

Scalar q -subresultants and Dickson matrices

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Abstract

Following the ideas of Ore and Li we study q -analogues of scalar subresultants and show how these results can be applied to determine the rank of an \mathbb{F}_q -linear transformation f of \mathbb{F}_{q^n} . As an application we show how certain minors of the Dickson matrix $D(f)$, associated with f , determine the rank of $D(f)$ and hence the rank of f .

Keywords: Dickson matrix, subresultant, linearized polynomial

1 Introduction

Let $f(x) = \sum_{i=0}^k a_i x^i$ and $g(x) = \sum_{i=0}^l b_i x^i$, with $a_k b_l \neq 0$, be two univariate polynomials with coefficients in the field \mathbb{K} ¹. In elimination theory, the classical resultant of f and g is

$$\text{Res}(f, g) = (-1)^{kl} b_l^k \prod_{i=1}^l f(\xi_i),$$

where $g(x) = b_l \prod_{i=1}^l (x - \xi_i)$ with $\xi_1, \xi_2, \dots, \xi_l \in \overline{\mathbb{K}}$ (where $\overline{\mathbb{K}}$ denotes the algebraic closure of \mathbb{K}). For $0 \leq m \leq \min\{k, l\}$ consider the following

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¹Note that in many of the cited literature a_0 and b_0 are used to denote the leading coefficients of f and g .

$(k + l - 2m) \times (k + l - 2m)$ matrix:

$$R_m(f, g) := \begin{pmatrix} a_k & a_{k-1} & a_{k-2} & \dots & a_{k-l+m+1} & \dots & a_{2m-l+2} & a_{2m-l+1} \\ 0 & a_k & a_{k-1} & \dots & a_{k-l+m+2} & \dots & a_{2m-l+3} & a_{2m-l+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & a_k & \dots & a_{m+1} & a_m \\ b_l & b_{l-1} & b_{l-2} & \dots & \dots & \dots & b_{2m-k+2} & b_{2m-k+1} \\ 0 & b_l & b_{l-1} & \dots & \dots & \dots & b_{2m-k+3} & b_{2m-k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & b_l & \dots & \dots & b_{m+1} & b_m \end{pmatrix},$$

where coefficients out of range are considered to be 0.

The determinant of $R_m(f, g)$ is also called the m -th *scalar subresultant* of f and g . Note that $|R_0(f, g)| = \text{Res}(f, g)$ and hence $\gcd(f, g) = 1$ if and only if $|R_0(f, g)| \neq 0$. This result has the following well-known generalization in elimination theory. For a proof we cite here the Appendix of [10] and the references therein, since the proof of Theorem 2.1 was motivated by the arguments found there.

Result 1.1. *The degree of $\gcd(f, g)$ is t if and only if $|R_0(f, g)| = \dots = |R_{t-1}(f, g)| = 0$ and $|R_t(f, g)| \neq 0$.*

The strength of the Result 1.1 is that it provides a way to study the number of common roots of f and g only by means of their coefficients.

Now let \mathbb{K} be a field of characteristic p , and let q be a power of p . A q -polynomial over \mathbb{K} with q -degree m is a polynomial of the form $f(x) = \sum_{i=0}^m a_i x^{q^i}$, with $a_m \neq 0$ and $a_0, a_1, \dots, a_m \in \mathbb{K}$. When $q = p$ prime, q -analogue of the classical resultant for q -polynomials was already mentioned in [14, Chapter 1, Section 7], however, an explicit formula was not given there. An explicit formula can be found for example in [17, page 59].

The subresultant theory was extended to Ore polynomials (cf. [15]) and hence also to the non-commutative ring of q -polynomials by Li in [11]. Here the non-commutative operation between two q -polynomials is composition, while addition is defined as usual. Note that this ring is a right-Euclidean domain with respect to the q -degree, cf. [14]. When $g = f \circ h$ then we will also say that h is a symbolic right divisor of g . Note that in the paper of Li the word subresultant is used to what is also known as *polynomial subresultant*. In the classical theory the m -th scalar subresultant is the leading coefficient of the m -th polynomial subresultant. See for example [1, Section 2] for a brief summary, where $S_m^{(m)}$ corresponds to what we (and

some other authors) call scalar subresultant. For the various notions consult with [9].

