Generalising the scattered property of subspaces

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Abstract

Let V be an r-dimensional $\mathbb{F}_{q^n}$-vector space. We call an $\mathbb{F}_q$-subspace U of V h-scattered if U meets the h-dimensional $\mathbb{F}_{q^n}$-subspaces of V in $\mathbb{F}_q$-subspaces of dimension at most h. In 2000 Blokhuis and Lavrauw proved that $\dim_{\mathbb{F}_q} U \leq rn/2$ when U is 1-scattered. Subspaces attaining this bound have been investigated intensively because of their relations with projective two-weight codes and strongly regular graphs. MRD-codes with a maximum idealiser have also been linked to $rn/2$-dimensional 1-scattered subspaces and to n-dimensional $(r-1)$-scattered subspaces.

In this paper we prove the upper bound $rn/(h+1)$ for the dimension of h-scattered subspaces, $h > 1$, and construct examples with this dimension. We study their intersection numbers with hyperplanes, introduce a duality relation among them, and study the equivalence problem of the corresponding linear sets.

1 Introduction

Let $V(n, q)$ denote an n-dimensional $\mathbb{F}_q$-vector space. A t-spread of $V(n, q)$ is a set $\mathcal{S}$ of t-dimensional $\mathbb{F}_q$-subspaces such that each vector of $V(n, q)\setminus\{\mathbf{0}\}$ is contained in exactly one element of $\mathcal{S}$. As shown by Segre in [26], a t-spread of $V(n, q)$ exists if and only if $t \mid n$.

Let V be an r-dimensional $\mathbb{F}_{q^n}$-vector space and let $\mathcal{S}$ be an n-spread of V, viewed as an $\mathbb{F}_q$-vector space. An $\mathbb{F}_q$-subspace U of V is called *scattered*

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w.r.t. $\mathcal{S}$ if it meets every element of $\mathcal{S}$ in an $\mathbb{F}_q$-subspace of dimension at most one, see [4]. If we consider V as an rn-dimensional $\mathbb{F}_q$-vector space, then it is well-known that the one-dimensional $\mathbb{F}_{q^n}$-subspaces of V, viewed as n-dimensional $\mathbb{F}_q$-subspaces, form an n-spread of V. This spread is called the *Desarguesian spread*. In this paper scattered will always mean scattered w.r.t. the Desarguesian spread. For such subspaces Blokhuis and Lavrauw showed that their dimension can be bounded by $rn/2$. After a series of papers, it is now known that when $2 \mid rn$ then there always exist scattered subspaces of this dimension [1, 3, 4, 11].

In this paper we introduce and study the following special class of scattered subspaces.

Definition 1.1. *Let V be an r-dimensional $\mathbb{F}_{q^n}$-vector space. An $\mathbb{F}_q$-subspace U of V is called h-scattered, $0 < h \leq r-1$, if $\langle U \rangle_{\mathbb{F}_{q^n}} = V$ and each h-dimensional $\mathbb{F}_{q^n}$-subspace of V meets U in an $\mathbb{F}_q$-subspace of dimension at most h. An h-scattered subspace of highest possible dimension is called a maximum h-scattered subspace.*

With this definition, the 1-scattered subspaces are the scattered subspaces generating V over $\mathbb{F}_{q^n}$. With $h = r$ the above definition would give the n-dimensional $\mathbb{F}_q$-subspaces of V defining subgeometries of $\mathrm{PG}(V, \mathbb{F}_{q^n})$. If $h = r-1$ and $\dim_{\mathbb{F}_q} U = n$, then U defines a scattered $\mathbb{F}_q$-linear set with respect to hyperplanes, introduced in [28, Definition 14]. A further generalisation of the concept of h-scattered subspaces can be found in the recent paper [2].

In this paper we prove that for an h-scattered subspace U of $V(r, q^n)$, if U does not define a subgeometry, then

$$\dim_{\mathbb{F}_q} U \leq \frac{rn}{h+1}, \tag{1}$$

cf. Theorem 2.3. Clearly, h-scattered subspaces reaching bound (1) are maximum h-scattered. When $h+1 \mid r$ then our examples prove that maximum h-scattered subspaces have dimension $rn/(h+1)$, cf. Theorem 2.6. In Theorem 2.7 we show that h-scattered subspaces of dimension $rn/(h+1)$ meet hyperplanes of $V(r, q^n)$ in $\mathbb{F}_q$-subspaces of dimension at least $rn/(h+1) - n$ and at most $rn/(h+1) - n + h$. Then we introduce a duality relation between maximum h-scattered subspaces of $V(r, q^n)$ reaching bound (1) and maximum $(n-h-2)$-scattered subspaces of $V(rn/(h+1) - r, q^n)$ reaching bound (1), which allows us to give some constructions also when $h+1$ is not a divisor of r, cf. Theorem 3.6.

Proposition 2.1 shows us that h-scattered subspaces are special classes of 1-scattered subspaces. In [28, Corollary 4.4] the $(r-1)$-scattered subspaces of $V(r,q^n)$ attaining bound (1), i.e. of dimension n, have been shown to be equivalent to MRD-codes of $\mathbb{F}_q^{n\times n}$ with minimum rank distance $n-r+1$ and with left or right idealiser isomorphic to $\mathbb{F}_{q^n}$. In Section 4 we study the $\mathbb{F}_q$-linear set L_U determined by an h-scattered subspace U. In contrast to the case of 1-scattered subspaces, it turns out that for any h-scattered $\mathbb{F}_q$-subspaces U and W of $V(r,q^n)$ with $h>1$, the corresponding linear sets L_U and L_W are $\mathrm{P\Gamma L}(r,q^n)$-equivalent if and only if U and W are $\Gamma\mathrm{L}(r,q^n)$-equivalent, cf. Theorem 4.5. For $r>2$ this result extends [28, Proposition 3.5] regarding the equivalence between MRD-codes and maximum $(r-1)$-scattered subspaces attaining bound (1) into an equivalence between MRD-codes and the corresponding linear sets, see [28, Remarks 4, 5].

2 The maximum dimension of an h-scattered subspace

We start this section by the following result.

Proposition 2.1. *For $h>1$ the h-scattered subspaces are also i-scattered for any $i<h$. In particular they are all 1-scattered.*

Proof. Let U be an h-scattered subspace of V. Suppose to the contrary that it is not i-scattered for some $i<h$. Therefore, there exists an i-dimensional $\mathbb{F}_{q^n}$-subspace S such that $\dim_{\mathbb{F}_q}(S\cap U)\geq i+1$. As $\langle U\rangle_{\mathbb{F}_{q^n}}=V$, there exist $\mathbf{u}_1,\ldots,\mathbf{u}_{h-i}\in U$ such that $\dim_{\mathbb{F}_{q^n}}\langle S,\mathbf{u}_1,\ldots,\mathbf{u}_{h-i}\rangle_{\mathbb{F}_{q^n}}=h$. Then

$$\dim_{\mathbb{F}_q}\left(U\cap\langle S,\mathbf{u}_1,\ldots,\mathbf{u}_{h-i}\rangle_{\mathbb{F}_{q^n}}\right)\geq(i+1)+(h-i)=h+1,$$

a contradiction. □

In the proof of the main result of this section we will need the following lemma.

Lemma 2.2. *For any integer i with $r\leq i\leq n$ in $V=V(r,q^n)$ there exists an $(r-1)$-scattered $\mathbb{F}_q$-subspace of dimension i.*

Proof. Fix an $\mathbb{F}_{q^n}$-basis of V, then the space V can be seen as $\mathbb{F}_{q^n}^r$. Consider the n-dimensional $\mathbb{F}_q$-subspace $U=\{(x,x^q,\ldots,x^{q^{r-1}}):x\in\mathbb{F}_{q^n}\}$ of V. Let

W be any i-dimensional $\mathbb{F}_q$-subspace of U. The intersection of W with a hyperplane $[a_0, a_1, \dots, a_{r-1}]$ of V is

$$\left\{ (x, x^q, \dots, x^{q^{r-1}}) : x \in \mathbb{F}_{q^n}, \sum_{j=0}^{r-1} a_j x^{q^j} = 0 \right\} \cap W,$$

which is clearly an $\mathbb{F}_q$-subspace of size at most $\deg \sum_{j=0}^{r-1} a_j x^{q^j} \leq q^{r-1}$. If $\langle W \rangle_{\mathbb{F}_{q^n}} \neq V$ then there was a hyperplane of V containing W, a contradiction, i.e. W is an $(r-1)$-scattered $\mathbb{F}_q$-subspace of V. $\square$

For $h = 1$, the following result was shown in [4].

Theorem 2.3. *Let V be an r-dimensional $\mathbb{F}_{q^n}$-vector space and U an h-scattered $\mathbb{F}_q$-subspace of V. Then either*

- $\dim_{\mathbb{F}_q} U = r$*, U defines a subgeometry of* $\mathrm{PG}(V, \mathbb{F}_{q^n})$ *and U is $(r-1)$-scattered, or*
- $\dim_{\mathbb{F}_q} U \leq rn/(h+1)$.

