CRITICAL PERCOLATION ON CERTAIN NON-UNIMODULAR GRAPHS

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Abstract. An important conjecture in percolation theory is that almost surely no infinite cluster exists in critical percolation on any transitive graph for which the critical probability is less than 1. Earlier work has established this for the amenable cases $\mathbb{Z}^2$ and $\mathbb{Z}^d$ for large d, as well as for all non-amenable graphs with unimodular automorphism groups. We show that the conjecture holds for the basic classes of non-amenable graphs with non-unimodular automorphism groups: for decorated trees and the non-unimodular Diestel-Leader graphs. We also show that the connection probability between two vertices decays exponentially in their distance. Finally, we prove that critical percolation on the positive part of the lamplighter group has no infinite clusters.

1. Introduction and Preliminaries

1.1. **Introduction.** We will focus on the following general conjecture of Benjamini and Schramm [BS96] on critical percolation (see Subsection 1.2 for definitions):

Conjecture 1.1. *Let G be a transitive graph. If $p_c < 1$, then almost surely critical percolation on G has no infinite clusters.*

Earlier work of Harris [Har60] and Kesten [Kes80] established that for the graph $\mathbb{Z}^2$ critical percolation almost surely has no infinite cluster at $p = p_c$. Later, Hara and Slade [HS94] established the same for $\mathbb{Z}^d$, when $d \geq 19$. However, Conjecture 1.1 remains open for $\mathbb{Z}^d$ where $3 \leq d \leq 18$, along with many other amenable graphs.

For regular trees, the conjecture is just the classical result that a critical Galton-Watson tree dies out. Wu [Wu93] showed the conjecture for products of regular trees and $\mathbb{Z}$, when the degree of the regular tree was large enough.

Benjamini, Lyons, Peres and Schramm, see [BLPS99a] or [BLPS99b], proved the conjecture for non-amenable graphs when the automorphism group of the graph is unimodular. However, their "Mass Transport Principle" does not adapt to the case when the action is not unimodular, leaving that case open, as well.

In this paper, we prove the conjecture for the best-known examples of transitive graphs with non-unimodular automorphism groups.

Section 2 deals with trees "decorated" by adding edges, where this decoration retains transitivity. A typical example is the "grandparent tree" (see Figure 2), due to Trofimov [Tro85], and also appearing in Soardi and Woess [SW90].

Section 3 proves the conjecture for a class of graphs due to Diestel and Leader [DL01], see Figure 3. Such a graph $\Gamma_{\alpha,\beta}$ is the "horocyclic product" of an $(\alpha+1)$-regular tree T_α and a $(\beta+1)$-regular tree T_β; see Subsection 3.1 for a formal definition.

When $\alpha \neq \beta$, the graph $\Gamma_{\alpha,\beta}$ is non-unimodular. Theorem 3.1 proves Conjecture 1.1 for these graphs $\Gamma_{\alpha,\beta}$.

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If $\alpha = \beta$, the graph $\Gamma_{\alpha,\beta}$ turns out to be a Cayley graph of the "lamplighter group" of Kaĭmanovich and Vershik (Example 6.1 of [KV83]). As such, it is unimodular, moreover, it is amenable, which means that we are unable to prove the conjecture in this case. However, in Section 4 we show that critical percolation on the positive part of this graph, and also of another natural Cayley graph of the lamplighter group, has no infinite components. This is analogous to the case of half-space percolation in $\mathbb{Z}^d$, see [BGN91].

We also show that, in critical percolation on any of our non-unimodular graphs, the connection probability between two vertices decays exponentially in their distance. The importance of such an exponential decay is discussed in [BS99]; for example, it might help in proving the existence of the non-uniqueness phase. For the Diestel-Leader graphs $\Gamma_{\alpha,\beta}$, the standard methods give the existence of this phase only when β is sufficiently large compared to α (or vice versa), see at the end of Section 3.

Note that the method of [AL91], see also [BLPS99b, Corollary 5.5], shows that if the automorphism group of a transitive graph is non-amenable (which is stronger than the non-amenability of the graph), then there can not be a unique infinite cluster in critical percolation. However, our examples have amenable automorphism groups, hence our proofs have to rule out the possibility of a unique infinite cluster, as well, which is non-trivial in the case of the Diestel-Leader graphs. A very recent preprint of Ádám Timár [Tim05], together with an unpublished result of Lyons, Peres and Schramm, show that there cannot exist infinitely many infinite clusters in critical percolation on any non-unimodular transitive graph.

1.2. **Background: amenability, unimodularity, and percolation.** Let $G = (V, E)$ be a locally finite infinite graph. Denote by $\operatorname{Aut} G$ its group of automorphisms, i.e. the group of bijective maps $g : V(G) \longrightarrow V(G)$ such that $\{u, v\} \in E(G)$ iff $\{gu, gv\} \in E(G)$. G is called *transitive* if any pair of vertices of G has an automorphism that maps the first vertex to the second one.

If we equip $\Gamma = \operatorname{Aut} G$ with the topology of pointwise convergence on G, then it becomes a locally compact topological group. Therefore it has both left- and right-invariant Haar measures, and we can consider the Banach space $L^\infty(\Gamma)$ of measurable essentially bounded real valued functions on Γ w.r.t. the left-invariant Haar measure. A linear functional $m : L^\infty(\Gamma) \longrightarrow \mathbb{R}$ is called an *invariant mean* if it maps nonnegative functions to nonnegative reals, the constant $\mathbf{1}$ function to 1, and $m(L_g\phi) = m(\phi)$ for any $g \in \Gamma$ and $\phi \in L^\infty(\Gamma)$, where $L_g(\phi)(h) := \phi(gh)$.

Definition 1.2.

- *The* edge-isoperimetric constant *of a graph G is*
$$\iota_E(G) := \inf\left\{\frac{|\partial_E K|}{|K|} : K \subset V(G), |K| < \infty\right\},$$
where $\partial_E K := \{\{u, v\} \in E(G) : u \in K, v \notin K\}$. *$G$ is* amenable *if* $\iota_E(G) = 0$.
- *A locally compact topological group Γ is* amenable *if it has an invariant mean. If Γ is finitely generated, then this is equivalent to saying that it has an amenable Cayley graph, see* [Pat88].
- *A locally compact topological group is called* unimodular *if its left- and right-invariant Haar-measures coincide.*

Schlichting [Sch79] and Trofimov [Tro85] give a combinatorial characterization of unimodularity, which is made explicit by Soardi and Woess [SW90] for the action of a group of graph automorphisms on the graph. According to this characterization, the action of a group of automorphisms Γ on a graph G is unimodular if and only if for any pair x, y of vertices, $|\mathrm{Stab}(x) \cdot y| = |\mathrm{Stab}(y) \cdot x|$, where $\mathrm{Stab}(x) = \{g \in \Gamma : gx = x\}$ is the *stabilizer* of x. We say that a transitive graph G is *unimodular* if the action of the full group $\operatorname{Aut} G$ is unimodular.

