Avoiding long Berge cycles, the missing cases $k = r + 1$ and $k = r + 2$

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Abstract

The maximum size of an r-uniform hypergraph without a Berge cycle of length at least k has been determined for all $k \geq r + 3$ by Füredi, Kostochka and Luo and for $k < r$ (and $k = r$, asymptotically) by Kostochka and Luo. In this paper, we settle the remaining cases: $k = r + 1$ and $k = r + 2$, proving a conjecture of Füredi, Kostochka and Luo.

Given a hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ and $E(\mathcal{H})$ denote the set of vertices and hyperedges of $\mathcal{H}$, respectively, and let $e(\mathcal{H}) = |E(\mathcal{H})|$. A hypergraph is called r-uniform if all of its hyperedges have size r. For convenience, we refer to an r-uniform hypergraph as an r-graph. Berge introduced the following definitions of a cycle and a path in a hypergraph.

Definition 1. *A* Berge cycle *of length l in a hypergraph is a set of l distinct vertices $\{v_1, \dots, v_l\}$ and l distinct hyperedges $\{e_1, \dots, e_l\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo l.*

A Berge path *of length l in a hypergraph is a set of $l+1$ distinct vertices $v_1, \dots, v_{l+1}$ and l distinct hyperedges $e_1, \dots, e_l$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq l$. We say that such a Berge path is between v_1 and v_{l+1}.*

Notation 1. *Let $\mathcal{H}$ be a hypergraph. Then its* 2-shadow, *$\partial_2\mathcal{H}$, is the collection of pairs that lie in some hyperedge of $\mathcal{H}$. Given a set $S \subseteq V(\mathcal{H})$, the subhypergraph of $\mathcal{H}$ induced by S is denoted by $\mathcal{H}[S]$.*

We say $\mathcal{H}$ is connected *if $\partial_2(\mathcal{H})$ is a connected graph. A hyperedge $h \in E(\mathcal{H})$ is called a* cut-hyperedge *of $\mathcal{H}$ if $\mathcal{H} \setminus \{h\} := (V(\mathcal{H}), E(\mathcal{H}) \setminus \{h\})$ is not connected.*

When we say D is a block of $\partial_2(\mathcal{H})$, we may either mean D is the vertex-set of the block, or D is the edge-set of the block depending on the context.

1 Background and our results

Győri, Katona and Lemons extended the well-known Erdős-Gallai theorem to hypergraphs by showing the following.

Theorem 1 (Győri, Katona, Lemons [8])**.** *Let $\mathcal{H}$ be an r-uniform hypergraph with no Berge path of length k. If $k > r+1 > 3$, we have*

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}.$$

If $r \geq k > 2$, we have

$$e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}.$$

For the case $k = r+1$, Győri, Katona and Lemons conjectured that the upper bound should have the same form as the $k > r+1$ case. This was settled by Davoodi, Győri, Methuku and Tompkins [1] who showed the following.

Theorem 2 (Davoodi, Győri, Methuku, Tompkins [1])**.** *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k. Then,*

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n.$$

The bounds in the above two theorems are sharp for each k and r for infinitely many n. Győri, Methuku, Salia, Tompkins and Vizer [9] proved a significantly smaller upper bound on the maximum number of hyperedges in an n-vertex r-graph with no Berge path of length k under the assumption that it is connected. Their bound is asymptotically exact when r is fixed and k and n are sufficiently large. The notion of Berge cycles and Berge paths was generalized to arbitrary Berge graphs F by Gerbner and Palmer in [5], and the (3-uniform) Turán number of Berge-$K_{2,t}$ was determined asymptotically in [6]. The general behaviour of the Turán number of Berge-F, as the uniformity increases, was studied in [7].

Recently, Füredi, Kostochka and Luo [3] proved exact bounds similar to Theorem 1 for hypergraphs avoiding long Berge cycles.

Theorem 3 (Füredi, Kostochka, Luo [3])**.** *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{k-1}$ and $\mathcal{H}[D] = K^r_{k-1}$.*

Moreover, Kostochka and Luo [10] found exact bounds for $k \leq r-1$ and asymptotic bounds for $k = r$. Let us remark that their asymptotic bound in the case $k = r$ also follows from Theorem 5 stated below. (More recently, extending [3], Füredi, Kostochka, Luo [4] proved exact bounds and determined the extremal examples for all n when $k \geq r+4$.)

The two cases $k = r+2$ and $k = r+1$ remained open. For the case $k = r+2$, Füredi, Kostochka and Luo conjectured [3] that a similar statement as that of Theorem 3 holds and mentioned the answer is unknown for the case $k = r+1$. In this paper, we prove this conjecture.

Theorem 4. *Let $r \geq 3$ and $n \geq 1$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length $r+2$ or longer. Then $e(\mathcal{H}) \leq \frac{r+1}{r}(n-1)$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D] = K^r_{r+1}$.*

In the case $k = r+1$, we prove the following exact result, and characterize the extremal examples.