Proof. Let k denote the dimension of U over $\mathbb{F}_q$. Since $\langle U \rangle_{\mathbb{F}_{q^n}} = V$, we have $k \geq r$ and in case of equality U defines a subgeometry of $\mathrm{PG}(V, \mathbb{F}_{q^n})$ which is clearly $(r-1)$-scattered. From now on we may assume $k > r$. First consider the case $h = r-1$. Fix an $\mathbb{F}_{q^n}$-basis in V and for $\mathbf{x} \in V$ denote the i-th coordinate w.r.t. this basis by x_i. Consider the following set of $\mathbb{F}_q$-linear maps from U to $\mathbb{F}_{q^n}$:

$$\mathcal{C}_U := \left\{ G_{a_0,\dots,a_{r-1}} : \mathbf{x} \in U \mapsto \sum_{i=0}^{r-1} a_i x_i : a_i \in \mathbb{F}_{q^n} \right\}.$$

First we show that the non-zero maps of $\mathcal{C}_U$ have rank at least $k-r+1$. Indeed, if $(a_0, \dots, a_{r-1}) \neq \mathbf{0}$, then $\mathbf{u} \in \ker G_{a_0,\dots,a_{r-1}}$ if and only if $\sum_{i=0}^{r-1} a_i u_i = 0$, i.e. $\ker G_{a_0,\dots,a_{r-1}} = U \cap H$, where H is the hyperplane $[a_0, a_1, \dots, a_{r-1}]$ of V. Since U is $(r-1)$-scattered, it follows that $\dim_{\mathbb{F}_q} \ker G_{a_0,\dots,a_{r-1}} \leq r-1$ and hence the rank of $G_{a_0,\dots,a_{r-1}}$ is at least $k-r+1$. Next we show that any two maps of $\mathcal{C}_U$ are different. Suppose to the contrary $G_{a_0,\dots,a_{r-1}} = G_{b_0,\dots,b_{r-1}}$, then $G_{a_0-b_0,\dots,a_{r-1}-b_{r-1}}$ is the zero map. If $(a_1 - b_1, \dots, a_r - b_r) \neq \mathbf{0}$, then U would be contained in the hyperplane $[a_0 - b_0, a_1 - b_1, \dots, a_{r-1} - b_{r-1}]$, a contradiction since $\langle U \rangle_{\mathbb{F}_{q^n}} = V$. Hence, $|\mathcal{C}_U| = q^{nr}$.

Suppose to the contrary $k > n$. The elements of $\mathcal{C}_U$ form a nr-dimensional $\mathbb{F}_q$-subspace of $\mathrm{Hom}_{\mathbb{F}_q}(U, \mathbb{F}_{q^n})$ and the non-zero maps of $\mathcal{C}_U$ have rank at least $k-r+1$. By Result 4.6 (Singleton-like bound) we get $q^{rn} \leq q^{k(n-k+r)}$ and hence $(k-n)(k-r) \leq 0$, which contradicts $k > r$.

From now on, we will assume $1 < h < r-1$, since the assertion has been proved in [4] for $h = 1$.

First we assume $n \geq h+1$. Then by Lemma 2.2, in $\mathbb{F}_{q^n}^h$ there exists an $(h-1)$-scattered $\mathbb{F}_q$-subspace W of dimension $h+1$.

Let G be an $\mathbb{F}_q$-linear transformation from V to itself with $\ker G = U$. Clearly, $\dim_{\mathbb{F}_q} \operatorname{Im} G = rn - k$. For each $(\mathbf{u}_1, \ldots, \mathbf{u}_h) \in V^h$ consider the $\mathbb{F}_{q^n}$-linear map

$$\tau_{\mathbf{u}_1,\ldots,\mathbf{u}_h} \colon (\lambda_1, \ldots, \lambda_h) \in W \mapsto \lambda_1 \mathbf{u}_1 + \ldots + \lambda_h \mathbf{u}_h \in V.$$

Consider the following set of $\mathbb{F}_q$-linear maps $W \to \operatorname{Im} G$

$$\mathcal{C} := \{G \circ \tau_{\mathbf{u}_1,\ldots,\mathbf{u}_h} : (\mathbf{u}_1, \ldots, \mathbf{u}_h) \in V^h\}.$$

Our aim is to show that these maps are pairwise distinct and hence $|\mathcal{C}| = q^{rnh}$. Suppose $G \circ \tau_{\mathbf{u}_1,\ldots,\mathbf{u}_h} = G \circ \tau_{\mathbf{v}_1,\ldots,\mathbf{v}_h}$. It follows that $G \circ \tau_{\mathbf{u}_1-\mathbf{v}_1,\ldots,\mathbf{u}_h-\mathbf{v}_h}$ is the zero map, i.e.

$$\lambda_1(\mathbf{u}_1 - \mathbf{v}_1) + \ldots + \lambda_h(\mathbf{u}_h - \mathbf{v}_h) \in \ker G = U \text{ for each } (\lambda_1, \ldots, \lambda_h) \in W. \quad (2)$$

For $i \in \{1, \ldots, h\}$, put $\mathbf{z}_i = \mathbf{u}_i - \mathbf{v}_i$, let $T := \langle \mathbf{z}_1, \ldots, \mathbf{z}_h \rangle_{q^n}$ and let $t = \dim_{q^n} T$. We want to show that $t = 0$. If $t = h$, then by (2)

$$\{\lambda_1 \mathbf{z}_1 + \ldots + \lambda_h \mathbf{z}_h : (\lambda_1, \ldots, \lambda_h) \in W\} \subseteq T \cap U,$$

hence $\dim_{\mathbb{F}_q}(T \cap U) \geq \dim_{\mathbb{F}_q} W = h+1$, which is not possible since T is an h-dimensional $\mathbb{F}_{q^n}$-subspace of V and U is h-scattered. Hence $0 \leq t < h$. Assume $t \geq 1$. Let $\Phi : \mathbb{F}_{q^n}^h \to T$ be the $\mathbb{F}_{q^n}$-linear map defined by the rule

$$(\lambda_1, \ldots, \lambda_h) \mapsto \lambda_1 \mathbf{z}_1 + \ldots + \lambda_h \mathbf{z}_h$$

and consider the map $\tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h}$. Note that $\tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h}$ is the restriction of Φ on the $\mathbb{F}_q$-vector subspace W of $\mathbb{F}_{q^n}^h$. It can be easily seen that

$$\dim_{\mathbb{F}_{q^n}} \ker \Phi = h - t, \quad (3)$$

$$\ker \tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h} = \ker \Phi \cap W, \quad (4)$$

and by (2)

$$\operatorname{Im} \tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h} \subseteq T \cap U. \quad (5)$$

Since $t \geq 1$, by Proposition 2.1 the $\mathbb{F}_q$-subspace W is $(h-t)$-scattered in $\mathbb{F}_{q^n}^h$ and hence taking (3) and (4) into account we get $\dim_{\mathbb{F}_q} \ker \tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h} \leq h-t$, which yields

$$\dim_{\mathbb{F}_q} \operatorname{Im} \tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h} \geq t+1. \tag{6}$$

By Proposition 2.1 the $\mathbb{F}_q$-subspace U is also a t-scattered subspace of V, thus by (5)

$$\dim_{\mathbb{F}_q} \operatorname{Im} \tau_{\mathbf{z}_1,\ldots,\mathbf{z}_h} \leq \dim_{\mathbb{F}_q}(T \cap U) \leq t,$$

contradicting (6). It follows that $t = 0$, i.e. $\mathbf{z}_i = 0$ for each $i \in \{1, \ldots h\}$ and hence $|\,\mathcal{C}\,| = q^{rnh}$. The trivial upper bound for the size of $\mathcal{C}$ is the size of $\mathbb{F}_q^{(h+1)\times(rn-k)}$, thus

$$q^{rnh} = |\,\mathcal{C}\,| \leq q^{(h+1)(rn-k)},$$

which implies

$$k \leq \frac{rn}{h+1}.$$

Now assume $n < h+1$. By Proposition 2.1 U is h'-scattered with $h' = n-1$. Since $h' < r-1$ and $n \geq h'+1$, we can argue as before and derive $k = \dim_{\mathbb{F}_q} U \leq rn/(h'+1) = r$, contradicting $k > r$. □

The previous proof can be adapted also for the $h = 1$ case without introducing the subspace W, cf. [30].

The following result is a generalisation of [3, Theorem 3.1].

Theorem 2.4. *Let $V = V_1 \oplus \ldots \oplus V_t$ where $V_i = V(r_i, q^n)$ and $V = V(r, q^n)$. If U_i is an h_i-scattered $\mathbb{F}_q$-subspace in V_i, then the $\mathbb{F}_q$-subspace $U = U_1 \oplus \ldots \oplus U_t$ is h-scattered in V, with $h = \min\{h_1, \ldots, h_t\}$. Also, if U_i is h-scattered in V_i and its dimension reaches bound* (1)*, then U is h-scattered in V and its dimension reaches bound* (1)*.*

Proof. Clearly, it is enough to prove the assertion for $t = 2$.

If $h = 1$, the result easily follows from Proposition 2.1 and from [3, Theorem 3.1]; hence, we may assume $h = h_1 \geq 2$.