There are basic connections between non-amenability of a graph and the non-amenability of its automorphism group. A useful lemma, see e.g. Lemma 3.3 of [BLPS99b], allows us to take invariant means on *any* appropriate graph, instead of on the group itself. If G is a countable graph, and Γ is a closed subgroup of $\operatorname{Aut} G$, then Γ acts on the Banach space $L^\infty(V(G))$ of real valued bounded functions by $L_g(\phi)(v) := \phi(gv)$, and we can define a Γ-invariant mean on G analogously to how we did above.

Lemma 1.3 (Characterization of group-amenability)**.** *Let G be a graph and Γ be a closed subgroup of* $\operatorname{Aut} G$*. Then Γ is amenable if and only if G has a Γ-invariant mean.* □

There is also a characterization of graph amenability in terms of the amenability of closed transitive groups of automorphisms, due to Soardi and Woess [SW90]. See also Theorem 3.4 of [BLPS99b].

Lemma 1.4 (Corollary 1 of [SW90])**.** *Let G be a graph and Γ a closed transitive subgroup of* $\operatorname{Aut} G$*. Then G is amenable if and only if Γ is amenable and its action is unimodular.* □

Given a graph G and $0 \le p \le 1$, *percolation* on G is a measure $\mathbb{P}_p\{\cdot\}$ on subsets $\mathcal{E} \subseteq E(G)$, where the events $\{e \in \mathcal{E}\}$, $e \in E(G)$, are all independent and occur with probability p. Edges $e \in \mathcal{E}$ are called *open*, and edges $e \in \mathcal{E}^c$ *closed*; paths shall be called *open* if all edges are open. The *cluster* of a vertex $o \in V(G)$ is

$$C(o) = \{v : o \longleftrightarrow v \text{ by an open path}\}.$$

By Kolmogorov's 0-1 law, for any value of p an infinite cluster exists with probability 0 or 1. So, define the *critical probability* p_c for percolation by

$$p_c = \inf\{p : \mathbb{P}_p\{\exists \infty \text{ cluster}\} = 1\}.$$

When the value of p is clear from the context, and especially when $p = p_c$, we write $\mathbb{P}\{\cdot\}$ for $\mathbb{P}_p\{\cdot\}$.

For non-amenable graphs with bounded degree it is known [BS96] that $0 < p_c < 1$, hence Conjecture 1.1 poses a real question in this case.

For non-amenable transitive graphs there is a second critical value of interest,

$$p_u = \inf\{p : \mathbb{P}_p\{\exists \text{ a unique } \infty \text{ cluster}\} = 1\}.$$

Another famous conjecture of [BS96] is the strict inequality $p_c < p_u$ for these graphs. The standard references for percolation are [Gri99] and [LP05].

1.3. **The general strategy.** The main steps of the proof are shared by all the examples we deal with. First, we shall use the tree structure underlying the graph to construct a Galton-Watson process and bound the expected number of vertices at level k that can be reached from a fixed vertex o at level 0 via certain restricted paths that stay in the "downwards half-graph" from o. Then a Fatou lemma argument will imply that the component is a.s. finite in this downwards half-graph. Moreover, as the combinatorial characterization of non-unimodularity suggests, the component of a vertex has more ways to grow "downwards" than "upwards", so the component cannot directly reach infinitely far upwards, either. In a decorated tree there is no "sideways" direction, so it follows easily that the entire component must be finite. For the Diestel-Leader graphs the specific combinatorial structure helps in showing that the "exponentially unlikely" upward growth makes it impossible that there is a cluster oscillating infinitely up and down.

2. Decorated trees

2.1. **Definition and examples.** Let T be a $d+1$-regular tree. T is a transitive non-amenable graph, $\operatorname{Aut} T$ is non-amenable, and its action on T is unimodular. We shall examine a class of non-amenable transitive graphs G derived from T by adding edges to it for which $\operatorname{Aut} G$ will be amenable (and therefore, by Lemma 1.4, will act on G in a non-unimodular manner).

Two rays (half-infinite simple paths) in T are called equivalent if they differ only in finitely many edges. An *end* of the tree is an equivalence class of rays. Pick an end ξ of T and direct all edges of T towards ξ. If there is an edge from v to u, we say that v is the *child* of u, and u is its *parent*. We shall use the terms *sibling*, *grandchild* and *grandparent* in their obvious meaning. We say that v is a *descendant* of u and that u is an *ancestor* of v if there is a directed path from v to u. The *downwards subtree* S_v of a vertex v is the graph on the vertices descended from v. Distinguishing some vertex $o \in T$, we may define a *level function* $\ell : V(T) \to \mathbb{Z}$ by $\ell(o) = 0$ and $\ell(v) = \ell(u) + 1$ whenever v is a child of u. Note that large values of this level function mean large depths in T, while negative values correspond to being higher than o. When considering the cluster of a given vertex o, we shall frequently make use of the *level sets* (relative to o), defined for $k \in \mathbb{Z}$ by $L_k := \{v : \ell(v) - \ell(o) = k\}$. For example, the visually clear expression that a path $v_1, \dots, v_n$ *does not go above level* L_k can be written as $\ell(v_i) \geq k$ for all $0 \leq i \leq n$.

Let $K = \{\alpha \in \operatorname{Aut} T : \alpha\xi = \xi\}$ be the group of ξ-preserving automorphisms of T. Then K is an *amenable* group (any Banach limit on ξ is a K-invariant mean, which suffices by Lemma 1.3), which acts on T transitively.

Now let L be some subgroup of K (possibly K itself) which acts transitively on T. Any locally finite graph $G = (V(T), E(T) \cup E')$ with $L \cdot E' = E'$ will be called a *decorated tree* (or *L-decorated tree*). The graph G itself is always non-amenable, since it results from the non-amenable graph T by adding edges. Considering the action of L on the vertices of G, we may regard it as a subgroup of $\operatorname{Aut} G$; however, $\operatorname{Aut} G$ might still be non-amenable.

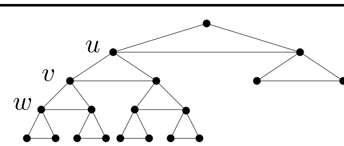

Figure 1. "Triangles" tree.

Example 2.1 ($\operatorname{Aut} G$ non-amenable, unimodular action)**.** *Take $d = 2$ and $L = K$, and let $E' = \{\{u, v\} : u,v$ are siblings$\}$, see Figure 1. Then $\operatorname{Aut} G$ is non-amenable, and its action is unimodular.*

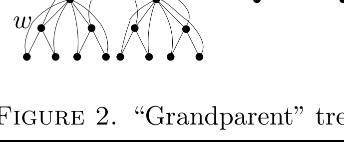

Figure 2. "Grandparent" tree.

Example 2.2 (Aut G amenable, non-unimodular action)**.** *Take $L = K$, and $E' = \{\{u, w\} : u$ is the grandparent of $w\}$, see Figure 2. The action of* Aut G *on G is not unimodular, and it is an amenable group.*

For the remainder of this section, we shall fix some graph G which is a decorated tree, and prove that critical percolation on G almost surely has no infinite components. While all the results hold for Aut G with unimodular action, this case is covered by [BLPS99b]; the result is new only for Aut G with non-unimodular action.

