Expositiones Mathematicae

HUTCHINSON WITHOUT BLASCHKE: AN ALTERNATIVE WAY TO FRACTALS

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ABSTRACT. The original approach of Hutchinson to fractals considers the defining equation as a fixed point problem, and then applies the Banach Contraction Principle. To do this, the Blaschke Completeness Theorem is essential. Avoiding Blaschke's result, this note presents an alternative way to fractals via the Kuratowski noncompactness measure. Moreover, our technique extends the existence part of Hutchinson's Theorem to condensing maps instead of contractions.

1. INTRODUCTION

In this note, fractals are considered through the Fixed Point Theorists's view, that is, as invariant objects of a given family of maps. More precisely, let $\mathscr{F}$ be a nonempty family of self-maps of a nonempty set X, and define the *invariance operator* $T\colon \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$T(H) = \bigcup_{f\in\mathscr{F}} f(H). \tag{1}$$

A set H is called $\mathscr{F}$*-invariant* if it is a fixed point of T, that is, $H = T(H)$ holds. In case of the weaker property $H \subset T(H)$ is valid, we speak about a *subinvariant* set. Let (X, d) be a metric space. Under an $\mathscr{F}$*-fractal* we mean a nonempty, compact, $\mathscr{F}$-invariant subset of X. Hutchinson's fundamental result [7] gives an existence and uniqueness property for fractals under some reasonable extra conditions: *If $\mathscr{F}$ is a finite family of contractions of a complete metric space, then there exists precisely one $\mathscr{F}$-fractal.* His approach is based on the fact that the invariance operator is a contraction in the Hausdorff–Pompeiu metric. Then the Banach Contraction Principle is applied. At this point of the argument, an extension of the original Blaschke Completeness Theorem [2] is essential.

Our main motivation is the next problem: *Can we prove Hutchinson's result without using the Blaschke Theorem?* The main results give a positive answer to this question. Moreover, besides an alternative approach, our method extends the classical fractal theorem.

The alternative way to fractals is based on the next concept [9]. Let (X, d) be a metric space. As usual, $U(x, \varepsilon)$ will stand for the open ball with center $x \in X$ and radius $\varepsilon > 0$. For an arbitrary set $H \subset X$, the (extended) real number

$$\chi(H) = \inf\bigl\{\varepsilon > 0 \mid \exists x_1, \dots, x_n \in X :\ H \subset U(x_1, \varepsilon) \cup \dots \cup U(x_n, \varepsilon)\bigr\}$$

is called the *Kuratowski noncompactness measure* of H. Clearly, $\chi(H) < +\infty$ if and only if H is bounded, and $\chi(H) = 0$ if and only if H is totally bounded. In the investigations, we need two additional properties of χ:

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- *If A, B are arbitrary subsets of X, then $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$.*
- *If X is complete, then $\chi(H) = 0$ if and only if H is relatively compact.*

Further important properties of the Kuratowski noncompactness measure with the hints of their proof can be found, for example, in the monograph of Dugundij and Granas [5].

Now the strategy of the alternative proof is the following. Assume that $\mathscr{F}$ is finite and consists of contractions. Take the fixed point of any member. Then, using the Kantorovich iteration [8] (in exactly the same way as in [1]), we can produce a nonempty $\mathscr{F}$-invariant set. Moreover, this set is bounded, since the Kantorovich iteration creates a Cauchy-sequence. Applying the properties of the Kuratowski noncompactness measure, the invariant set turns out to be a relatively compact one. Finally we show, that the closure of the invariant set (which is hence a nonempty and compact one), is $\mathscr{F}$-invariant, as well. This results in the existence part of Hutchinson's Theorem. Uniqueness is an immediate consequence of the facts that the invariance operator is a contraction in the fractal space and that a contraction can have at most one fixed point.

2. Auxiliary lemmas

Assume that X is a nonempty set, and let $H_1 \subset X$ be arbitrary. Under the *Kantorovich iteration* and its *limit* we mean the next recursion and union, respectively:

$$H_{n+1} = T(H_n), \qquad H = \bigcup_{n\in\mathbb{N}} H_n. \tag{2}$$

This iteration was applied by Kantorovich [8] to obtain order-theoretic fixed point results. Recently, it has also been used for 'minimalist fractal theory' (see [1]): The limit of the process represents a (nonempty) invariant set, suggesting the initial step of our alternative approach. Following the stages of the strategy described in the Introduction, we present here those auxiliary lemmas which are used to prove the main results.