Theorem 5. *Let $r \geq 3$ and $n \geq 1$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length $r+1$ or longer. Then $e(\mathcal{H}) \leq n-1$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of r hyperedges.*

Note that Theorem 5 easily implies Theorem 2. In fact, it gives the following stronger form. Here we quickly prove this implication.

Theorem 6. *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k. Then, $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n$. Moreover, equality holds if and only if each connected component D of $\partial_2(\mathcal{H})$ is K_{r+1}, and $\mathcal{H}[D] = K^r_{r+1}$.*

Proof. We proceed by induction on n. The base cases $n \leq r+1$ are easy to check. Let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length $k = r+1$ such that $e(\mathcal{H}) \geq n$. Then by Theorem 5, $\mathcal{H}$ contains a Berge cycle $\mathcal{C}$ of length $r+1$ or longer. $\mathcal{C}$ must be of length exactly $r+1$, otherwise it would contain a Berge path of length $r+1$. Let $v_1, \dots, v_{r+1}$ and $e_1, \dots, e_{r+1}$ be the vertices and edges of $\mathcal{C}$ where $\{v_i, v_{i+1}\} \subseteq e_i$ (indices are taken modulo $r+1$). For any i with $1 \leq i \leq r+1$, if e_i contains a vertex $v \notin \{v_1, \dots, v_{r+1}\}$, then $v_{i+1}e_{i+1}v_{i+2}e_{i+2} \dots e_{i-1}v_ie_iv$ forms a Berge path of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, all of the edges e_i (for $1 \leq i \leq r+1$) are contained in the set $S := \{v_1, \dots, v_{r+1}\}$. That is, $\mathcal{H}[S] = K^r_{r+1}$. It is easy to see that S forms a connected component in $\partial_2(\mathcal{H})$ because if any hyperedge h of $\mathcal{H}$ (with $h \notin \mathcal{C}$) contains a vertex of $\mathcal{C}$, then $\mathcal{C}$ can be extended to form a Berge path of length $r+1$.

Let $S_1, S_2, \dots, S_t$ be the vertex sets of connected components of $\partial_2(\mathcal{H})$. As noted before, one of them, say S_1, is equal to S. We delete the vertices of S_1 from $\mathcal{H}$ to form a new hypergraph $\mathcal{H}'$; note that $|V(\mathcal{H}')| = |V(\mathcal{H})| - (r+1)$ and $|E(\mathcal{H}')| = |E(\mathcal{H})| - (r+1)$ and the connected components of $\partial_2(\mathcal{H}')$ are $S_2, \dots, S_t$. By induction $|E(\mathcal{H}')| \le |V(\mathcal{H}')|$. Thus $|E(\mathcal{H})| = |E(\mathcal{H}')| + (r+1) \le |V(\mathcal{H}')| + (r+1) = |V(\mathcal{H})|$. Moreover, if $|E(\mathcal{H})| = |V(\mathcal{H})|$, then $|E(\mathcal{H}')| = |V(\mathcal{H}')|$, so by the induction hypothesis each connected component S_i ($i \ge 2$) of $\partial_2(\mathcal{H}')$ is K_{r+1}, and $\mathcal{H}'[S_i] = K^r_{r+1}$, proving the theorem. □

Structure of the paper: In Section 2, we prove some basic lemmas which are used in our proofs. In Section 3, we prove Theorem 4, and in Section 4, we prove Theorem 5.

2 Basic Lemmas

We will use the following two lemmas.

Lemma 1. *For any $r \ge 3$, if a set S of size $r+1$ contains r hyperedges of size r, then between any two vertices $u, v \in S$, there is a Berge path of length r consisting of these hyperedges.*

Proof. Let $\mathcal{H}$ be the hypergraph consisting of r hyperedges on $r+1$ vertices. First notice that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. (Indeed, there is at most one hyperedge that does not contain x and at most one hyperedge that does not contain y.) This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least 3 hyperedges contained in S. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2, \dots, x_{r+1}$ of length r in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_{r+1}$. We want to show that there are distinct hyperedges containing the pairs x_ix_{i+1} for each $1 \le i \le r$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \dots, x_rx_{r+1}\}$ in one class and the r hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We will show that Hall's condition holds: As noted before, every pair is contained in a hyperedge. Given any two distinct pairs x_ix_{i+1} and x_jx_{j+1}, there is at most one hyperedge that does not contain either of them; i.e., at least $r-1$ hyperedges contain one of them. Thus we need $2 \le r-1$ for Hall's condition to hold, but this is true as we assumed $r \ge 3$. Moreover, if we take any $3 \le j \le r$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \le r$, but this is true by assumption. This finishes the proof of the lemma. □