By way of contradiction suppose that there exists an h-dimensional $\mathbb{F}_{q^n}$-subspace W of V such that

$$\dim_{\mathbb{F}_q}(W \cap U) \geq h+1. \tag{7}$$

Clearly, W cannot be contained in V_1 since U_1 is h-scattered in V_1. Let $W_1 := W \cap V_1$ and $s := \dim_{\mathbb{F}_{q^n}} W_1$. Then $s < h$ and by Proposition 2.1, the $\mathbb{F}_q$-subspace U_1 is s-scattered in V_1, thus $\dim_{\mathbb{F}_q}(U_1 \cap W_1) \leq s$. Denoting $\langle U_1, W \cap U \rangle_{\mathbb{F}_q}$ by $\bar{U}_1$, the Grassmann formula and (7) yield

$$\dim_{\mathbb{F}_q} \bar{U}_1 - \dim_{\mathbb{F}_q} U_1 \geq h + 1 - s. \tag{8}$$

Consider the subspace $T := W + V_1$ of the quotient space $V/V_1 \cong V_2$. Then $\dim_{\mathbb{F}_{q^n}} T = h - s$ and T contains the $\mathbb{F}_q$-subspace

$$M := \bar{U}_1 + V_1.$$

Since M is also contained in the $\mathbb{F}_q$-subspace $U + V_1 = U_2 + V_1$, then M is h_2-scattered in V/V_1 and hence by $h - s \leq h \leq h_2$ and by Proposition 2.1, M is also $(h-s)$-scattered in V/V_1.

On the other hand,

$$\dim_{\mathbb{F}_q}(M \cap T) = \dim_{\mathbb{F}_q} M = \dim_{\mathbb{F}_q} \bar{U}_1 - \dim_{\mathbb{F}_q}(\bar{U}_1 \cap V_1) \geq$$

$$\dim_{\mathbb{F}_q} \bar{U}_1 - \dim_{\mathbb{F}_q}(U \cap V_1) = \dim_{\mathbb{F}_q} \bar{U}_1 - \dim_{\mathbb{F}_q} U_1,$$

and hence, by (8),

$$\dim_{\mathbb{F}_q}(M \cap T) \geq h - s + 1,$$

a contradiction.

The last part follows from $rn/(h+1) = \sum_{i=1}^{t} r_i n/(h+1)$. $\square$

Constructions of maximum 1-scattered $\mathbb{F}_q$-subspaces of $V(r, q^n)$ exist for all values of q, r and n, provided rn is even [1, 3, 4, 11]. For $r = 3$, $n \leq 5$ see [2, Section 5]. Also, there are constructions of maximum $(r-1)$-scattered $\mathbb{F}_q$-subspaces arising from MRD-codes (explained later in Section 4.1) for all values of q, r and n, cf. [28, Corollary 4.4]. In particular, the so called Gabidulin codes produce Example 2.5. One can also prove directly that these are maximum $(r-1)$-scattered subspaces by the same arguments as in the proof of Lemma 2.2.

Example 2.5. *In $\mathbb{F}_{q^n}^r$, if $n \geq r$, then the $\mathbb{F}_q$-subspace*

$$\{(x, x^q, x^{q^2}, \ldots, x^{q^{r-1}}) : x \in \mathbb{F}_{q^n}\}$$

is maximum $(r-1)$-scattered of dimension n.

Theorem 2.6. *If $h+1$ divides r and $n \geq h+1$, then in $V = V(r, q^n)$ there exist maximum h-scattered $\mathbb{F}_q$-subspaces of dimension $rn/(h+1)$.*

Proof. Put $r = t(h+1)$ and consider $V = V_1 \oplus \ldots \oplus V_t$, with V_i an $\mathbb{F}_{q^n}$-subspace of V with dimension $h+1$. For each i consider a maximum h-scattered $\mathbb{F}_q$-subspace U_i in V_i of dimension n which exists because of Example 2.5. By Theorem 2.4, $U_1 \oplus \ldots \oplus U_t$ is an h-scattered $\mathbb{F}_q$-subspace of V with dimension $tn = \frac{rn}{h+1}$. $\square$

In Theorem 2.6 we exhibit examples of maximum h-scattered subspaces of $V = V(r, q^n)$ whenever $h+1$ divides r. In Section 3 we introduce a method to construct such subspaces also when $h+1$ does not divide r. To do this, we will need an upper bound on the dimension of intersections of hyperplanes of V with a maximum h-scattered subspace of dimension $rn/(h+1)$. The proof of the following theorem is developed in Section 5.

Theorem 2.7. *If U is a maximum h-scattered $\mathbb{F}_q$-subspace of a vector space $V(r, q^n)$ of dimension $rn/(h+1)$, then for any $(r-1)$-dimensional $\mathbb{F}_{q^n}$-subspace W of $V(r, q^n)$ we have*

$$\frac{rn}{(h+1)} - n \leq \dim_{\mathbb{F}_q}(U \cap W) \leq \frac{rn}{(h+1)} - n + h.$$

The above theorem is a generalisation of [4, Theorem 4.2] and the first part of its proof relies on the counting technique developed in [4, Theorem 4.2].

3 Delsarte dual of an h-scattered subspace

Let U be a k-dimensional $\mathbb{F}_q$-subspace of a vector space $\Lambda = V(r, q^n)$, with $k > r$. By [21, Theorems 1, 2] (see also [20, Theorem 1]), there is an embedding of Λ in $\mathbb{V} = V(k, q^n)$ with $\mathbb{V} = \Lambda \oplus \Gamma$ for some $(k-r)$-dimensional $\mathbb{F}_{q^n}$-subspace Γ such that $U = \langle W, \Gamma \rangle_{\mathbb{F}_q} \cap \Lambda$, where W is a k-dimensional $\mathbb{F}_q$-subspace of $\mathbb{V}$, $\langle W \rangle_{\mathbb{F}_{q^n}} = \mathbb{V}$ and $W \cap \Gamma = \{\mathbf{0}\}$. Then the quotient space $\mathbb{V}/\Gamma$ is isomorphic to Λ and under this isomorphism U is the image of the $\mathbb{F}_q$-subspace $W + \Gamma$ of $\mathbb{V}/\Gamma$.

Now, let $\beta' \colon W \times W \to \mathbb{F}_q$ be a non-degenerate reflexive sesquilinear form on W with companion automorphism σ'. Then β' can be extended to a non-degenerate reflexive sesquilinear form $\beta \colon \mathbb{V} \times \mathbb{V} \to \mathbb{F}_{q^n}$. Indeed if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an $\mathbb{F}_q$-basis of W, since $\langle W \rangle_{\mathbb{F}_{q^n}} = \mathbb{V}$, for each $\mathbf{v}, \mathbf{w} \in \mathbb{V}$ we have

$$\beta(\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{k} a_i b_j^{\sigma} \beta'(\mathbf{u}_i, \mathbf{u}_j),$$

where $\mathbf{v} = \sum_{i=1}^{k} a_i \mathbf{u}_i$, $\mathbf{w} = \sum_{i=1}^{k} b_i \mathbf{u}_i$ and σ is an automorphism of $\mathbb{F}_{q^n}$ such that $\sigma_{|\mathbb{F}_q} = \sigma'$. Let $\perp$ and $\perp'$ be the orthogonal complement maps defined by β and β' on the lattice of $\mathbb{F}_{q^n}$-subspaces of $\mathbb{V}$ and of $\mathbb{F}_q$-subspaces of W, respectively. For an $\mathbb{F}_q$-subspace S of W the $\mathbb{F}_{q^n}$-subspace $\langle S \rangle_{\mathbb{F}_{q^n}}$ of $\mathbb{V}$ will be denoted by S^*. In this case $(S^*)^{\perp} = (S^{\perp'})^*$.

In this setting, we can prove the following preliminary result.

Proposition 3.1. *Let W, Λ, Γ, $\mathbb{V}$, $\perp$ and $\perp'$ be defined as above. If U is a k-dimensional $\mathbb{F}_q$-subspace of Λ with $k > r$ and*

$$\dim_{\mathbb{F}_q}(M \cap U) < k-1 \text{ holds for each hyperplane } M \text{ of } \Lambda, \qquad (\diamond)$$

then $W + \Gamma^{\perp}$ is a k-dimensional $\mathbb{F}_q$-subspace of the quotient space $\mathbb{V}/\Gamma^{\perp}$.

Proof. As described above, U turns out to be isomorphic to the $\mathbb{F}_q$-subspace $W + \Gamma$ of the quotient space $\mathbb{V}/\Gamma$. By $(\diamond)$, since each hyperplane of $\mathbb{V}/\Gamma$ is of form $H + \Gamma$ where H is a hyperplane of $\mathbb{V}$ containing Γ, it follows that

$$\dim_{\mathbb{F}_q}(H \cap W) < k-1 \text{ for each hyperplane } H \text{ of } \mathbb{V} \text{ containing } \Gamma. \qquad (\diamond\diamond)$$

To prove the assertion it is enough to prove

$$W \cap \Gamma^{\perp} = \{\mathbf{0}\}.$$

Indeed, by way of contradiction, suppose that there exists a nonzero vector $\mathbf{v} \in W \cap \Gamma^{\perp}$. Then the $\mathbb{F}_{q^n}$-hyperplane $\langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}}^{\perp}$ of $\mathbb{V}$ contains the subspace Γ and meets W in the $(k-1)$-dimensional $\mathbb{F}_q$-subspace $\langle \mathbf{v} \rangle_{\mathbb{F}_q}^{\perp'}$, which contradicts $(\diamond\diamond)$. $\square$

Definition 3.2. Let U be a k-dimensional $\mathbb{F}_q$-subspace of $\Lambda = V(r, q^n)$, with $k > r$ and such that $(\diamond)$ is satisfied. Then the k-dimensional $\mathbb{F}_q$-subspace $W + \Gamma^{\perp}$ of the quotient space $\mathbb{V}/\Gamma^{\perp}$ (cf. Proposition 3.1) will be denoted by $\bar{U}$ and we call it the *Delsarte dual* of U (w.r.t. $\perp$).