Theorem 2.3. *Let G be a decorated tree. Then critical percolation on G a.s. has no infinite components.*

Note 2.4. It is easy to check directly, and also follows from our proof of Theorem 2.3 below, that for any $p < 1$, percolation on a decorated tree satisfies $\mathbb{P}_p\{x \longleftrightarrow y\} \leq e^{-c\,\mathrm{dist}(x,y)}$ for some $c = c_p > 0$. The same result at $p = p_c$ for the Diestel-Leader graphs is not that easy, and will be proven in Section 3.

We will follow the strategy outlined in Subsection 1.3.

2.2. **Bounding branches.**

Definition 2.5. *Consider percolation on a decorated tree G. The* forward cluster $C^+(o)$ *of a vertex $o \in V(G)$ is defined by*

$$\begin{aligned} C^+(o) := \{v : v \longleftrightarrow o \textit{ by an open path } (o = v_0, v_1, \ldots, v_n = v) \\ \textit{inside the downwards subtree } S_o, \\ \textit{with } \ell(v_i) \leq \ell(v) \textit{ for all } 0 \leq i < n\}. \end{aligned}$$

Note that $C^+(o)$ is not necessarily connected. We start by showing that $C^+(o)$ is "narrow", in the sense that it contains few branches.

Lemma 2.6. *Consider critical percolation on a decorated tree G at $p_c(G)$. Let $o \in V(G)$ be a vertex, and define $V_k^+ := C^+(o) \cap L_k$. Then $e_k := \mathbb{E}\big|V_k^+\big| \leq 1$ for all $k \geq 0$.*

Proof. Suppose to the contrary that $e_{k_0} > 1$ for some $k_0 \geq 0$. We shall use this to find subsets of the vertices of $\{V^+_{j \cdot k_0}\}_{j=0}^{\infty}$ which will form a supercritical Galton-Watson process:

- The root of the process shall be the vertex $Y_{0,1} = \{o\}$.
- If at level $j-1$ of the process we picked vertices $v_{j-1,1}, \ldots, v_{j-1,N_{j-1}} \in V^+_{(j-1)k_0}$, we shall pick at level j as descendants of each $v_{j-1,i}$ the vertices $Y_{j,i} = C^+(v_{j-1,i}) \cap L_{jk_0}$.

Due to the construction, for any fixed j, if we condition on the previous generation $\{Y_{j-1,i} : i = 1, \ldots, N_{j-1}\}$, then the sets $Y_{j,i}$ are independent. Also, $|Y_{j,i}|$ has the same distribution as $|Y_{1,1}|$, so this is indeed a Galton-Watson process.

Since $Y_{1,1} = V^+_{k_0}$, this is a supercritical process. But e_{k_0} is a polynomial in p, and in particular is continuous. Thus, we may decrease p below p_c keeping $e_{k_0}(p) > 1$. This would give a positive probability for

$$|C(o)| \geq \sum_{j=0}^{\infty} \left|V^+_{j \cdot k_0}\right| = \infty,$$

contradicting criticality at p_c. □

2.3. **Clusters are finite.** Define r as the maximal length of a path in T connecting the two endpoints of an edge of G. Since G is locally finite and transitive, r is well-defined and finite. Furthermore, let D be the common degree of the vertices of G.

Lemma 2.7. *In critical percolation on a decorated tree G, for any $o \in V(G)$, the forward cluster $C^+(o)$ is a.s. finite.*

Proof. Consider the band of levels $H_k := \cup_{j=k}^{k+r} L_j$. Recall the random variables $|V_k^+|$ from Lemma 2.6. For a percolation configuration ω, let E_k be the set of edges in open paths leading from o to L_k, staying in S_o and not going below L_k. Define $F_k := \cup_{j=k}^{k+r} E_j$ and $W_k^+ := \cup_{j=k}^{k+r} V_j^+$. Then Lemma 2.6 and Fatou's lemma give us that

$$\mathbb{E}\big\{\liminf_{k\to\infty} |W_k^+|\big\} \leq \liminf_{k\to\infty} \mathbb{E}|W_k^+| \leq r+1.$$

Thus the random variable $\liminf_{k\to\infty} |W_k^+|$ is almost surely finite.

The event $\mathcal{A}_k(n) := \{|W_k^+| = n\}$ is clearly determined by F_k. Furthermore, given F_k and $\mathcal{A}_k(n)$ with some $n \geq 1$, the probability of the event $\mathcal{G}_k := \{$all edges incident to W_k^+ and not in F_k are closed$\}$ is at least $(1-p_c)^{Dn} > 0$. This means that if $\mathcal{A}_k(n)$ happens for infinitely many k values, then there is almost surely a K such that $\mathcal{G}_K$ occurs. But note that if $u \in V_m^+$ for some $m > k + r$, then any open simple path from o to u that shows this, when it first enters H_k, goes through some vertex $v \in W_k^+$, and it leaves W_k^+ the last time through an edge not in F_k. Hence, $\mathcal{G}_K$ implies that $V_m^+ = \emptyset$ for all $m > K + r$, which means that $\mathcal{A}_k(n)$ could not happen infinitely often.

Therefore, we must have $|W_k^+| = 0$ infinitely often a.s. But the above argument also shows that $W_k^+ = \emptyset$ implies $V_m^+ = \emptyset$ for all $m > k + r$, hence $|C^+(o)| < \infty$ a.s. □

In the case of a decorated tree it is particularly easy to use the tree-like structure of G to show that clusters cannot extend infinitely far "up" or "sideways".

Lemma 2.8. *In independent p-percolation with any $p < 1$, the cluster $C(o)$ is a.s. contained in some downwards subtree.*

Proof. Call the subtree S_v of any vertex v *isolated* if no open edges remain connecting S_v with $V(G) \setminus S_v$; define the events $I_v = \{S_v$ is isolated$\}$.

Recall the bound r on the "maximal length in T" of an edge of G. It follows that I_v depends only on a constant finite number of edges. Consider now the events $I_{v_1}, I_{v_2}, \ldots$ for some vertices $v_1, v_2, \ldots$ on the path upwards from o, which are sufficiently far apart so that these events are all independent. Then a.s. one (in fact, infinitely many) of the I_{v_i} will occur, and $C(o)$ is contained in the downwards subtree of this v_i. The probability that the distance of v_i from o is larger than t decays exponentially in t. □

Proof of Theorem 2.3. Almost surely, the conclusions of Lemmas 2.7 and 2.8 hold for all vertices of G. Similarly, it is enough to show that $C(o)$ is finite a.s.