Lemma 1. *If $\mathscr{F}$ is a nonempty family of self-maps of a nonempty set and H_1 is $\mathscr{F}$-subinvariant, then the Kantorovich itertaion* (2) *results an $\mathscr{F}$-invariant limit H.*

Proof. The subinvariant property of H_1 ensures that $H_1 \subset T(H_1)$; or equivalently, $H_1 \subset H_2$. On the other hand, T is a inclusion preserving map. Therefore, $T(H_1) \subset T(H_2)$ holds, yielding $H_2 \subset H_3$. Applying induction, finally we conclude that (H_n) is an increasing chain. Thus,

$$\begin{aligned} T(H) = T\Big(\bigcup_{n\in\mathbb{N}} H_n\Big) &= \bigcup_{f\in\mathscr{F}} f\Big(\bigcup_{n\in\mathbb{N}} H_n\Big) = \bigcup_{f\in\mathscr{F}} \bigcup_{n\in\mathbb{N}} f(H_n) \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{f\in\mathscr{F}} f(H_n) = \bigcup_{n\in\mathbb{N}} T(H_n) = \bigcup_{n\in\mathbb{N}} H_{n+1} = \bigcup_{n\in\mathbb{N}} H_n = H. \end{aligned}$$

This shows that H is an $\mathscr{F}$-invariant set. □

Lemma 2. *If $\mathscr{F}$ is a finite family of such self-maps of a complete metric space which decrease the Kuratowski noncompactness measure, then any bounded, $\mathscr{F}$-invariant set is relatively compact.*

Proof. Let $\mathscr{F} = \{f_1, \dots, f_n\}$. Assume to the contrary, that an $\mathscr{F}$-invariant set H is bounded, but not relatively compact. Then, $\chi(H)$ is positive and finite. Therefore, using the properties of the

Kuratowski noncompactness measure,

$$\begin{aligned}\chi(H) = \chi(T(H)) &= \chi(f_1(H) \cup \dots \cup f_n(H)) \\ &= \max\{\chi(f_1(H)), \dots, \chi(f_n(H))\} \\ &< \max\{\chi(H), \dots, \chi(H)\} = \chi(H)\end{aligned}$$

follows, which is a contradiction. □

Lemma 3. *If $\mathscr{F}$ is a finite family of continuous self-maps of a metric space and H is relatively compact $\mathscr{F}$-invariant set, then $\overline{H}$ is $\mathscr{F}$-invariant, as well.*

Proof. Let $\mathscr{F} = \{f_1, \dots, f_n\}$ and let $y \in T(\overline{H})$ be arbitrary. Then, $y \in f_k(\overline{H})$ for some suitable index $k \in \{1, \dots, n\}$. That is, $y = f_k(x)$, where $x \in \overline{H}$. Consider a sequence (x_m) from H such that $x_m \to x$. Since H is $\mathscr{F}$-invariant, $f_k(x_m) \in H \subset \overline{H}$ holds. The continuity of f_k guarantees that $y = f_k(x) \in \overline{H}$. This results in the inclusion $T(\overline{H}) \subset \overline{H}$. By the continuity of the members of $\mathscr{F}$ and the compactness of $\overline{H}$, the set $T(\overline{H})$ is compact, as well. In particular, it is closed. On the other hand, $H = T(H) \subset T(\overline{H})$ shows that H is a subset of the closed set $T(\overline{H})$. Therefore, we arrive at the reversed inclusion $\overline{H} \subset T(\overline{H})$. □

Similarly to the classical approach, we shall need the next concept. Given a metric space (X, d), denote the family of nonempty, bounded, and closed subsets of X by $\mathscr{F}(X)$. For $A, B \in \mathscr{F}(X)$, define

$$d_{HP}(A, B) := \inf\Big\{\varepsilon > 0 \mid A \subset \bigcup_{b \in B} U(b, \varepsilon),\ B \subset \bigcup_{a \in A} U(a, \varepsilon)\Big\}.$$

As the next lemma shows, d_{HP} turns out to be a metric on $\mathscr{F}(X)$. This metric was introduced by Pompeiu in his Ph.D. thesis [11] in the particular case when the underlying metric space is Euclidean. Hausdorff was the first, who realized the importance of Pompeiu's concept [6]. Although Hausdorff gave the precise quotations, Pompeiu was forgotten for a long time. According to these historical facts, we shall use the terminology *Hausdorff–Pompeiu distance*.