The term Delsarte dual comes from the Delsarte dual operation acting on MRD-codes, as pointed out in Theorem 4.12.

Theorem 3.3. *Let U be a maximum h-scattered $\mathbb{F}_q$-subspace of a vector space $\Lambda = V(r, q^n)$ of dimension $rn/(h+1)$, with $n \geq h+3$. Then the $\mathbb{F}_q$-subspace $\bar{U}$ of $\mathbb{V}/\Gamma^{\perp} = V(rn/(h+1) - r, q^n)$ obtained by the procedure of Proposition 3.1 is maximum $(n-h-2)$-scattered.*

Proof. Put $k := rn/(h+1)$. We first note that condition ($\diamond$) is satisfied for U since by Theorem 2.7 the hyperplanes of Λ meet U in $\mathbb{F}_q$-subspaces of dimension at most $rn/(h+1) - n + h < k-1$. Also, $k > r$ holds since $n \geq h+3$.

Hence we can apply the procedure of Proposition 3.1 to obtain the $\mathbb{F}_q$-subspace $\bar{U} = W + \Gamma^{\perp}$ of $\mathbb{V}/\Gamma^{\perp}$ of dimension k.

By way of contradiction, suppose that there exists an $(n-h-2)$-dimensional $\mathbb{F}_{q^n}$-subspace of $\mathbb{V}/\Gamma^{\perp}$, say M, such that

$$\dim_{\mathbb{F}_q}(M \cap \bar{U}) \geq n-h-1. \tag{9}$$

Then $M = H + \Gamma^{\perp}$, for some $(n+r-h-2)$-dimensional $\mathbb{F}_{q^n}$-subspace H of $\mathbb{V}$ containing $\Gamma^{\perp}$. For H, by (9), it follows that

$$\dim_{\mathbb{F}_q}(H \cap W) = \dim_{\mathbb{F}_q}(M \cap \bar{U}) \geq n-h-1.$$

Let S be an $(n-h-1)$-dimensional $\mathbb{F}_q$-subspace of W contained in H and let $S^* := \langle S \rangle_{\mathbb{F}_{q^n}}$. Then, $\dim_{\mathbb{F}_{q^n}} S^* = n-h-1$,

$$S^{\perp'} = W \cap (S^*)^{\perp} \quad \text{and} \quad S^{\perp'} \subset (S^*)^{\perp} = \langle S^{\perp'} \rangle_{\mathbb{F}_{q^n}}. \tag{10}$$

Since $S \subseteq H \cap W$ and $\Gamma^{\perp} \subset H$, we get $S^* \subset H$ and $H^{\perp} \subset \Gamma$, i.e.

$$H^{\perp} \subseteq \Gamma \cap (S^*)^{\perp}. \tag{11}$$

From (11) it follows that

$$\dim_{\mathbb{F}_{q^n}} \left(\Gamma \cap (S^*)^{\perp} \right) \geq \dim_{\mathbb{F}_{q^n}} H^{\perp} = k - (n+r-h-2).$$

This implies that

$$\dim_{\mathbb{F}_{q^n}} \langle \Gamma, (S^*)^{\perp} \rangle_{\mathbb{F}_{q^n}} = \dim_{\mathbb{F}_{q^n}} \Gamma + \dim_{\mathbb{F}_{q^n}} (S^*)^{\perp} - \dim_{\mathbb{F}_{q^n}} \left(\Gamma \cap (S^*)^{\perp} \right) \leq k-1$$

and hence $\langle \Gamma, (S^*)^{\perp} \rangle_{\mathbb{F}_{q^n}}$ is contained in a hyperplane T of $\mathbb{V}$ containing Γ. Also, $\dim_{\mathbb{F}_q}(S^{\perp'}) = \dim_{\mathbb{F}_q} W - \dim_{\mathbb{F}_q} S = k - (n-h-1)$ and, by (10), we get

$$S^{\perp'} = W \cap (S^*)^{\perp} \subseteq W \cap T.$$

Then $\hat{T} := T \cap \Lambda$ is a hyperplane of Λ and, by recalling $U = \langle W, \Gamma \rangle_{\mathbb{F}_q} \cap \Lambda$,

$$\dim_{\mathbb{F}_q}(\hat{T} \cap U) = \dim_{\mathbb{F}_q}(T \cap W) \geq \dim_{\mathbb{F}_q}(S^{\perp'}) = k - n + h + 1,$$

contradicting Theorem 2.7.

$\square$

In case of $h = r-1$, Theorem 3.3 follows from [28] and from the theory of MRD codes. Our theorem generalises this result to each value of h by using a geometric approach.

Corollary 3.4. *Starting from a maximum $(r-1)$-scattered $\mathbb{F}_q$-subspace U of $V(r,q^n)$ of dimension n, $n \geq r+2$, the $\mathbb{F}_q$-subspace $\bar{U}$ (cf. Definition 3.2) is a maximum $(n-r-1)$-scattered $\mathbb{F}_q$-subspace of $V(n-r,q^n)$ of dimension n.*

Corollary 3.5. *Starting from a maximum 1-scattered $\mathbb{F}_q$-subspace U of $V(r,q^n)$, rn even, $n \geq 4$, $\bar{U}$ (cf. Definition 3.2) is a maximum $(n-3)$-scattered $\mathbb{F}_q$-subspace of $V(r(n-2)/2,q^n)$ whose dimension attains bound (1).* □

Theorem 3.6. *If $n \geq 4$ is even and $r \geq 3$ is odd, then there exist maximum $(n-3)$-scattered $\mathbb{F}_q$-subspaces of $V(r(n-2)/2,q^n)$ which cannot be obtained from the direct sum construction of Theorem 2.6.*

Proof. By [1, 3, 4, 11] it is always possible to construct maximum 1-scattered $\mathbb{F}_q$-subspaces of $V(r,q^n)$. Then the result follows from Corollary 3.5 and from the fact that in this case $n-2$ does not divide $r(n-2)/2$. □

Remark 3.7. The Delsarte dual of an $\mathbb{F}_q$-subspace does not depend on the choice of the non-degenerate reflexive sesquilinear form on W.

Indeed, fix an $\mathbb{F}_q$-basis B of W, since $\langle W\rangle_{\mathbb{F}_{q^n}} = \mathbb{V}$, we can see W as $\mathbb{F}_q^k$ and $\mathbb{V}$ as $\mathbb{F}_{q^n}^k$. Let β_1' and β_2' be two non-degenerate reflexive sesquilinear forms on $\mathbb{F}_q^k$. Then, with respect to the basis B, the forms β_1' and β_2' are defined by the following rules:

$$\beta_i'((\mathbf{x},\mathbf{y})) = \mathbf{x}G_i\mathbf{y}_t^{\rho_i} \quad ^1,$$

where $G_i \in \mathrm{GL}(k,q)$ and ρ_i is an automorphism of $\mathbb{F}_q$ such that $\rho_i^2 = \mathrm{id}$ and $(G_i^{\rho_i})^t = G_i$, for $i \in \{1,2\}$. Now let β_1 and β_2 be their extensions over $\mathbb{F}_{q^n}^k$ defined by the rules

$$\beta_i((\mathbf{x},\mathbf{y})) = \mathbf{x}G_i\mathbf{y}_t^{\rho_i},$$

and let $\perp_1$ and $\perp_2$ be the orthogonal complement maps defined by β_1 and β_2 on the lattice of $\mathbb{F}_{q^n}$-subspaces of $\mathbb{F}_{q^n}^k$, respectively.

Again w.r.t. the basis B, the $\mathbb{F}_{q^n}$-subspace Γ described at the beginning of this section can be seen as a $(k-r)$-dimensional subspace of $\mathbb{F}_{q^n}^k$. Then, for $i \in \{1,2\}$ we have

$$\Gamma^{\perp_i} = \{\mathbf{x} : \mathbf{x}\,G_i\,\mathbf{y}_t^{\rho_i} = 0 \quad \forall\,\mathbf{y} \in \Gamma\}.$$

[1] Here $\mathbf{y}_t$ denotes the transpose of the vector $\mathbf{y}$.

Straightforward computations show that the invertible semilinear map

$$\varphi\colon \mathbf{x} \in \mathbb{F}_{q^n}^k \mapsto \mathbf{x}^{\rho_2^{-1}\rho_1} G_2^{\rho_2^{-1}\rho_1} G_1^{-1} \in \mathbb{F}_{q^n}^k,$$

leaves W invariant and maps $\Gamma^{\perp_2}$ to $\Gamma^{\perp_1}$. Then φ maps $W + \Gamma^{\perp_2}$ to $W + \Gamma^{\perp_1}$, i.e. φ maps the Delsarte dual of U calculated w.r.t β_2 to the Delsarte dual of U calculated w.r.t. β_1. See also [25, Section 2] and [27, Section 6.2].