Let $\mathbb{K} = \mathbb{F}_{q^n}$ and consider \mathbb{K} as an n -dimensional vector space over \mathbb{F}_q . Then there is an isomorphism between the ring of q -polynomials

$$\left\{ \sum_{i=0}^{n-1} a_i x^{q^i} : a_0, \dots, a_{n-1} \in \mathbb{F}_{q^n} \right\}$$

considered modulo $(x^{q^n} - x)$ and the ring of \mathbb{F}_q -linear transformations of \mathbb{F}_{q^n} . The set of roots of a q -polynomial form an \mathbb{F}_q -subspace and the dimension of this subspace is the dimension of the kernel of the corresponding \mathbb{F}_q -linear transformation. Thus $\deg \gcd(f(x), x^{q^n} - x) = q^{n-k}$, where k is the rank of the \mathbb{F}_q -linear transformation of \mathbb{F}_{q^n} defined by $f(x)$. When n is clear from the context, then we will say that k is the *rank* of f .

Result 1.2 (Ore [14, Theorem 2]). *The greatest common symbolic right divisor of two q -polynomials is the same as their ordinary greatest common divisor.*

It follows that the q -subresultant theory can be applied to determine $\gcd(f(x), x^{q^n} - x)$ and hence the rank of f . Our contribution to this theory is a direct proof to a q -analogue of Result 1.1 providing sufficient and necessary conditions which ensure that f has rank $n - k$ (cf. Theorem 2.1).

Recall that the Dickson matrix associated with $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ is

$$D(f) := \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1}^q & a_0^q & \dots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \dots & a_0^{q^{n-1}} \end{pmatrix}.$$

It is well-known that the rank of f equals the rank of $D(f)$, see for example [18, Proposition 4.4] or [13, Proposition 5]. In some recent constructions of maximum scattered subspaces and MRD-codes it was crucial to determine the rank of certain Dickson matrices (cf. [6, Section 7] and [7, Section 5]). In these papers this was done by considering certain minors of such matrices and excluding the possibility that their determinants vanish at the same time. On the other hand, in [4, Section 3] Dickson matrices were used to prove non-existence results of certain MRD-codes. This was done by proving that, for a certain choice of the parameters, all 6×6 submatrices of a 9×9 Dickson matrix have zero determinant. As an application of Theorem

2.1 we show that it is enough to investigate the nullity of the determinant of at most $k + 1$ well-defined minors to decide whether f has rank $n - k$. This result can significantly simplify the above mentioned arguments.

To state here the main result of this paper we introduce the notion $D_m(f)$ to denote the $(n - m) \times (n - m)$ matrix obtained from $D(f)$ after removing its first m columns and last m rows. Our main result is the following.

Theorem 1.3. $\dim_q(\ker f) = \mu$ if and only if

$$|D_0(f)| = |D_1(f)| = \dots = |D_{\mu-1}(f)| = 0 \quad (1)$$

and $|D_\mu(f)| \neq 0$.

Results in a similar direction have been obtained recently in [5] where for each q -polynomial f of q -degree k , k conditions were given, in terms of the coefficients of f , which are satisfied if and only if f has rank $n - k$ (there is a hidden $(k + 1)$ -th condition here as well, namely the assumption that the coefficient of x^{q^k} in f is non-zero). Independently, in [16] it was proved that the rank of f is $n - m$ if and only if a certain $k \times k$ matrix has rank $k - m$. If $m = k$, then this result gives back the main result of [5].

2 Scalar q -subresultants

Consider $f(x) = \sum_{i=0}^k a_i x^{q^i}$ and $g(x) = \sum_{i=0}^l b_i x^{q^i}$, two q -polynomials with coefficients in $\overline{\mathbb{F}}_q$ such that $a_k b_l \neq 0$. Put

$$q^\mu = \deg \gcd(f, g).$$

By Result 1.2, μ also equals the q -degree of the symbolic greatest common right divisor of f and g .

For $m \leq \min\{k, l\}$ we define the $(k + l - 2m) \times (k + l - 2m)$ matrix $R_{m,q}(f, g)$ as follows:

$$\begin{pmatrix} a_k^{q^{l-m-1}} & a_{k-1}^{q^{l-m-1}} & a_{k-2}^{q^{l-m-1}} & \dots & a_{k+m-l+1}^{q^{l-m-1}} & \dots & a_{2m-l+2}^{q^{l-m-1}} & a_{2m-l+1}^{q^{l-m-1}} \\ 0 & a_k^{q^{l-m-2}} & a_{k-1}^{q^{l-m-2}} & \dots & a_{k+m-l+2}^{q^{l-m-2}} & \dots & a_{2m-l+3}^{q^{l-m-2}} & a_{2m-l+2}^{q^{l-m-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & a_k & \dots & a_{m+1} & a_m \\ b_l^{q^{k-m-1}} & b_{l-1}^{q^{k-m-1}} & b_{l-2}^{q^{k-m-1}} & \dots & \dots & \dots & b_{2m-k+2}^{q^{k-m-1}} & b_{2m-k+1}^{q^{k-m-1}} \\ 0 & b_l^{q^{k-m-2}} & b_{l-1}^{q^{k-m-2}} & \dots & \dots & \dots & b_{2m-k+3}^{q^{k-m-2}} & b_{2m-k+2}^{q^{k-m-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & b_l & \dots & \dots & b_{m+1} & b_m \end{pmatrix}.$$

Note that $R_{m+1,q}(f, g)$ is obtained from $R_{m,q}(f, g)$ by removing its first and last columns, and its first and $(l - m + 1)$ -th rows.

We state here the q -analogue of Result 1.1.

Theorem 2.1. *The q -degree of $\gcd(f, g)$ is μ if and only if $|R_{0,q}(f, g)| = \dots = |R_{\mu-1,q}(f, g)| = 0$ and $|R_{\mu,q}(f, g)| \neq 0$.*

We prove this result directly by following the proof of the classical Result 1.1. Theorem 2.1 will easily follow from Proposition 2.3.

Proposition 2.2. *Recall $q^\mu = \deg \gcd(f, g)$ and let $m \leq \mu$. Let $c(x) = \sum_{i=0}^{k-m} c_i x^{q^i}$ and $d(x) = \sum_{i=0}^{l-m} d_i x^{q^i}$ be q -polynomials over $\overline{\mathbb{F}_q}$ with $c_{k-m} = a_k^{q^{l-m}}$, $d_{l-m} = b_l^{q^{k-m}}$, and their other coefficients are considered as unknowns. Then the set of solutions for these coefficients such that*

$$d \circ f - c \circ g = 0 \quad (2)$$

form a $(\mu - m)$ -dimensional affine $\overline{\mathbb{F}_q}$ -space.

Proof. First assume that f and g have only simple roots.

Let r be the greatest common monic symbolic right divisor of f and g and suppose that (2) holds for some c and d . Then $f = f_1 \circ r$ and $g = g_1 \circ r$ and (2) yields $d \circ f_1 = c \circ g_1$, thus d is zero on $f_1(\ker g_1)$ (in this proof the kernel is always taken over $\overline{\mathbb{F}_q}$) and c is zero on $g_1(\ker f_1)$. Since the greatest common symbolic right divisor of f_1 and g_1 is the identity map, it follows that $\gcd(f_1, g_1) = x$ and hence $\ker f_1 \cap \ker g_1 = \{0\}$. Thus $\dim_q f_1(\ker g_1) = \dim_q \ker g_1 = l - \mu$ and similarly $\dim_q g_1(\ker f_1) = k - \mu$. It follows that the unique q -polynomial d_1 of q -degree $l - \mu$ and with leading coefficient $b_l^{q^{k-\mu}}$ which vanishes on $f_1(\ker g_1)$ is a divisor of d . By Result 1.2 $\gcd(d, d_1) = d_1$ is also a symbolic right divisor of d , i.e. $d = d_2 \circ d_1$, for some monic d_2 with q -degree $(\mu - m)$. Similarly, the unique q -polynomial c_1 of q -degree $k - \mu$ and with leading coefficient $a_k^{q^{l-\mu}}$ which vanishes on $g_1(\ker f_1)$ is a symbolic right divisor of c , i.e. $c = c_2 \circ c_1$, for some monic c_2 with q -degree $(\mu - m)$.

Note that

$$d_1 \circ f_1 - c_1 \circ g_1$$

has q -degree $k + l - 2\mu - 1$ (the coefficient of $x^{q^{k+l-2\mu}}$ vanishes because of the assumptions on the leading coefficients of c and d) and it vanishes on $\ker f_1 \oplus \ker g_1$. Thus it is the zero polynomial.

Then

$$c_2 \circ c_1 \circ g_1 = c \circ g_1 = d \circ f_1 = d_2 \circ d_1 \circ f_1 = d_2 \circ c_1 \circ g_1$$

and hence $c_2 = d_2$. On the other hand, if $c_2 = d_2$, then we clearly have a solution since (2) becomes $d_2 \circ (d_1 \circ f_1 - c_1 \circ g_1) \circ r$ with the zero polynomial in the middle.