Lemma 4. *Under the notations and conventions above, $(\mathscr{F}(X), d_{HP})$ is a metric space.*

Proof. Observe first, that d_{HP} has finite values. Indeed, for arbitrary $A, B \in \mathscr{F}(X)$, there exist α, β positive numbers and $x, y \in X$ such that

$$A \subset U(x, \alpha) \quad \text{and} \quad B \subset U(y, \beta)$$

by boundedness. Thus $d(a, b) \le \alpha + d(x, y) + \beta$ remains true for all $a \in A$ and $b \in B$ due to the triangle inequality. Choosing $\varepsilon = \alpha + d(x, y) + \beta$, we get

$$a \in U(b, \varepsilon) \subset \bigcup_{b \in B} U(b, \varepsilon) \quad \text{and} \quad b \in U(a, \varepsilon) \subset \bigcup_{a \in A} U(a, \varepsilon).$$

Hence $d_{HP}(A, B) \le \varepsilon < +\infty$. If $A = B$, then $d_{HP}(A, B) = 0$ obviously holds. Conversely, assume that $d_{HP}(A, B) = 0$ for some $A, B \in \mathscr{F}(X)$. Let $a \in A$ be fixed. Then, for all $n \in \mathbb{N}$, there exists $b_n \in B$, such that $d(a, b) < 1/n$. Thus the sequence b_n tends to $a \in A$. Since B is closed, $a \in B$. However, $a \in A$ is arbitrary, consequently $A \subset B$. The other inclusion can be proved similarly, resulting in $A = B$.

The symmetry follows directly from the definition. Finally, we prove the triangle inequality. Let $A, B, C \in \mathscr{F}(X)$. Respectively, let $\varepsilon > d_{HP}(A, B)$ and $\delta > d_{HP}(B, C)$. If $a \in A$ if arbitrary, then there exists $b \in B$ and $c \in C$, such that $d(a,b) < \varepsilon$ and $d(b,c) < \delta$. So $d(a,c) < \varepsilon + \delta$. Since $a \in A$ is arbitrary, $a \in U(c, \varepsilon + \delta)$ follows. That is,

$$A \subset \bigcup_{c \in C} U(c, \varepsilon + \delta).$$

Interchanging the role of A and C, one can conclude $d_{HP}(A,C) < \varepsilon + \delta$ via the same reasoning. Taking the limits $\varepsilon \downarrow d_{HP}(A,B)$ and $\delta \downarrow d_{HP}(B,C)$, we get the triangle inequality. □

In the forthcomings, $\mathbb{R}_+$ denotes the nonnegative reals. Under a *comparison function* we mean an increasing, right-continuous function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ fulfilling $\varphi(t) < t$ for $t > 0$. Clearly, any comparison function vanishes at zero: $\varphi(0) = 0$. Since the composition of nondecreasing, right-continuous functions remains nondecreasing and right-continuous, the iterates of comparison functions are comparison functions, as well.

Let (X,d) be an arbitrary metric space. We say that the map $f\colon X \to X$ is a *Browder–Matkowski contraction* with comparison function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, if, for all elements $x, y \in X$, the next inequality holds:

$$d\big(f(x), f(y)\big) \le \varphi\big(d(x,y)\big).$$

For $q \in]0,1[$, the particular choice $\varphi(t) = qt$ shows that usual contractions are special Browder–Matkowski contractions. According to the result of Browder [3] and Matkowski [10], these generalized contractions have the same fixed point properties as classical ones: *Each Browder–Matkowski contraction of a complete metric space has exactly one fixed point.* Note also, that the same fixed point property remains true if we drop the assumption on right-continuity from the definition of comparison functions. However, in our aspect, this property turns out to be crucial: Besides several technical reasons, right-continuity of the comparison function provides the *continuity* of Browder–Matkowski contractions.