Lemma 2. *For any $r \ge 4$, if a set S of size $r+1$ contains $r-1$ hyperedges of size r, then between any two vertices $u, v \in S$, there is a Berge path of length $r-1$ consisting of these hyperedges.*

Proof. The proof is similar to that of Lemma 1. Let $\mathcal{H}$ be the hypergraph consisting of $r-1$ hyperedges on $r+1$ vertices. First notice that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least $r-1 \ge 3$ hyperedges contained in S. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2\dots x_r$ of length $r-1$ in the $\partial_2(\mathcal{H})$ connecting $u=x_1$ and $v=x_r$. We want to show that there are distinct hyperedges containing the pairs x_ix_{i+1} for each $1\le i\le r-1$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2,x_2x_3,\dots,x_{r-1}x_r\}$ in one class and the $r-1$ hyperedges $h\subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We show that Hall's condition holds: As noted before, every pair is contained in a hyperedge. Given any two distinct pairs x_ix_{i+1} and x_jx_{j+1}, there is at most one hyperedge that does not contain either of them; i.e., at least $r-2$ hyperedges contain one of them. Thus we need $2\le r-2$ for Hall's condition to hold, but this is true as we assumed $r\ge 4$. Moreover, if we take any $3\le j\le r-1$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j\le r-1$ for Hall's condition to hold, and this is true by assumption. This finishes the proof of the lemma. $\square$

3 Proof of Theorem 4 ($k=r+2$)

We will prove the theorem by induction on n. For the base cases, note that if $1\le n\le r$ then the statement of the theorem is trivially true. If $n=r+1$, the statement is true since there are at most $r+1$ hyperedges of size r on $r+1$ vertices. Moreover, equality holds if and only if $\mathcal{H}=K^r_{r+1}$.

We will show the statement is true for $n\ge r+2$ assuming it is true for all smaller values. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices having no Berge cycle of length $r+2$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S\subseteq V(\mathcal{H})$ of vertices, the number of hyperedges of $\mathcal{H}$ incident to the vertices of S is at least $|S|$.

Indeed, suppose there is a set $S\subseteq V(\mathcal{H})$ with fewer than $|S|$ hyperedges incident to the vertices of S. If $|S|=n$ we immediately have the required bound on $e(\mathcal{H})$, so assume $n>|S|$. We can delete the vertices of S from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n-|S|$ vertices. By induction, $\mathcal{H}'$ contains at most $\frac{r+1}{r}(n-|S|-1)$ hyperedges, so $\mathcal{H}$ contains less than $\frac{r+1}{r}(n-1-|S|)+|S|<\frac{r+1}{r}(n-1)$ hyperedges, as desired.

(2) There is no cut-hyperedge in $\mathcal{H}$.

Indeed, if $h\in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H}\setminus\{h\})$ is not a connected graph, so there are non-empty disjoint sets V_1 and V_2 such that $V(\mathcal{H})=V_1\cup V_2$, and there are no edges of $\partial_2(\mathcal{H}\setminus\{h\})$ between V_1 and V_2. So both hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge cycle of length $r+2$ or longer. By induction, $e(\mathcal{H}[V_1])\le \frac{r+1}{r}(|V_1|-1)$ and $e(\mathcal{H}[V_2])\le\frac{r+1}{r}(|V_2|-1)$. In total, $e(\mathcal{H})=e(\mathcal{H}[V_1])+e(\mathcal{H}[V_2])+1\le \frac{r+1}{r}(|V_1|+|V_2|-2)+1<\frac{r+1}{r}(|V(\mathcal{H})|-1)$, as desired.

Consider an auxiliary bipartite graph B consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ on the other class. Then property (1) shows that Hall's condition holds. Therefore, there is a perfect matching in B. In other words, there exists an injection $f: V(\mathcal{H}) \to E(\mathcal{H})$ such that $v \in f(v)$.

Given an injection $f: V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge path of the form $v_1 f(v_1) v_2 f(v_2) \dots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \leq i \leq l-1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f: V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi: V(\mathcal{H}) \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \dots v_{l-1} \phi(v_{l-1}) v_l$ is a longest path.

Claim 1. $\phi(v_l) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$.

Proof. First notice that if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \dots, v_{l-r-1}\}$, then the Berge cycle $v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \dots v_l \phi(v_l) v_i$ is of length $r+2$ or longer, a contradiction. Moreover, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then $\mathcal{P}_\phi$ can be extended to a longer path $v_1 \phi(v_1) v_2 \phi(v_2) \dots v_{l-1} \phi(v_{l-1}) v_l \phi(v_l) v$, a contradiction. This completes the proof of the claim. □

By Claim 1, we know that $\phi(v_l) = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\} \setminus \{v_j\}$ for some $l-r \leq j \leq l-1$.