4 Linear sets defined by h-scattered subspaces

Let V be an r-dimensional $\mathbb{F}_{q^n}$-vector space. A point set L of $\Lambda = \mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$ is said to be an $\mathbb{F}_q$-*linear set* of Λ of rank k if it is defined by the non-zero vectors of a k-dimensional $\mathbb{F}_q$-vector subspace U of V, i.e.

$$L = L_U := \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

One of the most natural questions about linear sets is their equivalence. Two linear sets L_U and L_W of $\mathrm{PG}(r-1, q^n)$ are said to be $\mathrm{P\Gamma L}$-*equivalent* (or simply *equivalent*) if there is an element φ in $\mathrm{P\Gamma L}(r, q^n)$ such that $L_U^{\varphi} = L_W$. In the applications it is crucial to have methods to decide whether two linear sets are equivalent or not. This can be a difficult problem and some results in this direction can be found in [9, 8, 12]. For $f \in \Gamma\mathrm{L}(r, q^n)$ we have $L_{U^f} = L_U^{\varphi_f}$, where φ_f denotes the collineation of $\mathrm{PG}(V, \mathbb{F}_{q^n})$ induced by f. It follows that if U and W are $\mathbb{F}_q$-subspaces of V belonging to the same orbit of $\Gamma\mathrm{L}(r, q^n)$, then L_U and L_W are equivalent. The above condition is only sufficient but not necessary to obtain equivalent linear sets. This follows also from the fact that $\mathbb{F}_q$-subspaces of V with different dimensions can define the same linear set, for example $\mathbb{F}_q$-linear sets of $\mathrm{PG}(r-1, q^n)$ of rank $k \geq rn - n + 1$ are all the same: they coincide with $\mathrm{PG}(r-1, q^n)$. Also, in [8, 12] for $r = 2$ it was pointed out that there exist maximum 1-scattered $\mathbb{F}_q$-subspaces of V on different orbits of $\Gamma\mathrm{L}(2, q^n)$ defining $\mathrm{P\Gamma L}$-equivalent linear sets of $\mathrm{PG}(1, q^n)$. It is then natural to ask for which linear sets can we translate the question of $\mathrm{P\Gamma L}$-equivalence into the question of $\Gamma\mathrm{L}$-equivalence of the defining subspaces. For further details on linear sets see [17, 18, 24].

In this section we study the equivalence issue of $\mathbb{F}_q$-linear sets defined by h-scattered linear sets for $h \geq 2$.

Definition 4.1. *If U is a (maximum) h-scattered $\mathbb{F}_q$-subspace of $V(r, q^n)$, then the $\mathbb{F}_q$-linear set L_U of $\mathrm{PG}(r-1, q^n)$ is called (maximum) h-scattered.*

The $(r-1)$-scattered $\mathbb{F}_q$-linear sets of rank n were defined also in [28, Definition 14] and following the authors of [28], we will call these $\mathbb{F}_q$-linear sets *maximum scattered with respect to hyperplanes.* Also, we will call 2-scattered $\mathbb{F}_q$-linear sets (of any rank) *scattered with respect to lines.*

Proposition 4.2 ([5, pg. 3 Eq. (6) and Lemma 2.1])**.** *Let V be a two-dimensional vector space over $\mathbb{F}_{q^n}$.*

1. *If U is an $\mathbb{F}_q$-subspace of V with $|L_U| = q+1$, then U has dimension 2 over $\mathbb{F}_q$.*
2. *Let U and W be two $\mathbb{F}_q$-subspaces of V with $L_U = L_W$ of size $q+1$. If $U \cap W \neq \{\mathbf{0}\}$, then $U = W$.*

Proposition 4.3. *If L_U is a scattered $\mathbb{F}_q$-linear set with respect to lines of $\mathrm{PG}(r-1, q^n) = \mathrm{PG}(V, \mathbb{F}_{q^n})$, then its rank is uniquely defined, i.e. for each $\mathbb{F}_q$-subspace W of V if $L_W = L_U$, then $\dim_{\mathbb{F}_q} W = \dim_{\mathbb{F}_q} U$.*

Proof. Let W be an $\mathbb{F}_q$-subspace of V such that $L_U = L_W$ and put $k = \dim_{\mathbb{F}_q} U$. Since U is a 1-scattered $\mathbb{F}_q$-subspace (cf. Proposition 2.1), $|L_U| = |L_W| = (q^k-1)/(q-1)$. It follows that $\dim_{\mathbb{F}_q} W \geq k$. Suppose that $\dim_{\mathbb{F}_q} W \geq k+1$, then there exists at least one point $P = \langle \mathbf{x} \rangle_{\mathbb{F}_{q^n}} \in L_W$ such that $\dim_{\mathbb{F}_q}(W \cap \langle \mathbf{x} \rangle_{\mathbb{F}_{q^n}}) \geq 2$. Let $Q = \langle \mathbf{y} \rangle_{\mathbb{F}_{q^n}} \in L_U = L_W$ be a point different from P, then $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{F}_{q^n}} \cap W$ has dimension at least 3 but the linear set defined by $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{F}_{q^n}} \cap W$ is $L_W \cap \langle P, Q \rangle$, thus it has size $q+1$, contradicting part 1 of Proposition 4.2. $\square$

Lemma 4.4. *Let L_U be a scattered $\mathbb{F}_q$-linear set with respect to lines in $\mathrm{PG}(r-1, q^n)$. If $L_U = L_W$ for some $\mathbb{F}_q$-subspace W, then $U = \lambda W$ for some $\lambda \in \mathbb{F}_{q^n}^*$.*

Proof. By Proposition 4.3, we have $\dim_{\mathbb{F}_q} W = \dim_{\mathbb{F}_q} U$ and hence, since U is 1-scattered, also W is 1-scattered. Let $P \in L_U$ with $P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$, then for some $\lambda \in \mathbb{F}_{q^n}^*$ we have $\mathbf{u} \in U \cap \lambda W$. Put $W' := \lambda W$ and note that $L_W = L_{W'}$. Our aim is to prove $W' \subseteq U$. Since U and W' are 1-scattered, we have $\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \cap U = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \cap W' = \langle \mathbf{u} \rangle_{\mathbb{F}_q}$.

What is left, is to show for each $\mathbf{w} \in W' \setminus \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}$ that $\mathbf{w} \in U$. To do this, consider the point $Q = \langle \mathbf{w} \rangle_{\mathbb{F}_{q^n}} \in L_{W'} = L_U$ and the line $\langle P, Q \rangle$ which meets L_U in $q+1$ points. By part 1 of Proposition 4.2, the $\mathbb{F}_q$-subspace $\big(\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbb{F}_{q^n}} \cap U\big)$ has dimension 2. Since $\big(\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbb{F}_{q^n}} \cap U\big) \cap \big(\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbb{F}_{q^n}} \cap W'\big) \neq \{\mathbf{0}\}$, by part 2 of Proposition 4.2 we get

$$\langle \mathbf{u}, \mathbf{w} \rangle_{\mathbb{F}_{q^n}} \cap U = \langle \mathbf{u}, \mathbf{w} \rangle_{\mathbb{F}_{q^n}} \cap W' = \langle \mathbf{u}, \mathbf{w} \rangle_{\mathbb{F}_q}.$$

Hence the assertion follows. $\square$

Theorem 4.5. *Consider two h-scattered linear sets L_U and L_W of $V(r, q^n)$ with $h \geq 2$. They are* $\mathrm{P\Gamma L}(r, q^n)$*-equivalent if and only if U and W are* $\Gamma\mathrm{L}(r, q^n)$*-equivalent.*

Proof. The if part is trivial. To prove the only if part assume that there exists $f \in \Gamma\mathrm{L}(r, q^n)$ such that $L_U^{\varphi_f} = L_W$, where φ_f is the collineation induced by f. Since $L_U^{\varphi_f} = L_{U^f}$, by Proposition 2.1 and Lemma 4.4, there exists $\lambda \in \mathbb{F}_{q^n}^*$ such that $\lambda U^f = W$ and hence U and W lie on the same orbit of $\Gamma\mathrm{L}(r, q^n)$. □

4.1 Scattered linear sets with respect to hyperplanes and MRD-codes

A *rank distance (or RD) code* $\mathcal{C}$ of $\mathbb{F}_q^{n\times m}$, $n \leq m$, can be considered as a subset of $\mathrm{Hom}_{\mathbb{F}_q}(U, V)$, where $\dim_{\mathbb{F}_q} U = m$ and $\dim_{\mathbb{F}_q} V = n$, with *rank distance* defined as $d(f, g) := \mathrm{rk}(f - g)$. The minimum distance of $\mathcal{C}$ is $d := \min\{d(f, g) : f, g \in \mathcal{C}, f \neq g\}$.

Result 4.6 ([13])**.** *If $\mathcal{C}$ is a rank distance code of $\mathbb{F}_q^{n\times m}$, $n \leq m$, with minimum distance d, then*

$$|\mathcal{C}| \leq q^{m(n-d+1)}. \tag{12}$$

Rank distance codes for which (12) holds with equality are called *maximum rank distance (or MRD) codes*.