Lemma 5. *If $\mathscr{F}$ is a finite family of Browder–Matkowski contractions of a metric space, then the invariance operator* (1) *is a Browder–Matkowski contraction in the Hausdorff–Pompeiu metric. In particular, its composite iterates creates a Cauchy sequence.*

Proof. Let $f_1, \dots, f_n\colon X \to X$ be Browder–Matkowski contractions of a metric space X with comparison functions $\varphi_1, \dots, \varphi_n$. First we show, that $\varphi := \max\{\varphi_1, \dots, \varphi_n\}$ is a comparison function, as well. Clearly, φ is right-continuous and

$$\varphi(t) = \max\{\varphi_1(t), \dots, \varphi_n(t)\} < \max\{t, \dots, t\} = t.$$

In the second step, we show that the invariance operator T is a Browder–Matkowski contraction with comparison function φ. Let $A, B \in \mathscr{F}(X)$ and choose $\varepsilon > 0$ such that $d_{HP}(A,B) < \varepsilon$. Then, for any $a \in A$ there exists $b \in B$ such that $a \in U(b, \varepsilon)$. Hence

$$d(f_k(a), f_k(b)) \le \varphi_k\big(d(a,b)\big) \le \varphi\big(d(a,b)\big) \le \varphi(\varepsilon).$$

This yields

$$f_k(a) \in U(f_k(b), \varphi(\varepsilon)) \subset \bigcup_{y \in T(B)} U(y, \varphi(\varepsilon)),$$

and consequently

$$T(A) \subset \bigcup_{y \in T(B)} U(y, \varphi(\varepsilon)).$$

Similar arguments result in

$$T(B) \subset \bigcup_{y \in T(A)} U(y, \varphi(\varepsilon)).$$

Thus $d_{HP}(T(A), T(B)) \leq \varphi(\varepsilon)$. Taking the limit $\varepsilon \downarrow d_{HP}(A, B)$ and applying the right-continuity of φ, we arrive at the desired contraction property of T. The second statement is a well-known consequence of this property. $\square$

3. THE MAIN RESULTS

Our main results present fractal theorems when the invariance operator consists of Browder–Matkowski contractions or so-called condensing maps, respectively. The first one generalizes the result of Hutchinson:

Theorem 1. *If $\mathscr{F}$ is a finite family of Browder–Matkowski contractions of a complete metric space, then there exists precisely one $\mathscr{F}$-fractal.*

Proof. In what follows, let (X, d) be the underlying complete metric space and $\mathscr{F} = \{f_1, \dots, f_n\}$ be the family of Browder–Matkowski contractions with comparison functions $\varphi_1, \dots, \varphi_n$. By the Browder–Matkowski Fixed Point Theorem, each member of $\mathscr{F}$ has exactly one fixed point in X. Let x_0 be an arbitrary one, and let $H_1 = \{x_0\}$. Then, H_1 is a nonempty $\mathscr{F}$-subinvariant set, and hence the Kantorovich iteration produces a nonempty $\mathscr{F}$-invariant limit H by Lemma 1.

Consider the sequence of sets (H_n) defined in (2). By Lemma 5, this is a Cauchy sequence, and hence it is bounded in the Hausdorff–Pompeiu metrics. In particular, there exists $r > 0$ such that, for all $n \in \mathbb{N}$, the inequality $d_{HP}(H_1, H_n) < r$ holds. That is,

$$H_n \subset \bigcup_{x \in H_1} U(x, r) = U(x_0, r).$$

This implies $H \subset U(x_0, r)$, showing the boundedness of H given in (2).

Now we prove that, $f \colon X \to X$ is a Browder–Matkowski contraction with comparison function φ, the inequality holds

$$\chi(f(H)) \leq \varphi(\chi(H))$$

whenever H is a bounded subset of X. Fix $\varepsilon > \chi(H)$. Then, there exists a finite ε-net $E \subset X$ for H. If $x \in H$, then there exists $h \in E$ fulfilling $d(x, h) < \varepsilon$. Therefore,

$$d(f(x), f(h)) \leq \varphi(d(x, h)) \leq \varphi(\varepsilon),$$

yielding that $\{f(h) \mid h \in E\}$ is a finite $\varphi(\varepsilon)$-net for $f(H)$. Thus, $\chi(f(H)) \leq \varphi(\varepsilon)$. Taking the limit $\varepsilon \downarrow \chi(H)$ and using the right-continuity of φ, we arrive at the desired estimation. In particular, the properties of comparison functions guarantee $\varphi(\chi(H)) < \chi(H)$. Thus any Browder–Matkowski contraction decreases the Kuratowski noncompactness measure. Hence, by Lemma 2 and by the previous part, the $\mathscr{F}$-invariant limit H is relatively compact.