Claim 2. *For any $i \in \{l-r, l-r+1, \dots, l\} \setminus \{j\}$, we have $\phi(v_i) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$.*

Proof. When $i = l$, we know the statement is true. Suppose $i \in \{l-r, l-r+1, \dots, l-1\} \setminus \{j\}$. Let us define a new injection $\psi: V(\mathcal{H}) \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{v_1, v_2, \dots, v_l\}$, and for every $v \in \{v_1, v_2, \dots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \geq k \geq i+1$.

Now consider the Berge path $v_1 \phi(v_1) v_2 \phi(v_2) \dots v_i \phi(v_l) v_l \phi(v_{l-1}) \dots v_{i+2} \phi(v_{i+1}) v_{i+1}$, equivalently $v_1 \psi(v_1) v_2 \psi(v_2) \dots v_i \psi(v_i) v_l \psi(v_l) \dots v_{i+2} \psi(v_{i+2}) v_{i+1}$. This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, notice that the sets of last $r+1$ vertices of both paths are the same. Thus we can apply Claim 1 to conclude that $\phi(v_i) = \psi(v_{i+1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$, as desired. □

Claim 2 shows that there are r hyperedges (each of size r) contained in the set $S := \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ of size $r+1$. We will apply Lemma 1 to S.

Claim 3. *The set $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ induces a block of $\partial_2(\mathcal{H})$.*

Proof. Since the set $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ contains $r \geq 3$ hyperedges every pair $x, y \in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block D of $\partial_2(\mathcal{H})$ containing S.

Suppose D contains a vertex $t \notin S$. Then since D is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between t and S, which are vertex-disjoint besides t. Let $V(P_1) \cap S = \{u\}$ and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v.

On the other hand, by Lemma 1, there is a Berge path $\mathcal{P}'$ of length r between u and v consisting of the r hyperedges contained in S. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in S, while hyperedges of $\mathcal{P}'$

are contained in S). Therefore, $\mathcal{P} \cup \mathcal{P}'$ forms a Berge cycle of length $r+2$ or longer unless $\mathcal{P}$ consists of only one hyperedge, say h. Note that h contains a vertex $x \notin S$ and $u, v \in h$; moreover by property (2), h is not a cut-hyperedge of $\mathcal{H}$. So after deleting h from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{h\}$ is still connected – so there is a (shortest) Berge path $\mathcal{Q}$ in $\mathcal{H} \setminus \{h\}$ between x and a vertex $s \in S$ (note that the hyperedges of $\mathcal{Q}$ are not contained in S). The vertex s is different from either u or v, say $s \neq u$ without loss of generality. By Lemma 1, there is a Berge path $\mathcal{Q}'$ of length r between s and u (consisting of hyperedges contained in S). Then, $\mathcal{Q}, \mathcal{Q}'$ and h form a Berge cycle of length at least $r+2$, a contradiction. Therefore, D contains no vertex outside S; thus S induces a block of $\partial_2(\mathcal{H})$, as required. $\square$

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 3 shows that one of these blocks, say D_1, is induced by S. Let us contract the vertices of S to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length $r+2$ or longer, as well; moreover $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$. Thus, by induction, we have $e(\mathcal{H}') \leq \frac{r+1}{r}(|V(\mathcal{H}')|-1)$. Therefore,

$$e(\mathcal{H}) \leq \frac{r+1}{r}(|V(\mathcal{H}')|-1) + (r+1) = \frac{r+1}{r}(|V(\mathcal{H})|-r-1) + (r+1) = \frac{r+1}{r}(|V(\mathcal{H})|-1).$$

Now if $e(\mathcal{H}) = \frac{r+1}{r}(|V(\mathcal{H})|-1)$, then we must have $e(\mathcal{H}') = \frac{r+1}{r}(|V(\mathcal{H}')|-1)$ and S must contain all $r+1$ subsets of size r (i.e., $\mathcal{H}[S] = \mathcal{H}[D_1] = K_{r+1}^r$). Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block D_i (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i] = K_{r+1}^r$. This means that for every block D of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D] = K_{r+1}^r$, completing the proof.

4 Proof of Theorem 5 ($k = r+1$)

The proof is similar to that of Theorem 4 but there are many important differences.

We use induction on n. For the base cases, notice that the statement of the theorem is trivially true if $1 \leq n \leq r$. Moreover, if $n = r+1$, then $e(\mathcal{H}) \leq r$ because otherwise, $\mathcal{H} = K_{r+1}^r$ and then it is easy to see that there is a (Hamiltonian) Berge cycle of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, $e(\mathcal{H}) \leq r = n-1$. Moreover, equality holds if and only if $\partial_2(\mathcal{H}) = K_{r+1}$ and $\mathcal{H}$ consists of r hyperedges.