Assume that $C(o)$ is infinite. Let v be a vertex such that $C(o) \subseteq S_v$. There are finitely many (no more than $(d^{\ell(o)-\ell(v)+1} - 1)/(d-1)$) vertices u_i in S_v such that $\ell(u_i) \leq \ell(o)$, and the downwards cluster $C^+(u_i)$ of each such vertex is finite a.s. On the other hand, $C(o)$ can be infinite only if o is connected to vertices w on arbitrarily deep levels in S_v. If we consider an open path from o to such a w, then the first vertex on this path which is on the level of w is actually an element of $C^+(u_i)$ for one of the vertices u_i. But this is impossible if w is located deep enough, hence $C(o)$ must be finite. □

3. Diestel-Leader graphs

3.1. **Definition.** Diestel and Leader [DL01] give the following example of a graph with non-unimodular automorphism group. They conjecture that this transitive graph is not quasi-isometric to any Cayley graph.

Fix integers $\alpha, \beta \geq 2$. Let T_α and T_β be an $(\alpha+1)$-regular and a $(\beta+1)$-regular tree, respectively. Choose an end of T_α and an end of T_β, and orient the edges of each tree towards its distinguished end. Now construct the graph $\Gamma'_{\alpha,\beta} = (V'_{\alpha,\beta}, E'_{\alpha,\beta})$ with vertices $V'_{\alpha,\beta} = V(T_\alpha) \times V(T_\beta)$ and edges

$$E'_{\alpha,\beta} = \Big\{ \{(u_1,u_2),(v_1,v_2)\} \,\Big|\, \{u_1,v_1\} \in E(T_\alpha) \text{ and } \{u_2,v_2\} \in E(T_\beta) \\ \text{are oriented in opposite directions.} \Big\}.$$

Note that if $\ell_1 : T_\alpha \to \mathbb{Z}$ and $\ell_2 : T_\beta \to \mathbb{Z}$ are level functions for T_α and T_β respectively, then $\ell_1(u_1)+\ell_2(u_2) = \ell_1(v_1)+\ell_2(v_2)$ for any edge $\{(u_1,u_2),(v_1,v_2)\}$ of $\Gamma'_{\alpha,\beta}$. Thus, $\Gamma'_{\alpha,\beta}$ has infinitely many connected components, all isomorphic. Define $\Gamma_{\alpha,\beta}$ to be one such connected component.

Figure 3 illustrates a portion of $\Gamma_{\alpha,\beta}$ when $\alpha = 3$ and $\beta = 2$, along with a path in it.

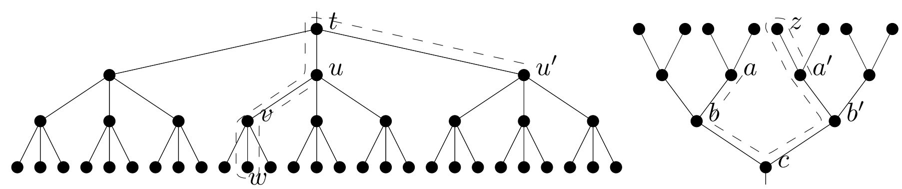

FIGURE 3. The Diestel-Leader graph $\Gamma_{3,2}$

Nodes in $\Gamma_{\alpha,\beta}$ are pairs of vertices at *the same* level; edges must follow both trees' edges.
Sample path: $(u,a),(v,b),(w,c),(v,b'),(u,a'),(t,z),(u',a')$.

We also define the two projections onto the first and second components of $V_{\alpha,\beta}$, labelled $\pi_1 : \Gamma_{\alpha,\beta} \to T_\alpha$ and $\pi_2 : \Gamma_{\alpha,\beta} \to T_\beta$, and note that if $\{x,y\}$ is an edge of $\Gamma_{\alpha,\beta}$, then $\{\pi_1(x),\pi_1(y)\}$ and $\{\pi_2(x),\pi_2(y)\}$ are edges of T_α and T_β respectively. Also, define a level function $\ell := \ell_1 \circ \pi_1$. The level sets L_k for $k \in \mathbb{Z}$ are defined relative to this function ℓ. We shall refer to an edge in $\Gamma_{\alpha,\beta}$ from x to y as going *up* if $\ell(y) = \ell(x) - 1$, and going *down* if $\ell(y) = \ell(x) + 1$.

Here are some standard facts concerning $\Gamma_{\alpha,\beta}$, which appear in [DL01] and [Woe00].

- $\Gamma_{\alpha,\beta}$ is clearly a transitive graph.
- $\operatorname{Aut}\Gamma_{\alpha,\beta}$ is unimodular iff $\alpha = \beta$, as the combinatorial characterization is easily checked.
- $\operatorname{Aut}\Gamma_{\alpha,\beta}$ is always amenable. This is because $\operatorname{Aut}\Gamma_{\alpha,\beta}$ is a subgroup of the direct product of the groups of those automorphims of T_α and T_β that preserve the distinguished end. As we have seen, these two groups are amenable, and group amenability is preserved by direct sums and by going to a subgroup.
- $\Gamma_{\alpha,\beta}$ is amenable iff $\alpha = \beta$. This follows from the facts above and Lemma 1.4.

We will prove the following:

Theorem 3.1. *If $\alpha > \beta$, then critical percolation on $\Gamma_{\alpha,\beta}$ almost surely has no infinite clusters. Furthermore, $\mathbb{P}_{p_c}\{x \longleftrightarrow y\} \leq C\rho^{\operatorname{dist}(x,y)}$ for any $\rho \in \left(\frac{\beta}{\alpha}, 1\right)$ and suitable $C = C(\rho) < \infty$.*

Many of the lemmas used in the proof are primarily combinatorial, and hold also when $\alpha = \beta$. So we shall not assume $\alpha \neq \beta$ unless it is explicitly stated in a lemma. The unimodular case $\alpha = \beta$ will be discussed in Section 4. The main help in the proofs will be that the geometry of the graph $\Gamma_{\alpha,\beta}$ has some similarities with that of a tree:

Note 3.2. *Let $x_0, x_1, \ldots$ be a path in $\Gamma_{\alpha,\beta}$, such that the edge from x_0 to x_1 goes down, and $\forall i \geq 0 : \ell(x_i) \geq \ell(x_0)$. Then the path $\pi_1(x_0), \pi_1(x_1), \ldots$ stays within the downwards subtree of $\pi_1(x_0)$.*

This motivates the following definitions:

Definition 3.3. *The* forward subcluster *of a vertex $o \in V(\Gamma_{\alpha,\beta})$ is the set*

$$C^+(o) = C^+(o,\omega) := \{v : o \longleftrightarrow v \text{ by an open path } (o = v_0, v_1, \ldots, v_n = v) \\ \text{such that } \ell(o) \leq \ell(v_i) \leq \ell(v) \text{ for all } i\}.$$

Furthermore, the downwards subcluster *of o is*

$$C'(o) = C'(o,\omega) := \{v : o \longleftrightarrow v \text{ by an open path } (o = v_0, v_1, \ldots, v_n = v) \\ \text{such that } \ell(o) \leq \ell(v_i) \text{ for all } i\}.$$

3.2. **Finiteness downwards.** As before, the first step of the proof is to bound the rate of growth of the forward part of the critical cluster, and to conclude that the cluster cannot directly go down infinitely deeply.