From now on, we will only consider $\mathbb{F}_q$-linear MRD-codes of $\mathbb{F}_q^{n\times n}$, i.e. those which can be identified with $\mathbb{F}_q$-subspaces of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$. Since $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$ is isomorphic to the ring of q-polynomials over $\mathbb{F}_{q^n}$ modulo $x^{q^n} - x$, denoted by $\mathcal{L}_{n,q}$, with addition and composition as operations, we will consider $\mathcal{C}$ as an $\mathbb{F}_q$-subspace of $\mathcal{L}_{n,q}$. Given two $\mathbb{F}_q$-linear MRD codes, $\mathcal{C}_1$ and $\mathcal{C}_2$, they are equivalent if and only if there exist φ_1, $\varphi_2 \in \mathcal{L}_{n,q}$ permuting $\mathbb{F}_{q^n}$ and $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ such that

$$\varphi_1 \circ f^\rho \circ \varphi_2 \in \mathcal{C}_2 \text{ for all } f \in \mathcal{C}_1,$$

where $\circ$ stands for the composition of maps and $f^\rho(x) = \sum a_i^\rho x^{q^i}$ for $f(x) = \sum a_i x^{q^i}$. For a rank distance code $\mathcal{C}$ given by a set of linearized polynomials, its left and right idealisers can be written as:

$$L(\mathcal{C}) = \{\varphi \in \mathcal{L}_{n,q} : \varphi \circ f \in \mathcal{C} \text{ for all } f \in \mathcal{C}\},$$

$$R(\mathcal{C}) = \{\varphi \in \mathcal{L}_{n,q} : f \circ \varphi \in \mathcal{C} \text{ for all } f \in \mathcal{C}\}.$$

By [19, Section 2.7] and [28] the next result follows. We give a proof of the first part for the sake of completeness.

Result 4.7. *$\mathcal{C}$ is an $\mathbb{F}_q$-linear MRD-code of $\mathcal{L}_{n,q}$ with minimum distance $n-r+1$ and with left-idealiser isomorphic to $\mathbb{F}_{q^n}$ if and only if up to equivalence*

$$\mathcal{C} = \langle f_1(x), \ldots, f_r(x) \rangle_{\mathbb{F}_{q^n}}$$

for some $f_1, f_2, \ldots, f_r \in \mathcal{L}_{n,q}$ and the $\mathbb{F}_q$-subspace

$$U_{\mathcal{C}} = \{(f_1(x), \ldots, f_r(x)) : x \in \mathbb{F}_{q^n}\}$$

is a maximum $(r-1)$-scattered $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^r$.

Proof. Let $T = \{\omega_a : a \in \mathbb{F}_{q^n}\}$, where for each $a \in \mathbb{F}_{q^n}$, $\omega_a(x) = ax \in \mathcal{L}_{n,q}$ and let L denote the left-idealiser of $\mathcal{C}$. Since T and L are Singer cyclic subgroups of $\mathrm{GL}(\mathbb{F}_{q^n}, \mathbb{F}_q)$ and any two such groups are conjugate (cf. [16, pg. 187]) it follows that there exists an invertible q-polynomial g such that $g \circ L \circ g^{-1} = T$. Then for each $h \in \mathcal{C}' := g^{-1} \circ \mathcal{C}$ it holds that $\omega_a \circ h \in \mathcal{C}'$ for each $a \in \mathbb{F}_{q^n}$, which proves the first statement. For the second part see [28, Corollary 4.4]. $\square$

Remark 4.8. *The* adjoint *of a q-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$, with respect to the bilinear form $\langle x, y \rangle := \mathrm{Tr}_{q^n/q}(xy)$ ([2]), is given by*

$$\hat{f}(x) := \sum_{i=0}^{n-1} a_i^{q^{n-i}} x^{q^{n-i}}.$$

If $\mathcal{C}$ is a rank distance code given by q-polynomials, then the adjoint code *$\mathcal{C}^\top$ of $\mathcal{C}$ is $\{\hat{f} : f \in \mathcal{C}\}$. The code $\mathcal{C}$ is an MRD if and only if $\mathcal{C}^\top$ is an MRD and also $L(\mathcal{C}) \cong R(\mathcal{C}^\top)$, $R(\mathcal{C}) \cong L(\mathcal{C}^\top)$. Thus Result 4.7 can be translated also to codes with right-idealiser isomorphic to $\mathbb{F}_{q^n}$.*

The next result follows from [28, Proposition 3.5].

Result 4.9. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathbb{F}_q$-linear MRD-codes of $\mathcal{L}_{n,q}$ with minimum distance $n-r+1$ and with left-idealisers isomorphic to $\mathbb{F}_{q^n}$. Then $U_{\mathcal{C}}$ and $U_{\mathcal{C}'}$ are $\Gamma\mathrm{L}(r, q^n)$-equivalent if and only if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent.*

By Theorem 4.5, for $r > 2$ we can extend Result 4.9 in the following way.

Theorem 4.10. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathbb{F}_q$-linear MRD-codes of $\mathcal{L}_{n,q}$ with minimum distance $n-r+1$, $r > 2$, and with left-idealisers isomorphic to $\mathbb{F}_{q^n}$. Then the linear sets $L_{U_{\mathcal{C}}}$ and $L_{U_{\mathcal{C}'}}$ are $\mathrm{P}\Gamma\mathrm{L}(r, q^n)$-equivalent if and only if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent.*

[2]Where $\mathrm{Tr}_{q^n/q}(x) = x + x^q + \ldots + x^{q^{n-1}}$ denotes the $\mathbb{F}_{q^n} \to \mathbb{F}_q$ trace function.

In the following we motivate why we used the term "Delsarte dual" in Definition 3.2. In particular, we prove that the duality of Section 3 corresponds to the Delsarte duality on MRD-codes when $(r-1)$-scattered $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}^r$ are considered.

First recall that in terms of linearized polynomials, the Delsarte dual of a rank distance code $\mathcal{C}$ of $\mathcal{L}_{n,q}$ introduced in [13] and in [14] can be interpreted as follows

$$\mathcal{C}^\perp = \{f \in \mathcal{L}_{n,q} : b(f,g) = 0 \ \ \forall g \in \mathcal{C}\},$$

where $b(f,g) = \mathrm{Tr}_{q^n/q}\left(\sum_{i=0}^{n-1} a_i b_i\right)$ for $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ and $g(x) = \sum_{i=0}^{n-1} b_i x^{q^i} \in \mathcal{L}_{n,q}$.

Remark 4.11. Let $\mathcal{C}$ be an $\mathbb{F}_q$-linear MRD-code of $\mathcal{L}_{n,q}$ with minimum distance $n-r+1$ and with left-idealiser isomorphic to $\mathbb{F}_{q^n}$. By Result 4.7 and by [10, Theorem 2.2], there exist r distinct integers in $\{0,\ldots,n-1\}$ such that, up to equivalence,

$$\mathcal{C} = \langle h_0(x),\ldots,h_{r-1}(x)\rangle_{\mathbb{F}_{q^n}},$$

where

$$h_i(x) = x^{q^{t_i}} + \sum_{j \notin \{t_0,\ldots,t_{r-1}\}} g_{i,j} x^{q^j} \tag{13}$$

and $g_{i,j} \in \mathbb{F}_{q^n}$.
Also, let $\{s_0, s_1, \ldots, s_{n-r-1}\} := \{0,\ldots,n-1\} \setminus \{t_0,\ldots,t_{r-1}\}$. Then it is easy to see that the Delsarte dual of $\mathcal{C}$ is

$$\mathcal{C}^\perp = \langle h'_0(x),\ldots,h'_{n-r-1}(x)\rangle_{\mathbb{F}_{q^n}},$$

where

$$h'_i(x) = x^{q^{s_i}} - \sum_{j \in \{t_0,\ldots,t_{r-1}\}} g_{j,s_i} x^{q^j}. \tag{14}$$

Theorem 4.12. *Let $\mathcal{C}$ be an $\mathbb{F}_q$-linear MRD-code of $\mathcal{L}_{n,q}$ with minimum distance $n-r+1$ and with left-idealiser isomorphic to $\mathbb{F}_{q^n}$. Then there exist $h_0(x),\ldots,h_{r-1}(x), h'_0(x),\ldots,h'_{n-r-1}(x) \in \mathcal{L}_{n,q}$ such that, up to equivalence,*

- $\mathcal{C} = \langle h_0(x),\ldots,h_{r-1}(x)\rangle_{\mathbb{F}_{q^n}}$,
- $\mathcal{C}^\perp = \langle h'_0(x),\ldots,h'_{n-r-1}(x)\rangle_{\mathbb{F}_{q^n}}$,
- *the Delsarte dual of $U_{\mathcal{C}} = \{(h_0(x),\ldots,h_{r-1}(x)) : x \in \mathbb{F}_{q^n}\}$ is the $\mathbb{F}_q$-subspace $U_{\mathcal{C}^\perp} = \{(h'_0(x),\ldots,h'_{n-r-1}(x)) : x \in \mathbb{F}_{q^n}\}$.*