We will show the statement is true for n assuming it is true for all smaller values. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices having no Berge cycle of length $r+1$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S \subseteq V(\mathcal{H})$ with $|S| \leq |V(\mathcal{H})|-1 = n-1$, the number of hyperedges of $\mathcal{H}$ incident to the vertices of S is at least $|S|$.

Indeed, suppose there is a set $S \subset V(\mathcal{H})$ (i.e., $|S| \leq |V(\mathcal{H})|-1$) with fewer than $|S|$ hyperedges incident to the vertices of S. We delete the vertices of S from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n-|S|$ vertices. By induction, $\mathcal{H}'$ contains at most $(n-|S|-1)$ hyperedges, so $\mathcal{H}$ contains less than $(n-1-|S|)+|S| = (n-1)$ hyperedges, as required.

(2) There is no cut-hyperedge in $\mathcal{H}$.

Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are disjoint non-empty sets V_1 and V_2 such that $V(\mathcal{H}) = V_1 \cup V_2$ and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between V_1 and V_2. So the hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge cycle of length $r+1$ or longer. Therefore, by induction, $e(\mathcal{H}[V_1]) \leq |V_1|-1$ and $e(\mathcal{H}[V_2]) \leq |V_2|-1$. In total, $e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \leq (|V_1|+|V_2|-2) + 1 = |V(\mathcal{H})|-1$, as desired.

Moreover, we claim that the equality $e(\mathcal{H}) = |V(\mathcal{H})|-1$ cannot hold in this case (i.e., if there is a cut-hyperedge). Indeed, if equality holds, then we must have $e(\mathcal{H}[V_1]) = |V_1|-1$ and $e(\mathcal{H}[V_2]) = |V_2|-1$. Notice that since $r \geq 3$, the hyperedge h either contains at least two vertices $x, y \in V_1$ or two vertices $x, y \in V_2$. Without loss of generality, assume the former is true. By induction, $\partial_2(\mathcal{H}[V_1])$ is connected and for every block D of $\partial_2(\mathcal{H}[V_1])$, we have $D = K_{r+1}$ and the subhypergraph induced by D consists of r hyperedges. So by Lemma 1, there is a Berge path of length r (consisting of the r hyperedges induced by D) between any two vertices of a block D. Then it is easy to see that since $\partial_2(\mathcal{H}[V_1])$ is connected, there is a Berge path $\mathcal{P}$ of length at least r between any two vertices of V_1, so in particular between x and y. Then $\mathcal{P}$ together with h forms a Berge cycle of length $r+1$ in $\mathcal{H}$, a contradiction.

Consider an auxiliary bipartite graph B consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ on the other class. Then property (1) shows that Hall's condition holds for all subsets of $V(\mathcal{H})$ of size up to $|V(\mathcal{H})|-1$. Therefore, there is a matching in B that matches all the vertices in $V(\mathcal{H})$, except at most one vertex, say x. In other words, there exists an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ such that for every $v \in V(\mathcal{H}) \setminus \{x\}$, we have $v \in f(v)$. Given an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge path of the form $v_1 f(v_1) v_2 f(v_2) \ldots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \leq i \leq l-1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \ldots v_{l-1} \phi(v_{l-1}) v_l$ is a longest path.

Because of the way $\mathcal{P}_\phi$ was constructed, it is also clear that $x \notin \{v_1, v_2, \ldots, v_{l-1}\}$. We consider two cases depending on whether v_l is equal to x or not.

Case 1: $v_l \neq x$**.** Our aim is to get a contradiction, and show that this case is impossible.

Claim 4. *If $v_l \neq x$, then $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$.*

Proof. If $v_l \neq x$, then we claim $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$. Indeed, if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \ldots, v_{l-r}\}$, then the Berge cycle $v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \ldots v_l \phi(v_l) v_i$ is of length $r+1$ or longer, a contradiction. Moreover, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \ldots, v_l\}$, then P_ϕ can be extended to a longer path $v_1 \phi(v_1) v_2 \phi(v_2), \ldots, v_{l-1} \phi(v_{l-1}) v_l \phi(v_l) v$, a contradiction again, proving that $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$. □

Fix some $i \in \{l-r+1, l-r+2, \ldots, l-1\}$. Let us define a new injection $\psi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{x, v_1, v_2, \ldots, v_l\}$, and for every $v \in \{v_1, v_2, \ldots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \geq k \geq$

$i+1$. Now consider the Berge path $v_1\phi(v_1)v_2\phi(v_2)\ \ldots\ v_i\phi(v_l)v_l\phi(v_{l-1})\ldots v_{i+2}\phi(v_{i+1})v_{i+1}$ $= v_1\psi(v_1)v_2\psi(v_2)\ \ldots\ v_i\psi(v_i)v_l\psi(v_l)\ldots v_{i+2}\psi(v_{i+2})v_{i+1}$. This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, $v_{i+1}\neq x$, so we can apply Claim 4 to conclude that $\psi(v_{i+1}) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\} = \phi(v_i)$. But then $\phi(v_i) = \phi(v_l)$, a contradiction to the fact that ϕ was an injection.