Lemma 3.4. *Let $o \in V(\Gamma_{\alpha,\beta})$ and consider critical percolation on $\Gamma_{\alpha,\beta}$. Define $V_k^+ = V_k^+(o) := \{x : x \in C^+(o), \ell(x) - \ell(o) = k\}$. Then the values $e_k = e_k(p_c) := \mathbb{E}\left|V_k^+\right|$ satisfy $e_k \leq 1$.*

Proof. We can copy the proof of Lemma 2.6. The only difference is that the independence of the number of offspring of any two vertices on the same level of the Galton-Watson process we are building is now provided by Note 3.2, as opposed to the earlier explicit restriction that the paths in $C^+(o)$ should stay inside the subtree S_o. □

Lemma 3.5. *In critical percolation on $\Gamma_{\alpha,\beta}$, we have that $\pi_1[C'(o)]$ is finite a.s.*

Proof. Exactly as in the proof of Lemma 2.7, we can use Fatou's lemma and the sequence of events $\mathcal{G}_k$ to conclude that there is a random integer K such that $V_k^+(o)$ is empty for all $k > K$. In other words, $\pi_1(C^+(o))$ is finite almost surely. Now note that, unlike $C^+(o)$, the set $C'(o)$ is necessarily connected. Hence, if $\pi_1(C'(o))$ was infinite, then for any $k > 0$, there would be a simple open path in $C'(o)$ between o and some vertex of L_k. The first time this path enters L_k, at vertex, say, $v \in L_k$, then $v \in C^+(o)$ would also hold. Since k was arbitrary, $\pi_1(C^+(o))$ would be infinite, too. □

The last result can be strengthened by the following simple lemma, which shows that the structure of $\Gamma_{\alpha,\beta}$ ensures that a connected set C with finite $\pi_1(C)$ cannot be infinite.

Lemma 3.6. *Let C be a connected component of $\Gamma_{\alpha,\beta}$ such that there exists some $k \in \mathbb{Z}$ with $\ell(q) \leq k$ for any $q \in C$. Then for all $u \in T_\alpha$, the projection π_1 maps only finitely many elements of C to u, and indeed $\left|\pi_1^{-1}[u] \cap C\right| \leq \beta^{k-\ell_1(u)}$.*

Proof. If $\ell_1(u) > k$, then $\pi_1^{-1}[u] \cap C = \emptyset$. Now suppose that this set is non-empty, with $k - \ell_1(u) = j \geq 0$, and take some $(u,a) \in \pi_1^{-1}[u] \cap C$, with $a \in T_\beta$. Consider the unique ancestor $b \in T_\beta$ of a which has $\ell_2(b) = \ell_2(a) - j$, and let $T \subset T_\beta$ be the infinite subtree of descendants of b. Denote the jth descendants of b by $a_1 = a, a_2, \ldots, a_{\beta^j} \in T$. Note that any open path $\gamma \subseteq C$ starting from (u,a), because of $\ell(q) \leq k$ for all $q \in \gamma$, satisfies $\pi_2(\gamma) \subseteq T$. Hence $\pi_1^{-1}[u] \cap C \subseteq \{(u,a_1), \ldots, (u,a_{\beta^j})\}$, and the claim follows. □

3.3. **Finiteness upwards.** Next we prove that almost surely no vertex can connect to vertices unboundedly "upwards" of it in the tree.

Lemma 3.7. *Consider critical percolation on $\Gamma_{\alpha,\beta}$, and fix a vertex o. For all $k \in \mathbb{N}$, define*

$$V_k^-(o) := \big\{x : \ell(o) - \ell(x) = k,\ o \in C^+(x)\big\}.$$

Then $\mathbb{E}\big|V_k^-\big| \le \left(\frac{\beta}{\alpha}\right)^k$.

Proof. There are β^k vertices in $\Gamma_{\alpha,\beta}$ that have a positive probability to appear in V_k^-, and all these probabilities are the same, also equaling to $q_i := \mathbb{P}_{p_c}\{\exists u \in C^+(o) \text{ with } \pi_1(u) = a_i\}$, where the a_i, $i = 1, \dots, \alpha^k$, are the k'th generation descendants of $\pi_1(o)$ in T_α. Now, rewriting e_k from Lemma 3.4 as

$$1 \ge e_k = \sum_{i=1}^{\alpha^k} q_i = \alpha^k q_1$$

gives $q_i \le \alpha^{-k}$, and the desired bound follows from the linearity of expectation. □

Lemma 3.8. *Suppose $\alpha > \beta$, and for all $k \in \mathbb{N}$, define*

$$U_k^-(o) = \{x : \ell(o) - \ell(x) = k,\ o \in C'(x)\}.$$

Then $a_k := \mathbb{E}\big|U_k^-\big| < \infty$.

Proof. First we prove that $a_0 \le \frac{\alpha}{\alpha-\beta} < \infty$, then that $a_{k+1} \le a_k(\beta/\alpha)a_0 < \infty$ for all $k \in \mathbb{N}$.

Consider a simple open path γ connecting o to x and showing $x \in U_0^-$. Let y be the *last* lowest vertex on the path. Write $j = \ell(y) - \ell(o)$. Then the portion of γ between y and x shows that $x \in V_j^-(y)$, with the definition of Lemma 3.7, while the portion of γ between y and o shows that $y \in C^+(o)$. Now, such an open path γ, going through these vertices o, y, x, though with y not being necessarily the *last* lowest vertex, exists if and only if both events $\{x \in V_j^-(y)\}$ and $\{y \in C^+(o)\}$ occur, due to disjoint sets of open edges; i.e. iff $\{x \in V_j^-(y)\} \,\square\, \{y \in C^+(o)\}$ happens, with the notation of the van den Berg – Kesten inequality, see [vdBK85] or [Gri99].

This BK inequality says that for increasing measurable events $\mathcal{A}$ and $\mathcal{B}$ in independent p-percolation, $\mathbb{P}_p\{\mathcal{A}\square\mathcal{B}\} \le \mathbb{P}_p\{\mathcal{A}\}\mathbb{P}_p\{\mathcal{B}\}$. Therefore, at p_c,

$$\begin{aligned} a_0 = \mathbb{E}|U_0^-| &\le \sum_{j=0}^{\infty} \sum_x \sum_y \mathbb{P}\{\{x \in V_j^-(y)\} \,\square\, \{y \in C^+(o)\}\} \\ &\le \sum_{j=0}^{\infty} \sum_x \sum_y \mathbb{P}\{x \in V_j^-(y)\}\mathbb{P}\{y \in C^+(o)\} \\ &\le \sum_{j=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^j \cdot 1 = \frac{\alpha}{\alpha - \beta} < \infty, \end{aligned}$$

where the sums are over $\{x : \ell(x) = \ell(o)\}$ and $\{y : \ell(y) = \ell(o) + j\}$, and we used Lemmas 3.7 and 3.4 to get the third line.