Proof. By Remark 4.11, up to equivalence, $\mathcal{C} = \langle h_0(x), \ldots, h_{r-1}(x)\rangle_{\mathbb{F}_{q^n}}$, for some $h_0(x), \ldots, h_{r-1}(x)$ as in (13), and $\mathcal{C}^\perp = \langle h'_0(x), \ldots, h'_{n-r-1}(x)\rangle_{\mathbb{F}_{q^n}}$, for some $h'_0(x), \ldots, h'_{n-r-1}(x)$ as in (14). Note that, since $\mathcal{C}$ is an MRD-code, the linearized polynomials $h_0(x), \ldots, h_{r-1}(x)$ have no common roots other than 0 since otherwise the code would not contain invertible maps, see e.g. [22, Lemma 2.1]. Our aim is to show that applying the duality introduced in Section 3 to $U_{\mathcal{C}} = \{(h_0(x), \ldots, h_{r-1}(x)) : x \in \mathbb{F}_{q^n}\}$ we get the $\mathbb{F}_q$-subspace $U_{\mathcal{C}^\perp} = \{(h'_0(x), \ldots, h'_{n-r-1}(x)) : x \in \mathbb{F}_{q^n}\}$. By Result 4.7 we have that $U_{\mathcal{C}}$ is a maximum $(r-1)$-scattered $\mathbb{F}_q$-subspace of $\mathbb{F}^r_{q^n}$. If $n > r$, i.e. $\mathcal{C}$ has minimum distance greater than one, we can embed $\Lambda = \langle U_{\mathcal{C}}\rangle_{\mathbb{F}_{q^n}}$ in $\mathbb{F}^n_{q^n}$ in such a way that

$$\Lambda = \left\{(x_0, x_1, \ldots, x_r, \ldots, x_{n-1}) \in \mathbb{F}^n_{q^n} : x_j = 0 \;\; j \notin \{t_0, \ldots, t_{r-1}\}\right\},$$

and hence the vector $(h_0(x), \ldots, h_{r-1}(x))$ of $U_{\mathcal{C}}$ is extended to the vector $(a_0, a_1, \ldots, a_{n-1})$ of $\mathbb{F}^n_{q^n}$ as follows

$$a_i = \begin{cases} h_i(x) & \text{if } i \in \{t_0, \ldots, t_{r-1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let Γ be the $\mathbb{F}_{q^n}$-subspace of $\mathbb{F}^n_{q^n}$ of dimension $n-r$ represented by the equations

$$\Gamma : \begin{cases} x_{t_0} = -\displaystyle\sum_{j \notin \{t_0, \ldots, t_{r-1}\}} g_{0,j} x_j \\ \vdots \\ x_{t_{r-1}} = -\displaystyle\sum_{j \notin \{t_0, \ldots, t_{r-1}\}} g_{r-1,j} x_j \end{cases}$$

and let $W = \{(x, x^q, \ldots, x^{q^{n-1}}) : x \in \mathbb{F}_{q^n}\}$. It can be seen that $\Gamma \cap W = \{\mathbf{0}\}$, otherwise the polynomials $h_0(x), \ldots, h_{r-1}(x)$ would have a common root. Also

$$U_{\mathcal{C}} = \langle W, \Gamma\rangle_{\mathbb{F}_q} \cap \Lambda.$$

Let $\beta \colon \mathbb{F}^n_{q^n} \times \mathbb{F}^n_{q^n} \to \mathbb{F}_{q^n}$ be the standard inner product, i.e. $\beta((\mathbf{x}, \mathbf{y})) = \sum_{i=0}^{n-1} x_i y_i$ where $\mathbf{x} = (x_0, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, \ldots, y_{n-1})$. Also, the restriction of β over $W \times W$ is $\beta\big|_{W\times W}((x, x^q, \ldots, x^{q^{n-1}}), (y, y^q, \ldots, y^{q^{n-1}})) = \mathrm{Tr}_{q^n/q}(xy)$. Furthermore, with respect to the orthogonal complement operation $\perp$ defined by β we have that

$$\Gamma^\perp : x_j = \sum_{\ell=0}^{r-1} g_{j,\ell} x_{t_\ell} \qquad j \notin \{t_0, \ldots, t_{r-1}\}.$$

Then the Delsarte dual $\bar{U}_{\mathcal{C}}$ of $U_{\mathcal{C}}$ is the $\mathbb{F}_q$-subspace $W+\Gamma^\perp$ of the quotient space $\mathbb{F}_{q^n}^n/\Gamma^\perp$ isomorphic to $U' := \langle W, \Gamma^\perp\rangle_{\mathbb{F}_q} \cap \Lambda'$, where Λ' is the $\mathbb{F}_{q^n}$-subspace of $\mathbb{F}_{q^n}^n$ of dimension $n-r$ represented by the following equations

$$\Lambda' : x_{t_0} = \ldots = x_{t_{r-1}} = 0.$$

By identifying Λ' with $\mathbb{F}_{q^n}^{n-r}$, direct computations show that U' can be seen as the $\mathbb{F}_q$-subspace $U_{\mathcal{C}^\perp} = \{(h_0'(x), \ldots, h_{n-r-1}'(x)) : x \in \mathbb{F}_{q^n}\}$ of dimension n of $\mathbb{F}_{q^n}^{n-r}$, i.e. $U' = U_{\mathcal{C}^\perp}$. $\square$

5 Intersections of maximum h-scattered subspaces with hyperplanes

This section is devoted to prove

Theorem 2.7 *If U is a maximum h-scattered $\mathbb{F}_q$-subspace of a vector space $V(r,q^n)$ of dimension $rn/(h+1)$, then for any $(r-1)$-dimensional $\mathbb{F}_{q^n}$-subspace W of $V(r,q^n)$ we have*

$$\frac{rn}{(h+1)} - n \leq \dim_{\mathbb{F}_q}(U \cap W) \leq \frac{rn}{(h+1)} - n + h.$$

As we already mentioned, the theorem above is a generalization of [4, Theorem 4.2], which is the $h=1$ case of our result. In that paper, the number of hyperplanes meeting a 1-scattered subspace of dimension $rn/2$ in a subspace of dimension $rn/2-n$ or $rn/2-n+1$ has been determined as well. Subsequently to this paper, in [29] (see also [23] for the $h=2$ case), such values have been determined for every h.

5.1 Preliminaries on Gaussian binomial coefficients

The Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined as the number of the k-dimensional subspaces of the n-dimensional vector space $\mathbb{F}_q^n$. Hence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} 1 & \text{if } k=0 \\ \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\ldots(1-q)} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n. \end{cases} \tag{15}$$

Recall the following properties of the Gaussian binomial coefficients.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q, \tag{16}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q . \tag{17}$$

$$\prod_{j=0}^{n-1}(1+q^j t) = \sum_{j=0}^{n} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q t^j, \tag{18}$$

Definition 5.1. *The q-Pochhammer symbol is defined as*

$$(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1}).$$

Theorem 5.2 (q-binomial theorem [15, pg. 25, Exercise 1.3 (i)])**.**

$$(ab;q)_n = \sum_{k=0}^{n} b^k \begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k (b;q)_{n-k}, \tag{19}$$

$$(ab;q)_n = \sum_{k=0}^{n} a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k (b;q)_{n-k}. \tag{20}$$

Corollary 5.3. *In* (19) *and* (20) *put $a = q^{-nr/s}$ and $b = q^{nr/s-n}$ to obtain*

$$(q^{-n};q)_s = \sum_{j=0}^{s} q^{j(nr/s-n)} \begin{bmatrix} s \\ j \end{bmatrix}_q (q^{-nr/s};q)_j (q^{nr/s-n};q)_{s-j}, \tag{21}$$

$$(q^{-n};q)_s = q^{-nr} \sum_{j=0}^{s} q^{jnr/s} \begin{bmatrix} s \\ j \end{bmatrix}_q (q^{-nr/s};q)_j (q^{nr/s-n};q)_{s-j}, \tag{22}$$

respectively.

The l-th elementary symmetric function of the variables $x_1, x_2, \dots, x_n$ is the sum of all distinct monomials which can be formed by multiplying together l distinct variables.

Definition 5.4. *Denote by $\sigma_{k,l}$ the l-th elementary symmetric polynomial in $k+1$ variables evaluated in $1, q, q^2, \dots, q^k$.*

Lemma 5.5 ([6, Proposition 6.7 (b)])**.**

$$\sigma_{k,l} = q^{l(l-1)/2} \begin{bmatrix} k+1 \\ l \end{bmatrix}_q .$$

We will also need the following q-binomial inverse formula of Carlitz.

Theorem 5.6 ([7, special case of Theorem 2, pg. 897 (4.2) and (4.3)])**.** *Suppose that $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are two sequences of complex numbers. If $a_k = \sum_{j=0}^{k}(-1)^j q^{j(j-1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q b_j$, then $b_k = \sum_{j=0}^{k}(-1)^j q^{j(j+1)/2-jk} \begin{bmatrix} k \\ j \end{bmatrix}_q a_j$ and vice versa.*

5.2 Double counting

Put $s = h+1 \mid rn$ and let U be an rn/s-dimensional $\mathbb{F}_q$-subspace of $V(r, q^n)$ such that for each $(s-1)$-dimensional $\mathbb{F}_{q^n}$-subspace W, we have $\dim_{\mathbb{F}_q}(W \cap U) \le s-1$.

