq \frac{\gamma(\frac{d}{2},1)\gamma(1-\frac{d}{2}(p-1),1)}{p^{\frac{d}{2}}\Gamma(\frac{d}{2})(2\pi\kappa)^{\frac{d}{2}(p-1)}\beta^{1-\frac{d}{2}(p-1)}}.\end{aligned} \tag{5.39}$$

Thus, $\beta \geq C\kappa^{-\frac{p-1}{1+2/d-p}}$, proving the lower bound in (4.2).

Now let us explain why the proof of the lower bounds, for both $p \to 1+2/d$ and $\kappa \to 0$, remains essentially unchanged for $b < 0$ or $b > 0$. Indeed, if σ is given by (1.12), Proposition 3.11 implies that we have to multiply g by a factor $e^{b\sigma_0 t}$. But under the truncation $\mathbb{1}_{\{g(t,x)>\epsilon\}}$ (resp. $\mathbb{1}_{\{g(1;t,x)>\epsilon\}}$ when $\kappa \to 0$ is considered), we have $t < T$ where $T = (2\pi\epsilon^{2/d})^{-1}$ is independent of p (resp. κ). In particular, g and $ge^{b\sigma_0 t}$ differ at most by a multiplicative constant $e^{b\sigma_0 T}$ on $[0,T]$, which is irrelevant for the calculations above.

(2) The upper bound for $\overline{\lambda}(p)$ in (4.3) as $p \to 1+2/d$ follows from (1) because we have (5.8). For the upper bound in (4.4), observe from (5.8) that $\overline{\lambda}(p) \leq \beta_0/c$ where β_0 was introduced in the proof of Proposition 2.3. Upon inspection of formula (2.4), we see that β_0 must satisfy

$$\frac{C}{\beta_0 - \frac{1}{2}\kappa c^2 d} + \frac{C'}{\kappa^{\frac{d(p-1)}{2p}}(\beta_0 - \frac{1}{2}\kappa c^2 d)^{\frac{2-d(p-1)}{2p}}} + \frac{C''}{\kappa^{\frac{1}{4}}(\beta_0 - \frac{1}{2}\kappa c^2)^{\frac{1}{4}}}\mathbb{1}_{\{d=1,\ p\geq 2\}} \leq 1.$$

As long as $\lambda \not\equiv 0$, the second summand is the dominant one for small κ, so β_0 as a function of κ behaves in this case like

$$\frac{1}{2}\kappa c^2 d + C\kappa^{-\frac{p-1}{1+2/d-p}}.$$

Consequently, if we optimize the resulting bound for $\overline{\lambda}(p)$ over c, we get

$$\overline{\lambda}(p) \leq \inf_{c\geq 0}\left(\frac{1}{2}\kappa c d + Cc^{-1}\kappa^{-\frac{p-1}{1+2/d-p}}\right) = C'\kappa^{\frac{1+1/d-p}{1+2/d-p}},$$

which implies the upper bound in (4.4).

In order to establish the lower bounds in (4.3) and (4.4), it suffices by the same reason as in (1) to take $b = 0$. In this case, for fixed ϵ and κ, we bound (5.29) from below by

$$C\frac{\tilde{\alpha}^{-2(1-(p-1)\frac{d}{2})}}{1+\frac{2}{d}-p},$$

where $C > 0$ does not depend on p. As a result,

$$\underline{\lambda}(p) \geq \left(\frac{C}{1+\frac{2}{d}-p}\right)^{\frac{1}{2(1-(p-1)d/2)}},$$

which is the lower bound in (4.3). For $\kappa \to 0$, we repeat the argument given in the proof of Theorem 3.6, but use the truncation $\mathbb{1}_{\{g(1;t,x)>\epsilon\}}$ instead of $\mathbb{1}_{\{g(t,x)>\epsilon\}}$ in (5.14). Hence, instead of $\tilde{h}$ in (5.24), the function of interest is

$$h'(t) = \int_{|x|\geq\tilde{\alpha}t} g^p(t,x)\mathbb{1}_{\{g(1;t,x)>\epsilon\}}\,\mathrm{d}x.$$

If we redo the calculations from (5.25) to (5.29), then instead of (5.26), we should consider $R' = \tilde{\alpha}^2/(\kappa\epsilon^{2/d})$ so that in the end, we obtain exactly the same lower bound for $\int_0^\infty h'(t)\,\mathrm{d}t$ as in (5.29), but under the new condition $\tilde{\alpha}^2\epsilon^{-2/d} \geq 2\pi\kappa$. Hence, we can make $\int_0^\infty h'(t)\,\mathrm{d}t$ arbitrarily large if we take

$$\tilde{\alpha} = C\kappa^{\frac{1+1/d-p}{1+2/d-p}},$$

and a large value for $C > 0$. This choice of $\tilde{\alpha}$ satisfies $\tilde{\alpha}^2 \epsilon^{-2/d} \geq 2\pi\kappa$ for all κ small enough, so the lower bound in (4.4) follows. Note that at this part it is enough if $f(x) = O(e^{-c|x|})$ holds for some fixed $c > 0$.

□

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