Now take a simple open path γ from o to x and showing $x \in U_{k+1}^-(o)$. Let z be the *first* vertex on this path that lies in $U_{k+1}^-(o)$, and let t be previous vertex on the path. Then the portion of γ between o and t shows that $t \in U_k^-(o)$, while the portion of γ between z and x shows that $x \in U_0^-(z)$. Similarly as above, such an open path γ, going through these vertices o, t, z, x, exists if and only if the three events $\{t \in U_k^-(o)\}$, $\{(t, z) \text{ is open}\}$

and $\{x \in U_0^-(z)\}$ occur on disjoint sets of open edges. Hence the BK inequality now gives

$$a_{k+1} \le a_k \cdot (\beta p_c) \cdot a_0.$$

Since T_α is a subgraph of $\Gamma_{\alpha,\beta}$, we have $p_c \le 1/\alpha$. Therefore, by induction, $a_k \le (\beta/\alpha)^k a_0^{k+1} \le \left(\frac{\beta}{\alpha-\beta}\right)^k \frac{\alpha}{\alpha-\beta} < \infty$. $\square$

Now plugging the finiteness of the a_k's into a similar, but more refined argument, we get for all $\alpha > \beta$ that $a_k \to 0$ exponentially, as $k \to \infty$.

Lemma 3.9. *Suppose $\alpha > \beta$. Define the event*

$$\mathcal{A}_k = \{o \textit{ is connected by an open path to a vertex } k \textit{ levels above it}\}.$$

Then, in critical percolation, $\lim_{k\to\infty} \mathbb{P}\{\mathcal{A}_k\} = 0$, decaying exponentially.

Proof. Note that $\mathcal{A}_k = \left\{U_k^- \neq \emptyset\right\}$. Thus, by Markov's inequality, it is enough to show that for $a_k := \mathbb{E}\left|U_k^-\right|$ we have $\lim_{k\to\infty} a_k = 0$ exponentially quickly.

Consider a simple open path γ connecting o to x and showing $x \in U_k^-$. Let y be the *last* lowest vertex on the path. Write $j = \ell(y) - \ell(o)$. Then the portion of γ between y and x shows that $x \in V_{j+k}^-(y)$, with the definition of Lemma 3.7. Now let z be the *last* highest vertex on the portion of γ between o and y, and write $i = \ell(o) - \ell(z)$. Clearly, $0 \le i \le k$, and $z \in U_i^-(o)$. The path γ also shows that $y \in C^+(z)$.

The existence of such an open path γ, going through these vertices x, y, z, is equivalent to the occurrence of the three events $\{x \in V_{j+k}^-(y)\}$, $\{y \in C^+(z)\}$ and $\{z \in U_i^-(o)\}$ on disjoint edge sets. Hence the BK inequality gives that, at p_c,

$$\begin{aligned}
\mathbb{E}|U_k^-| &\le \sum_{i=0}^{k}\sum_{j=0}^{\infty}\sum_x\sum_y\sum_z \mathbb{P}\{\{x \in V_{j+k}^-(y)\} \,\square\, \{y \in C^+(z)\} \,\square\, \{z \in U_i^-(o)\}\} \\
&\le \sum_{i=0}^{k}\sum_{j=0}^{\infty}\sum_x\sum_y\sum_z \mathbb{P}\{x \in V_{j+k}^-(y)\}\mathbb{P}\{y \in C^+(z)\}\mathbb{P}\{z \in U_i^-(o)\} \\
&\le \sum_{i=0}^{k}\sum_{j=0}^{\infty} a_i \cdot 1 \cdot \left(\frac{\beta}{\alpha}\right)^{j+k} = \frac{\alpha}{\alpha-\beta}\left(\frac{\beta}{\alpha}\right)^k \sum_{i=0}^{k} a_i,
\end{aligned}$$

where the sums are over $\{x : \ell(x) = \ell(o) - k\}$, $\{y : \ell(y) = \ell(o) + j\}$, and $\{z : \ell(z) = \ell(o) - i\}$, and we used the definition of a_i and Lemmas 3.7 and 3.4 to get the third line.

The finiteness of the a_i's is known from Lemma 3.8. Now suppose $\limsup_{i\to\infty} a_i \ge \delta > 0$. Because of the exponential decay of the factor $\left(\frac{\beta}{\alpha}\right)^k$ in the previous inequality, this can happen only if $\limsup_{i\to\infty} a_i = \infty$. But then there are infinitely many indices m for which $a_m = \max\{a_0, a_1, \dots, a_m\}$, and for such an m our inequality implies $a_m \le \frac{\alpha}{\alpha-\beta}(m+1)a_m\left(\frac{\beta}{\alpha}\right)^m$. But this is impossible if m is large enough. Hence $\lim_{k\to\infty} a_k = 0$. Moreover, convergence implies boundedness, $a_k \le A$ for some A, hence we actually have $a_k \le \frac{\alpha}{\alpha-\beta}(k+1)A\left(\frac{\beta}{\alpha}\right)^k$, which is less than $B\rho^k$ for $\frac{\beta}{\alpha} < \rho < 1$ and $B > 0$ large enough. $\square$

Iterating further our argument gives the following:

Lemma 3.10. *Suppose $\alpha > \beta$, and for all $k \in \mathbb{N}$, define*

$$W_k^-(o) := L_{-k} \cap C(o).$$

Then, in critical percolation, $\mathbb{E}\left|W_k^-(o)\right| < \infty$, and they decay exponentially in k.

Proof. Consider an open path γ from o to $x \in W_k^-$. Let w_1 be the last highest vertex on γ, and v_1 be the last lowest vertex on the portion of γ from w_1 to x. Then, for $t \geq 2$, let w_t be the last highest vertex on the portion of γ from v_{t-1} to x, and v_t be the last lowest vertex on the portion of γ from w_t to x. We make these definitions for all $t \geq 1$, but there will certainly be a smallest $T \geq 1$ such that $w_t = v_t = x$ for all $t \geq T$. Writing $j_t = \ell(x) - \ell(w_t)$ and $i_t = \ell(v_t) - \ell(x)$, we have $j_1 > j_2 > \cdots > j_{T-1} > j_T = j_{T+1} = \cdots = 0$ and $i_1 > i_2 > \cdots > i_{T-1} \geq i_T = i_{T+1} = \cdots = 0$. Note that $w_1 \in U^-_{k+j_1}(o)$ and $w_{t+1} \in V^-_{i_t+j_{t+1}}(v_t)$, while $v_t \in C^+(w_t)$. The BK inequality now gives