Let h_i denote the number of $(r-1)$-dimensional $\mathbb{F}_{q^n}$-subspaces meeting U in an $\mathbb{F}_q$-subspace of dimension i. It is easy to see that

$$h_i = 0 \text{ for } i < \frac{rn}{s} - n.$$

In $\mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$, the integer h_i coincides with the number of hyperplanes $\mathrm{PG}(W, \mathbb{F}_{q^n})$ such that $\dim_{\mathbb{F}_q}(W \cap U) = i$. Also, the number of hyperplanes is $(q^{rn}-1)/(q^n-1)$, which is the same as $\sum_i h_i$, thus

$$\sum_i h_i (q^n - 1) = q^{rn} - 1. \tag{23}$$

For $k \in \{0, 1, \ldots, s-1\}$ we can double count the set

$$\{(H, (P_1, P_2, \ldots, P_{k+1})) : H \text{ is a hyperplane, } P_1, P_2, \ldots, P_{k+1} \in H \cap L_U \\ \text{and } \langle P_1, P_2, \ldots, P_{k+1} \rangle \cong \mathrm{PG}(k, q)\}.$$

By Proposition 2.1 this gives

$$\sum_i h_i \left(\frac{q^i - 1}{q-1}\right)\left(\frac{q^i - q}{q-1}\right)\cdots\left(\frac{q^i - q^k}{q-1}\right) =$$
$$\left(\frac{q^{rn/s} - 1}{q-1}\right)\left(\frac{q^{rn/s} - q}{q-1}\right)\cdots\left(\frac{q^{rn/s} - q^k}{q-1}\right)\left(\frac{q^{(r-k-1)n} - 1}{q^n-1}\right),$$

or equivalently

Lemma 5.7.
$$\sum_i h_i (q^n-1)(q^i-1)(q^i-q)(q^i-q^2)\ldots(q^i-q^k) =$$
$$(q^{rn/s}-1)(q^{rn/s}-q)(q^{rn/s}-q^2)\ldots(q^{rn/s}-q^k)(q^{(r-k-1)n}-1).$$

□

Our aim is to prove

$$A := \sum_i h_i (q^n-1)(q^i - q^{n(r-s)/s})\ldots(q^i - q^{n(r-s)/s+s-1}) = 0.$$

This would clearly yield $h_i = 0$, for $i > n(r-s)/s + s - 1$, and hence Theorem 2.7.

5.3 Expressing A

First for $k \in \{0, \dots, s-1\}$ we will express

$$\alpha_k := \sum_i h_i(q^n - 1)q^{ki}.$$

Put $\beta_0 := \alpha_0 = q^{rn} - 1$ (cf. (23)), and

$$\beta_k := \sum_i h_i(q^n - 1)(q^i - 1)(q^i - q)\dots(q^i - q^{k-1}),$$

where the values of β_k are known due to Lemma 5.7.
Recall

$$\sigma_{k,l} = \sum_{0 \le i_1 < \ldots < i_l \le k} q^{i_1 + \ldots + i_l}.$$

Then it is easy to see that

$$\alpha_k = \beta_k + \sum_{j=0}^{k-1} (-1)^{k-j-1} \alpha_j \sigma_{k-1,k-j},$$

and hence, using also Lemma 5.5,

$$\beta_k = \sum_{j=0}^{k} (-1)^{k-j} q^{(k-j)(k-j-1)/2} \alpha_j \begin{bmatrix} k \\ k-j \end{bmatrix}_q,$$

or equivalently, by (17),

$$\beta_k q^{-k(k-1)/2}(-1)^k = \sum_{j=0}^{k} q^{j(j+1)/2 - jk} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j \alpha_j.$$

Then Theorem 5.6 applied to the sequences $\{a_k = \alpha_k\}_k$ and $\{b_k = \beta_k q^{-k(k-1)/2}(-1)^k\}_k$ gives

$$\alpha_k = \sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q \beta_j q^{-j(j-1)/2}(-1)^j = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \beta_j. \qquad (24)$$

It is easy to see that

$$A = \sum_{j=0}^{s} (-1)^{s-j} \alpha_j q^{(s-j)n(r-s)/s} \sigma_{s-1,s-j}$$

and hence by Lemma 5.5

$$A = \sum_{j=0}^{s} (-1)^{s-j} \alpha_j q^{(s-j)n(r-s)/s+(s-j)(s-j-1)/2} \begin{bmatrix} s \\ s-j \end{bmatrix}_q . \qquad (25)$$

By Lemma 5.7 we have

$$\beta_k = (q^{(r-k)n} - 1) \prod_{j=0}^{k-1} (q^{rn/s} - q^j) =$$

$$(q^{(r-k)n} - 1) q^{k(k-1)/2} (-1)^k \prod_{j=0}^{k-1} (1 - q^{rn/s-j}).$$

By (18) with $t = -q^{rn/s-k+1}$

$$\prod_{j=0}^{k-1} (1-q^{rn/s-j}) = \prod_{j=0}^{k-1} (1-q^{rn/s-k+1} q^j) = \sum_{j=0}^{k} q^{j(j-1)/2+rnj/s-(k-1)j} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j,$$

thus

$$\beta_k = \sum_{j=0}^{k} (q^{(r-k)n} - 1) q^{k(k-1)/2} q^{j(j-1)/2+rnj/s-(k-1)j} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^{j+k} =$$

$$\sum_{j=0}^{k} (q^{(r-k)n} - 1) q^{(k-j)(k-j-1)/2+rnj/s} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^{j+k} =$$

$$\sum_{t=0}^{k} (q^{(r-k)n} - 1) q^{t(t-1)/2+rn(k-t)/s} \begin{bmatrix} k \\ t \end{bmatrix}_q (-1)^t .$$

Hence by (24) and (25)

$$A = \sum_{k=0}^{s} \sum_{j=0}^{k} \sum_{t=0}^{j} (q^{(r-j)n} - 1) q^{t(t-1)/2+rn(j-t)/s+(s-k)n(r-s)/s+(s-k)(s-k-1)/2} \begin{bmatrix} s \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} j \\ t \end{bmatrix}_q (-1)^{t+k+s} .$$

5.4 Proof of $A = 0$

Since q-binomial coefficients out of range are defined as zero, cf. (15), it is enough to prove that the following expression is zero:

$$\sum_{k=0}^{s}\sum_{j=0}^{s}\sum_{t=0}^{s}(q^{rn-jn}-1)q^{rnj/s-rnt/s+(s-k)n(r-s)/s+\frac{1}{2}(s-k-1)(s-k)+\frac{1}{2}(t-1)t}\begin{bmatrix}s\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}j\\t\end{bmatrix}_q(-1)^{t+k}.$$

It is clearly equivalent to prove $a_s = b_s$, where

$$a_s = \sum_{j=0}^{s} q^{nr-nj}\sum_{k=0}^{s}\sum_{t=0}^{s} q^{rnj/s-rnt/s+(s-k)n(r-s)/s+\frac{1}{2}(s-k-1)(s-k)+\frac{1}{2}(t-1)t}\begin{bmatrix}s\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}j\\t\end{bmatrix}_q(-1)^{t+k},$$

$$b_s = \sum_{j=0}^{s}\sum_{k=0}^{s}\sum_{t=0}^{s} q^{rnj/s-rnt/s+(s-k)n(r-s)/s+\frac{1}{2}(s-k-1)(s-k)+\frac{1}{2}(t-1)t}\begin{bmatrix}s\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}j\\t\end{bmatrix}_q(-1)^{t+k}.$$

Proposition 5.8.

$$a_s = q^{nr}(-1)^s(q^{-n};q)_s.$$

Proof. Clearly, it is enough to prove

$$(-1)^s(q^{-n};q)_s =$$
$$\sum_{j=0}^{s} q^{-nj}\sum_{k=0}^{s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q q^{nrj/s+(s-k)n(r-s)/s+\frac{1}{2}(-k+s-1)(s-k)}\sum_{t=0}^{s} q^{-nrt/s+\frac{1}{2}(t-1)t}\begin{bmatrix}j\\t\end{bmatrix}_q(-1)^t,$$

where by (18)

$$\sum_{t=0}^{s} q^{-nrt/s+\frac{1}{2}(t-1)t}\begin{bmatrix}j\\t\end{bmatrix}_q(-1)^t = (q^{-nr/s};q)_j,$$

thus the triple sum can be reduced to

$$\sum_{j=0}^{s} q^{nrj/s-nj}(q^{-nr/s};q)_j\sum_{k=0}^{s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q q^{(s-k)n(r-s)/s+\frac{1}{2}(-k+s-1)(s-k)}.$$

By (16) and (17) this can be written as

$$\sum_{j=0}^{s} q^{nrj/s-nj}(q^{-nr/s};q)_j\begin{bmatrix}s\\j\end{bmatrix}_q\sum_{k=0}^{s}(-1)^k\begin{bmatrix}s-j\\s-k\end{bmatrix}_q q^{(s-k)n(r-s)/s+\frac{1}{2}(-k+s-1)(s-k)}, \tag{26}$$

where again by (18)

$$\sum_{k=0}^{s}(-1)^k \begin{bmatrix} s-j \\ s-k \end{bmatrix}_q q^{(s-k)n(r-s)/s+\frac{1}{2}(-k+s-1)(s-k)} =$$

$$(-1)^s \sum_{z=0}^{s}(-1)^z \begin{bmatrix} s-j \\ z \end{bmatrix}_q q^{zn(r-s)/s+z(z-1)/2} =$$

$$(-1)^s(q^{nr/s-n};q)_{s-j}.$$

By (26) we have

$$(-1)^s \sum_{j=0}^{s} q^{j(nr/s-n)} \begin{bmatrix} s \\ j \end{bmatrix}_q (q^{-nr/s};q)_j(q^{nr/s-n};q)_{s-j},$$

which by (21) equals $(-1)^s(q^{-n};q)_s$. □

Proposition 5.9.

$$b_s = q^{nr}(-1)^s(q^{-n};q)_s.$$

Proof. As before, b_s can be written as

$$q^{nr}(-1)^s \sum_{j=0}^{s} q^{jnr/s-nr} \begin{bmatrix} s \\ j \end{bmatrix}_q (q^{-nr/s};q)_j(q^{nr/s-n};q)_{s-j}.$$

Then the assertion follows from (22). □

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