Since we can choose the first $(\mu - m)$ coefficients of $d_2(x) = \sum_{i=0}^{\mu-m} \hat{d}_i x^{q^i}$ arbitrarily, the assertion follows. More precisely, if $d_1(x) = \sum_{j=0}^{\mu-m} \bar{d}_j x^{q^j}$ with $\bar{d}_{l-\mu} = b_l^{q^{k-\mu}}$ and with coefficients out of range defined as 0, then $d(x)$ is of the form

$$\sum_{i=0}^{k-m} \sum_{j=0}^i \hat{d}_{i-j} \bar{d}_j^{q^{i-j}} x^{q^i},$$

with $\hat{d}_k \in \overline{\mathbb{F}_q}$ for $0 \leq k \leq \mu - m - 1$, $\hat{d}_{\mu-m} = 1$ and $\hat{d}_l = 0$ for $l > \mu - m$. These polynomials form a $(\mu - m)$ -dimensional affine $\overline{\mathbb{F}_q}$ -space and as we have seen, any such $d(x)$ uniquely defines a $c(x)$ for which (2) holds.

Now consider the case when f and g may have multiple roots. Let $f = x^{q^{k_1}} \circ \tilde{f}$ and $g = x^{q^{l_1}} \circ \tilde{g}$ where \tilde{f} and \tilde{g} have only simple roots. W.l.o.g. assume $l_1 \leq k_1$. We want to find the dimension of the solutions of

$$d \circ x^{q^{k_1}} \circ \tilde{f} = c \circ x^{q^{l_1}} \circ \tilde{g},$$

under the given assumptions on the degrees and leading coefficients of c and d . Clearly, the multiplicities of the roots of the left hand side and the right hand side have to coincide and hence $c = c' \circ x^{q^{k_1-l_1}}$. Let \tilde{d} and \tilde{c}' denote the q -polynomials whose coefficients are the q^{-k_1} -th roots of the coefficients of d and c , respectively. Then the solutions of the previous system correspond to the solutions of

$$x^{q^{k_1}} \circ \tilde{d} \circ \tilde{f} = x^{q^{k_1}} \circ \tilde{c}' \circ \tilde{g}$$

and hence to those of

$$\tilde{d} \circ \tilde{f} = \tilde{c}' \circ \tilde{g},$$

where the q -degree of \tilde{d} is $(l - l_1) - (m - l_1)$ and the q -degree of \tilde{c}' is $(k - k_1) - (m - l_1)$. The roots of the q -polynomials \tilde{f} and \tilde{g} are simple, thus we can apply the first part of this proof for these polynomials. The leading coefficients of \tilde{d} and \tilde{c}' are $b_l^{q^{k-m-k_1}}$ and $a_k^{q^{l-m-k_1}}$, respectively; the leading coefficients of \tilde{f} and \tilde{g} are $a_k^{q^{-k_1}}$ and $b_l^{q^{-l_1}}$, respectively. Since $b_l^{q^{k-m-k_1}} = b_l^{q^{-l_1} \deg \tilde{c}'}$ and $a_k^{q^{l-m-k_1}} = a_k^{q^{-k_1} \deg \tilde{d}}$, the conditions on the leading coefficients also hold. Note that the q -degree of $\gcd(\tilde{f}, \tilde{g})$ is $\mu - l_1$. Then the dimension of the solutions of this system is $(\mu - l_1) - (m - l_1) = \mu - m$. \square

Proposition 2.3. *Suppose $m \leq \mu$. Then the nullity of the matrix $R_{m,q}(f, g)$ is $\mu - m$.*

Proof. Let f, g, c, d be defined as before, then

$$\begin{aligned} d \circ f - c \circ g &= \sum_{i=0}^{l-m} d_i \sum_{j=0}^k a_j^{q^i} x^{q^{j+i}} - \sum_{i=0}^{k-m} c_i \sum_{j=0}^l b_j^{q^i} x^{q^{j+i}} = \\ &= \sum_{i=0}^{k+l-m} \left(\sum_{j=0}^i d_{i-j} a_j^{q^{i-j}} - c_{i-j} b_j^{q^{i-j}} \right) x^{q^i}. \end{aligned}$$

The q -degree of $r := \gcd(f, g)$ is $\mu \geq m$ and $r \mid d \circ f - c \circ g$, thus d and c form a solution to $d \circ f - c \circ g = 0$ if and only if the q -degree of $d \circ f - c \circ g$ is less than m . In another words, we only have to concentrate on the coefficients of terms with q -degree $i \in \{m, m+1, \dots, k+l-m\}$ in $d \circ f - c \circ g$.