$$\begin{aligned}
\mathbb{E}|W_k^-| &\leq \sum_{\substack{j_1>j_2>\dots \\ i_1>i_2>\dots}} \sum_{\substack{w_1,w_2,\dots \\ v_1,v_2,\dots}} \mathbb{P}\{w_1 \in U^-_{k+j_1}(o)\} \prod_{t=1}^{\infty} \Big(\mathbb{P}\{w_{t+1} \in V^-_{i_t+j_{t+1}}(v_t)\} \mathbb{P}\{v_t \in C^+(w_t)\} \Big) \\
&\leq \sum_{\substack{j_1>j_2>\dots \\ i_1>i_2>\dots}} B\rho^{k+j_1} \prod_{t=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^{i_t+j_{t+1}} \\
&= B\rho^k \frac{1}{1-\rho} \left(\sum_{j_2>j_3>\dots} \left(\frac{\beta}{\alpha}\right)^{j_2+j_3+\dots} \right) \left(\sum_{i_1>i_2>\dots} \left(\frac{\beta}{\alpha}\right)^{i_1+i_2+\dots} \right) \\
&= B\rho^k \frac{1}{1-\rho} \left(\sum_{n=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^n q(n) \right)^2,
\end{aligned}$$

where, to get the second line, we again used Lemmas 3.7 and 3.4 and wrote $a_k \leq B\rho^k$ from the proof of Lemma 3.9, while, in the last line, we wrote $q(n)$ for the number of partitions of $n \in \mathbb{N}$ with all distinct parts, with the convention $q(0) = 1$. It is easy to see that $q(n)$ has subexponential growth,

$$q(n) \leq \sum_{k=1}^{\sqrt{2n}} \binom{n}{k} \leq \exp(C\sqrt{n}\log n),$$

but very precise estimates exist: it is well-known [And76] that $q(n)$ is also the number of partitions of n into odd parts, and we have

$$q(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4 \cdot 3^{1/4} n^{3/4}},$$

see [Ise61, HJ63]. We thus conclude that the last infinite series converges to a finite value $Q_{\alpha,\beta}$ for any $\beta < \alpha$. That is,

$$\mathbb{E}|W_k^-| \leq B\rho^k \frac{1}{1-\rho} Q^2_{\alpha,\beta},$$

and the proof is complete. □

Proof of Theorem 3.1. Consider a component $C = C(o)$ of critical percolation on $\Gamma_{\alpha,\beta}$. In view of Lemma 3.6, it suffices to show that a.s. $\pi_1(C)$ is finite to conclude that a.s. C is finite.

By Lemma 3.9, a.s. every component C has a highest level, which contains a finite number of vertices. By Lemmas 3.5 and 3.6, a.s. each vertex has a finite downwards subcluster. But C is just the union of the downwards subclusters of its vertices at the highest level, hence is (a.s.) finite.

The exponential decay of the connection probabilities follows immediately from Lemma 3.10 and the fact that as a function of $t \in \mathbb{N}$, there is an exponentially large number of vertices v with the properties that $\mathrm{dist}(o, v) = t$, all v's are on the same level of $\Gamma_{\alpha,\beta}$, and, moreover, their connection probabilities to o are the same. □

The characterization of p_u due to Schonmann [Sch99] and the amenability of $\Gamma_{\beta,\beta}$ imply that

$$p_u(\Gamma_{\alpha,\beta}) \leq p_u(\Gamma_{\beta,\beta}) = p_c(\Gamma_{\beta,\beta}) \leq 1/\beta$$

for $\beta \leq \alpha$. A condition on α and β for $p_c(\Gamma_{\alpha,\beta}) < p_u(\Gamma_{\alpha,\beta})$ can be easily given using a result of Schramm, see [LP05, Theorem 6.28]. If we denote by $a_n(G)$ the number of simple loops of length n containing a fixed vertex $o \in V(G)$, and $\gamma(G) := \limsup_n a_n(G)^{1/n}$, then $p_u(G) \geq 1/\gamma(G)$. For $G = \Gamma_{\alpha,\beta}$ it is not difficult to see that

$$\gamma \leq \sqrt{\alpha\beta} + \sqrt{(\alpha-1)(\beta-1)},$$

by the following argument.

First of all, $\Gamma_{\alpha,\beta}$ is a bipartite graph, so $a_{2n+1}(\Gamma_{\alpha,\beta}) = 0$, while $a_{2n}(\Gamma_{\alpha,\beta})$ is bounded from above by the number of simple non-backtracking paths of length $2n$ ending on the starting level. (Note that in this estimate we do not lose much by relaxing the loop-condition; however, excluding immediate backtracks is quite far from ensuring that the path be simple.) In such a path, we have n upwards and n downwards moves, in an arbitrary order, with k instances of changing direction from upwards to downwards, where $k \in \{0, 1, \ldots, n\}$. Then, the number of changes in direction from downwards to upwards is between $k-1$ and $k+1$. The number of such sequences with a given k value is at most $Cn\binom{2n}{2k}$. When such a path changes direction, to avoid backtracking, it has $\alpha - 1$ or $\beta - 1$ ways to continue; when it does not change direction, it has α or β ways. Therefore,

$$\begin{aligned}
a_{2n}(\Gamma_{\alpha,\beta}) &\leq \sum_{k=0}^{n} C'n\binom{2n}{2k}\alpha^{n-k}\beta^{n-k}(\alpha-1)^k(\beta-1)^k \\
&< C'n(\alpha\beta)^n \sum_{k=0}^{2n}\binom{2n}{k}x^k, \qquad \text{with} \quad x = \sqrt{\frac{(\alpha-1)(\beta-1)}{\alpha\beta}}, \\
&= C'n\left(\sqrt{\alpha\beta}\right)^{2n}(1+x)^{2n}.
\end{aligned}$$

Taking the $(2n)$th root of the last line gives the claimed bound on γ.

On the other hand, it is clear that $p_c \leq 1/\alpha$. (By considering small cycles, this inequality, as well as the above bound on γ, can be improved.) Hence

$$\sqrt{\alpha\beta} + \sqrt{(\alpha-1)(\beta-1)} \leq \alpha \qquad \text{implies} \qquad p_c(\Gamma_{\alpha,\beta}) < p_u(\Gamma_{\alpha,\beta}).$$

This is the case e.g. for $\Gamma_{6,2}$, and for $\alpha \geq 4\beta$, in general.

If one could deduce from the uniform exponential decay of connection probabilities at p_c (which we have verified for all $\alpha < \beta$) that for some $p > p_c$, the connection probabilities still tend to 0, it would follow that $p_c(\Gamma_{\alpha,\beta}) < p_u(\Gamma_{\alpha,\beta})$ by the Harris-FKG inequality.

4. The lamplighter group

Recall that when $\alpha = \beta$, the graph $\Gamma_{\alpha,\beta}$ is amenable and unimodular. The first half of our proof of Theorem 3.1 still holds, but the bound of Lemma 3.7 does not mean exponential decay, and so this method brakes down.