Finally, as we have already mentioned, Browder–Matkowski contractions are continuous. Thus, by Lemma 3, the set $\overline{H}$ is a nonempty, compact, $\mathscr{F}$-invariant set. The uniqueness is a direct consequence of Lemma 5 and the fact that a contraction may have at most one fixed point. This completes the proof. $\square$

Given a metric space X, a map $f \colon X \to X$ is called *condensing*, if it is continuous and decreases the Kuratowski noncompactness measure, that is, $\chi(f(H)) < \chi(H)$ holds whenever $H \subset X$ is bounded. The result of Darbo [4] and Sadovskiĭ [12] claims that *if X is a Banach-space, K is a* *nonempty, bounded, closed subset, then each condensing map $f\colon K \to K$ has at least one fixed point.* Using this fixed point property, our method generalizes the existence part of Hutchinson's Theorem, when the invariance operator consists of condensing maps:

Theorem 2. *If $\mathscr{F}$ is a finite family of condensing self-maps of a nonempty, bounded, closed, convex subset in a Banach space, then there exists at least one $\mathscr{F}$-fractal.*

Proof. Let K be a nonempty, bounded, closed, convex subset of a Banach space X and let $\mathscr{F} = \{f_1, \dots, f_n\}$ be the family of condensing self-maps of K. According to the Darbo–Sadovskiĭ Fixed Point Theorem, there exist a fixed point for each member of $\mathscr{F}$. Having such a fixed point x_0, the set $H_1 = \{x_0\}$ generates a nonempty $\mathscr{F}$-invariant set H via (2) by Lemma 1. Note that H is bounded since $H \subset K$. Thus Lemma 2 and Lemma 3 complete the proof. □

On compact domains, continuous and condensing maps coincide. Hence, as an immediate consequence of Theorem 2, we arrive at the next result.

Theorem 3. *If $\mathscr{F}$ is a finite family of continuous self-maps of a nonempty, compact, convex subset in a Banach space, then there exists at least one $\mathscr{F}$-fractal.*

Observe, that Theorem 1 can also be proved via the original approach of Hutchinson. Indeed, Lemma 5 guarantees that the invariance operator induces a Browder–Matkowski contraction in the space $(\mathscr{F}(X), d_{HP})$. This space becomes complete if the underlying space is complete by the Blaschke Theorem. These facts enable us to use the Browder–Matkowski Fixed Point Theorem directly, and we can conclude to the uniqueness and existence of a nonempty, bounded and closed invariant set. After some extra efforts, this invariant set turns out to be compact, as well.

However, the classical approach cannot be followed to prove Theorem 2: Since the continuous image of a closed set is not necessarily closed, the invariance operator may not be a self-map of the space of nonempty, closed, bounded subsets.

Let us emphasize, that the Kantorovich iteration enables to approximte $\mathscr{F}$-fractals once a fixed point of any member of $\mathscr{F}$ is known. This approximation works even in those cases, when the invariance operator T may not allow the usual Banach–Piccard iteration.

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References

[1] M. Bessenyei and E. Pénzes, *Fractals for minimalists*, Aequat. Math. (2019), to appear.
[2] W. Blaschke, *Kreis und Kugel (1916)*, Chelsea Publishing Co., New York, 1949.
[3] F. E. Browder, *Remarks on fixed point theorems of contractive type*, Nonlinear Anal. **3** (1979), 657–661.
[4] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92.
[5] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.
[6] F. Hausdorff, *Grundzuege der mengenlehre*, Viet, Leipzig, 1914.
[7] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747.
[8] L. Kantorovitch, *The method of successive approximations for functional equations*, Acta Math. **71** (1939), 63–97.
[9] K. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.
[10] J. Matkowski, *Integrable solutions of functional equations*, Dissertationes Math. **127** (1975), 1–68.
[11] D. Pompeiu, *Sur la continuité des fonction de variables complexes (ph.d. thesis)*, Ann. Fac. Sci. de Toulouse **7** (1905), 264–315.

[12] B. N. Sadovskiĭ, *On a fixed point principle*, Funkcional. Anal. i Priložen. **1** (1967), no. 2, 74–76.

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