Case 2: $v_l = x$**.**

Claim 5. $\phi(v_{l-1}) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_l\}$.

Proof. If $\phi(v_{l-1})$ contains a vertex $v\notin\{v_1, v_2, \ldots, v_l\}$, then we consider the Berge path $v_1\phi(v_1)v_2\phi(v_2),\ldots,v_{l-1}\phi(v_{l-1})v$. Since $v\neq x$, we get a contradiction by Case 1. Moreover, if $\phi(v_{l-1})$ contains a vertex v_i with $i\in\{1,2,\ldots,l-r-1\}$, then the Berge cycle $v_i\phi(v_i)v_{i+1}\phi(v_{i+1})\ldots v_{l-1}\phi(v_{l-1})v_i$ is of length $r+1$ or longer, a contradiction. This finishes the proof of the claim. □

By Claim 5, we know that $\phi(v_{l-1}) = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}\setminus\{v_j\}$ for some j with $l-r\leq j\leq l-2$. (From now, in the rest of the proof we fix this j.)

Claim 6. *For any* $i\in\{l-r, l-r+1, \ldots, l-1\}\setminus\{j\}$*, we have* $\phi(v_i)\subset\{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$*.*

Proof. When $i = l-1$, we know the statement is true by Claim 5.

Suppose $i\in\{l-r, l-r+1, \ldots, l-2\}\setminus\{j\}$. Let us define a new injection $\psi : V(\mathcal{H})\setminus\{x\}\to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v\notin\{v_1, v_2, \ldots, v_l\}$, and for every $v\in\{v_1, v_2, \ldots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_{l-1})$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l-1\geq k\geq i+1$. Now consider the Berge path $v_1\phi(v_1)v_2\phi(v_2)\ \ldots\ v_i\phi(v_{l-1})v_{l-1}\phi(v_{l-2})\ldots v_{i+1} = v_1\psi(v_1)v_2\psi(v_2)\ \ldots\ v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\ldots v_{i+1}$. (Note that when $i = l-2$, the Berge path is simply $v_1\phi(v_1)v_2\phi(v_2)\ \ldots\ v_i\phi(v_{l-1})v_{l-1} = v_1\psi(v_1)v_2\psi(v_2)\ldots v_i\psi(v_i)v_{l-1}$.)

If $\psi(v_{i+1})$ contains a vertex $v\notin\{v_1, v_2, \ldots, v_l\}$, then the Berge path $v_1\psi(v_1)v_2\psi(v_2)\ \ldots\ v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\ldots v_{i+2}\psi(v_{i+2})v_{i+1}\psi(v_{i+1})v$ has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, since $v\neq x$, we get a contradiction by Case 1.

If $\psi(v_{i+1})$ contains a vertex $v_k\in\{v_1, v_2, \ldots, v_{l-r-1}\}$ then one can see that the Berge cycle $v_k\psi(v_k)v_{k+1}\psi(v_{k+1})\ \ldots\ v_{l-1}\psi(v_{l-1})v_k$ is of length $r+1$ or longer, a contradiction. Therefore, we have $\psi(v_{i+1})\subset\{v_{l-r}, v_{l-r+1}, \ldots, v_l\}$. But we defined $\psi(v_{i+1}) = \phi(v_i)$, proving the claim. □

Note that Claim 6 shows that $r-1$ hyperedges of $\mathcal{H}$ are contained in a set $S := \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$ of size $r+1$. The following claim shows that if we can find one more hyperedge of $\mathcal{H}$ contained in S, then S must induce a block of $\partial_2(\mathcal{H})$.

Claim 7. *Suppose* $r\geq 3$*. If a set* S *of size* $r+1$ *contains* r *hyperedges of* $\mathcal{H}$ *then it induces a induces a block of* $\partial_2(\mathcal{H})$*.*

Proof. Since the set S contains at least 3 hyperedges every pair $x,y\in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block D of $\partial_2(\mathcal{H})$ containing S.

Suppose D contains a vertex $t\notin S$. Then since D is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between t and S, which are vertex-disjoint besides t. Let $V(P_1)\cap S = \{u\}$

and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v.

On the other hand, by Lemma 1, there is a Berge path $\mathcal{P}'$ of length r between u and v consisting of the r hyperedges contained in S. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in S, while hyperedges of $\mathcal{P}'$ are contained in S). Therefore, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length $r+1$ or longer, a contradiction. Therefore, D contains no vertex outside S; thus S induces a block of $\partial_2(\mathcal{H})$, as required. □

We will use the above claim several times later. At this point we need to distinguish the cases $r = 3$ and $r \geq 4$, since Lemma 2 only applies in the latter case.