Take the "positive part" $\Gamma^+_{\alpha,\alpha}$ defined by taking the subgraph induced by the vertices

$$V(\Gamma^+_{\alpha,\alpha}) = \{v \in V(\Gamma_{\alpha,\alpha}) : \pi_1(v) \text{ is a descendant of } \pi_1(o)\}.$$

Clearly, $p_c(\Gamma_{\alpha,\alpha}) \leq p_c(\Gamma^+_{\alpha,\alpha})$, and our proof above shows that $p_c(\Gamma^+_{\alpha,\alpha})$-percolation on $\Gamma^+_{\alpha,\alpha}$ has no infinite clusters. This remains true for $p_c(\Gamma_{\alpha,\alpha})$-percolation on $\Gamma^+_{\alpha,\alpha}$, so any infinite path in $p_c(\Gamma_{\alpha,\alpha})$-percolation on $\Gamma_{\alpha,\alpha}$ would have to cross the plane $\{v : \ell(v) = \ell(o)\}$ infinitely many times.

A special interest in the graphs $\Gamma_{\alpha,\alpha}$ comes from the fact that they also arise as Cayley graphs of the so-called "lamplighter groups", introduced by Kaĭmanovich and Vershik (Example 6.1 of [KV83]), and further studied from a probabilistic point of view e.g. by Lyons, Pemantle and Peres [LPP96] and Woess [Woe05].

Definition 4.1 (Example 6.1 of [KV83])**.** *Consider the direct sum $\sum_{\mathbb{Z}^k} \mathbb{Z}_2$, which can also be viewed as the additive group $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ of finitely supported $\{0,1\}$-configurations on $\mathbb{Z}^k$, with the operation of pointwise addition* mod *2. The value of a configuration $f \in F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ on an element $x \in \mathbb{Z}^k$ will be denoted by $f(x)$ and the* support $\left\{x \in \mathbb{Z}^k : f(x) \neq 0\right\}$ *of f by* $\operatorname{supp} f$*. Let*

$$G_k = \mathbb{Z}^k \ltimes F_0(\mathbb{Z}^k, \mathbb{Z}_2)$$

be the semidirect *product of the groups $\mathbb{Z}^k$ and $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$, where the lattice $\mathbb{Z}^k$ acts naturally on $F_0(\mathbb{Z}^k, \mathbb{Z}_2)$ by shifts.*

The group G_1 was named the *lamplighter group* because of the following interpretation. Imagine a lamplighter standing on an infinite street, with lamps at every integer coordinate. Any element (j, f) describes a configuration: the lamplighter is next to lamp j, and f is the indicator function of the finite set F of lamps which are lit. For convenience, we shall also denote this element by (j, F). Define the *left* and *right flag* functions by $L((j,F)) := \min F$ and $R((j,F)) := \max F$, with $\min\{\} := +\infty$, $\max\{\} := -\infty$ for the empty set $\{\}$, and the *lamplighter position* by $\ell((j,F)) := j$. See Figure 4.

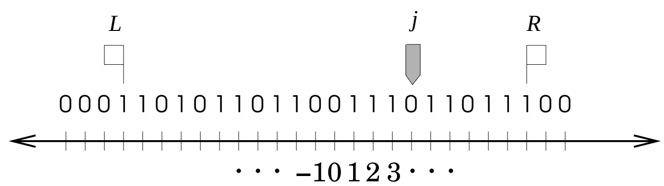

FIGURE 4. A configuration of lamps and the lamplighter in G_1.

The group operation is given by $(j, F) \cdot (j', F') = (j + j', F \triangle (j + F'))$, where $\triangle$ is symmetric set difference: the lamplighter flips the lamps F' relative to her current position, and advances j' lamps.

Recall the construction of the graph $\Gamma_{2,2}$ by orienting two 3-regular trees in opposite directions. Label the edges of each tree '0' or '1', to satisfy these conditions:

(1) The two "downwards" edges from each vertex are labelled '0' and '1';
(2) The edges of every "upwards" path $v_0, v_1, \ldots$ are eventually all labelled '0'.

Then, given any vertex v at level $\ell(v) = k$, we may identify v with the element (k, f) of G_1 as follows: Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be the labels of the edges along the paths upwards from $\pi_1(v)$ and $\pi_2(v)$, respectively. For $j \geq 0$, define $f(k+j) = b_j$ and $f(k-1-j) = a_j$. Then f has finite support, so (k, f) is in G_1 indeed. In fact, $\Gamma_{2,2}$ is the Cayley graph of the lamplighter group G_1 with generators $\{(\pm 1, \{0\}), (\pm 1, \{\})\}$.

Another natural Cayley graph G is given by the generators $(0, \{0\})$ (the lamplighter flips the state of the current lamp and stays in place) and $(\pm 1, \{\})$ (the lamplighter advances one lamp). Consider again the "positive half" G^+ of G, defined by taking only the vertices

$$\{(j, F) : j \geq 0, \forall k \in F : k \geq 0\}.$$

This is the portion of the graph accessible to the lamplighter if she is limited to the non-negative portion of the street. The subset of $V(G^+)$ given by $\{v : R(v) \leq \ell(v)\}$ induces a tree $\mathcal{F}$, the so-called Fibonacci tree, identified in [LPP96]. See Figure 5.

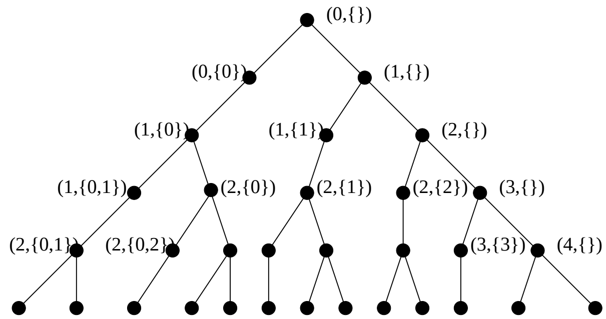

FIGURE 5. The Fibonacci tree in the lamplighter group.

We again have $p_c(G) \le p_c(G^+)$, and will consider $p_c(G^+)$-percolation on G^+. For a vertex $o \in V(\mathcal{F})$, define the *forward cluster* $C^+(o) = C^+(o,\omega)$ as the set of vertices $v \in V(\mathcal{F})$ accessible by open paths $o = v_0, v_1, \dots, v_n = v$ inside G^+ (not necessarily inside $\mathcal{F}$) in a $p_c(G^+)$-percolation configuration ω, such that $\ell(o) \le \ell(v_i) \le \ell(v)$, and the lamp at $R(o)$ is never adjusted in the path.

It is easy to see that we have the required independence in order to make our usual Galton-Watson argument work, therefore $e_k(p_c) := \mathbb{E}_{p_c}|C^+(o) \cap \{v : \ell(v) = k\}| \le 1$ for all $k \ge 0$. Again, as in Lemmas 2.6 and 3.5, we can conclude that $C^+(o)$ must be finite. Moreover, any open infinite simple path from o in G^+ would have infinitely many vertices inside $C^+(o)$, therefore the whole cluster of o is almost surely finite in critical percolation on G^+.

We have shown two transitive amenable graphs for which we know that critical percolation on the "positive part" almost surely has no infinite clusters, but we cannot prove this for the whole graph. Analogously, Barsky, Grimmett and Newman [BGN91] proved that critical percolation on the half-space graphs of the integer lattices $\mathbb{Z}^d$ has no infinite clusters.

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