The case $r \geq 4$

Since $r \geq 4$, by Claim 6 and Lemma 2 there is a Berge path of length $r-1$ between any two vertices of $S = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$. This will allow us to show the following.

Claim 8. $\phi(v_j) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\} = S$

Proof. Suppose for a contradiction that $\phi(v_j)$ contains a vertex $v \notin S$. The hyperedge $\phi(v_j)$ contains at least two vertices from S, namely v_j and v_{j+1}. By property (2), $\phi(v_j)$ is not a cut-hyperedge of $\mathcal{H}$. So after deleting $\phi(v_j)$ from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{\phi(v_j)\}$ is still connected – so there is a (shortest) Berge path $\mathcal{Q}$ in $\mathcal{H} \setminus \{\phi(v_j)\}$ between v and a vertex $s \in S$ (note that the hyperedges of $\mathcal{Q}$ are not contained in S). The vertex s is different from either v_j or v_{j+1}, say $s \neq v_j$, without loss of generality. By Lemma 2, there is a Berge path $\mathcal{Q}'$ of length $r-1$ between s and v_j (consisting of the hyperedges contained in S). Then $\mathcal{Q}, \mathcal{Q}'$ and $\phi(v_j)$ form a Berge cycle of length at least $r+1$ in $\mathcal{H}$, a contradiction. □

Claim 6 and Claim 8 together show that there are at least r hyperedges of $\mathcal{H}$ contained in S. If all $r+1$ subsets of S of size r are hyperedges of $\mathcal{H}$, then S induces K_{r+1}^r and it is easy to show that it contains a Berge cycle of length $r+1$, a contradiction. This means S contains exactly r hyperedges of $\mathcal{H}$. Then by Claim 7, we know that S induces a block of $\partial_2(\mathcal{H})$.

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 7 shows that one of these blocks, say D_1, is induced by S. Let us contract the vertices of S to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length $r+1$ or longer, as well; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$ and $e(\mathcal{H}') = e(\mathcal{H}) - r$. By induction, we have $e(\mathcal{H}') \leq |V(\mathcal{H}')| - 1$. Therefore,

$$e(\mathcal{H}) = e(\mathcal{H}') + r \leq (|V(\mathcal{H}')| - 1) + r = (|V(\mathcal{H})| - r - 1) + r = |V(\mathcal{H})| - 1.$$

If $e(\mathcal{H}) = |V(\mathcal{H})| - 1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')| - 1$ and S must contain exactly r hyperedges. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block D_i (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i]$ contains exactly r hyperedges. This means that for every block D of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D]$ contains exactly r hyperedges, completing the proof in the case $r \geq 4$.

The case $r = 3$

Recall that using Claim 6 we can find a set S of size 4 which contains 2 hyperedges of $\mathcal{H}$. Let $S = \{x, y, a, b\}$ and the two hyperedges be xab and yab. By property (2), xab is not a cut-hyperedge of $\mathcal{H}$. So after deleting xab from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{xab\}$ is still connected – so there is a (shortest) Berge path $\mathcal{Q}$ between x and $\{y, a, b\}$. If $\mathcal{Q}$ is of length at least 2, then it is easy to see that $\mathcal{Q}$ together with yab and xab form a Berge cycle of length at least 4, a contradiction. So $\mathcal{Q}$ consists of only one hyperedge, say h.

Our goal is to find a set of vertices which induces a block of $\partial_2(\mathcal{H})$, so that we can apply induction.

If $|h \cap \{y, a, b\}| = 2$ then h, xab, yab are 3 hyperedges of $\mathcal{H}$ contained in S, so by Claim 7, we can conclude that S induces a block of $\partial_2(\mathcal{H})$. (Notice that S contains exactly $|S|-1 = 3$ hyperedges of $\mathcal{H}$, otherwise it is easy to find a Berge cycle of length 4; this will be useful later.) So we can suppose $|h \cap \{y, a, b\}| = 1$. We consider two cases depending on whether h is either xat or xbt, or whether h is xyt for some $t \notin S$.

Case 1. First suppose without loss of generality that $h = xat$ for some $t \notin S$. Consider the set $\mathcal{D}$ of all hyperedges of $\mathcal{H}$ containing the pairs xa, ab or xb and let D be the set of vertices spanned by them. For each pair of vertices $i, j \in \{x, a, b\}$, let $V_{ij} = \{v \mid ijv \in \mathcal{H}\} \setminus \{x, a, b\}$. We claim that the sets V_{xa}, V_{ab}, V_{xb} are pairwise disjoint. Suppose for the sake of a contradiction that $t' \in V_{xa} \cap V_{ab}$. Then the hyperedges xat', abt', xab are contained in a set of 4 vertices $\{x, a, b, t'\}$. Thus by Claim 7, this set induces a block of $\partial_2(\mathcal{H})$ and we are done (we found the desired block!). Thus we can suppose $V_{xa} \cap V_{ab} = \emptyset$. Similarly $V_{ab} \cap V_{xb} = \emptyset$ and $V_{xa} \cap V_{xb} = \emptyset$. This shows that $|D| = 3 + |V_{xa}| + |V_{xb}| + |V_{ab}|$. On the other hand, $\mathcal{D}$ consists of $1 + |V_{xa}| + |V_{xb}| + |V_{ab}|$ hyperedges, so $|\mathcal{D}| = |D| - 2$.

We will now show that D induces a block of $\partial_2(\mathcal{H})$. Let D' be a (maximal) block of $\partial_2(\mathcal{H})$ containing D and suppose for the sake of a contradiction that it contains a vertex $p \notin D$. Then since D' is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between p and D, which are vertex-disjoint besides p. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v. If $uv \notin \{xa, ab, xb\}$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 between u and v consisting of the hyperedges of $\mathcal{D}$. Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand if $uv \in \{xa, ab, xb\}$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$ but then puv should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair uv); moreover, it is easy to check that between u and v there is a Berge path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, D' contains no vertex outside D; so D

induces a block of $\partial_2(\mathcal{H})$ (which contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Case 2. Finally suppose $h = xyt$ for some $t \notin S$. Let $\mathcal{D}$ be the set of all hyperedges of $\mathcal{H}$ containing the pair xy plus the hyperedges xab and yab, and let D be the set of vertices spanned by the hyperedges of $\mathcal{D}$. Let $V_{xy} = \{v \mid xyv \in \mathcal{H}\}$. We claim that $a \notin V_{xy}$ and $b \notin V_{xy}$. Indeed suppose for the sake of a contradiction that $a \in V_{xy}$. Then the hyperedges xab, yab, xya are contained in a set of 4 vertices $\{x, y, a, b\}$. So by Claim 7, this set induces a block of $\partial_2(\mathcal{H})$, and we are done. So $a \notin V_{xy}$. Similarly, we can conclude $b \notin V_{xy}$. Therefore, $|D| = |V_{xy}| + 4$. On the other hand, $|\mathcal{D}| = |V_{xy}| + 2$, so $|\mathcal{D}| = |D| - 2$.

We claim that D induces a block of $\partial_2(\mathcal{H})$. The proof is very similar to that of **Case 1**, we still give it for completeness. Let D' be a (maximal) block of $\partial_2(\mathcal{H})$ containing D and suppose for the sake of a contradiction that it contains a vertex $p \notin D$. Then since D' is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between p and D, which are vertex-disjoint besides p. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v.

If $uv \neq xy$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 or 4 between u and v consisting of the hyperedges of $\mathcal{D}$. (Indeed if $u, v \in V_{xy}$, then $\mathcal{P}'$ is of length 4, otherwise it is of length 3.) Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand if $uv = xy$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$ but then puv should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair uv); moreover, it is easy to check that between u and v there is a Berge path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, D' contains no vertex outside D; so D induces a block of $\partial_2(\mathcal{H})$ (and contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. In **Case 1** and **Case 2** we showed that one of these blocks, (say) $D_1 = D$ is such that $\mathcal{H}[D_1]$ contains $|D_1|-2$ hyperedges of $\mathcal{H}$, otherwise, D_1 is a set of 4 vertices such that $\mathcal{H}[D_1]$ contains exactly $|D_1| - 1 = 3$ hyperedges of $\mathcal{H}$. In all these cases, note that $e(\mathcal{H}[D_1]) \leq |D_1| - 1$.

Let us contract the vertices of D_1 to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length 4 or longer, as well; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - |D_1| + 1$ and $e(\mathcal{H}') = e(\mathcal{H}) - e(\mathcal{H}[D_1])$. By induction, we have $e(\mathcal{H}') \leq |V(\mathcal{H}')| - 1$. Therefore,

$$e(\mathcal{H}) = e(\mathcal{H}') + e(\mathcal{H}[D_1]) \leq |V(\mathcal{H}')| - 1 + |D_1| - 1 = (|V(\mathcal{H})| - |D_1| + 1) - 1 + |D_1| - 1 = |V(\mathcal{H})| - 1.$$

If $e(\mathcal{H}) = |V(\mathcal{H})| - 1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')| - 1$ and $\mathcal{H}[D_1]$ must contain exactly $|D_1| - 1$ hyperedges. As noted before, this is only possible if D_1 has 4 vertices and induces exactly 3 hyperedges of $\mathcal{H}$. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block D_i (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_4$ and $\mathcal{H}'[D_i]$ contains exactly 3 hyperedges. This means for every block D of $\partial_2(\mathcal{H})$, we have $D = K_4$ and $\mathcal{H}[D]$ contains exactly 3 hyperedges of $\mathcal{H}$, completing the proof in the case $r = 3$.

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