Note that the coefficient of q^{k+l-m} is $d_{l-m} a_k^{q^{l-m}} - c_{k-m} b_l^{q^{k-m}}$ (coefficients out of range are considered to be 0), which is 0 because of our assumptions on c and d . Now let

$$\mathbf{v} = (d_{l-m-1}, d_{l-m-2}, \dots, d_0, -c_{k-m-1}, -c_{k-m-2}, \dots, -c_0)$$

and

$$\mathbf{b} = (b_l^{q^{k-m}} a_{k-1}^{q^{l-m}} - a_k^{q^{l-m}} b_{l-1}^{q^{k-m}}, \dots, b_l^{q^{k-m}} a_{2m-l}^{q^{l-m}} - a_k^{q^{l-m}} b_{2m-k}^{q^{k-m}}).$$

We claim that

$$\mathbf{v} R_{m,q}(f, g) = -\mathbf{b} \quad (3)$$

holds if and only if

$$\sum_{j=0}^i d_{i-j} a_j^{q^{i-j}} - c_{i-j} b_j^{q^{i-j}} = 0 \quad (4)$$

for all $m \leq i \leq k+l-m-1$. To see this we show that the $(k+l-2m-t)$ -th coordinates in the vectors at the left and right hand side of (3) coincide if and only if (4) holds with $i = m+t$. Indeed, in

$$\sum_{j=0}^{m+t} d_{m+t-j} a_j^{q^{m+t-j}} - c_{m+t-j} b_j^{q^{m+t-j}} \quad (5)$$

$d_{m+t-j} \neq 0$ only if $j \in \{m+t, m+t-1, \dots, 2m+t-l\}$ and $c_{m+t-j} \neq 0$ only if $j \in \{m+t, m+t-1, \dots, 2m+t-k\}$. Thus, after changing indices in the summation, (5) equals

$$\sum_{j=0}^{l-m} d_{l-m-j} a_{2m+t-l+j}^{q^{l-m-j}} - \sum_{j=0}^{k-m} c_{k-m-j} b_{2m+t-k+j}^{q^{k-m-j}}. \quad (6)$$

Since $d_{l-m} = b_l^{q^{k-m}}$ and $c_{k-m} = a_k^{q^{l-m}}$, the $(k+l-2m-t)$ -th coordinates on the left and right hand side of (3) coincide if and only if

$$\sum_{j=0}^{l-m-1} d_{l-m-1-j} a_{2m-l+1+t+j}^{q^{l-m-1-j}} - \sum_{s=0}^{k-m-1} c_{k-m-1-s} b_{2m-k+1+t+j}^{q^{k-m-1-s}} =$$

$$d_{l-m} a_{2m-l+t}^{q^{l-m}} - c_{k-m} b_{2m-k+t}^{q^{k-m}},$$

and this happens if and only if (6) equals zero.

Thus the dimension of the kernel of the $\overline{\mathbb{F}_q}$ -linear transformation of $\overline{\mathbb{F}_q}^{k+l-2m}$ defined by $\mathbf{x} \mapsto \mathbf{x} R_{m,q}(f, g)$ is the same as the dimension of the set of solutions of (2) and this finishes the proof. \square

Corollary 2.4. *Let f be a q -polynomial over \mathbb{F}_{q^n} and put $g(x) = x^{q^n} - x$. Then $\dim_q(\ker f) = \mu$ if and only if*

$$|R_{0,q}(f, g)| = |R_{1,q}(f, g)| = \dots = |R_{\mu-1,q}(f, g)| = 0 \quad (7)$$

and $|R_{\mu,q}(f, g)| \neq 0$.

As an illustration, the $(n+k) \times (n+k)$ matrix $R_{0,q}(f, g)$ in the particular case when $g(x) = x^{q^n} - x$ and $f(x) = \sum_{i=0}^k a_i x^{q^i}$ has the following form:

$$\begin{pmatrix} a_k^{q^{n-1}} & a_{k-1}^{q^{n-1}} & \dots & a_0^{q^{n-1}} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_k^{q^{n-2}} & \dots & a_1^{q^{n-2}} & a_0^{q^{n-2}} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & a_k & a_{k-1} & \dots & a_0 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 & -1 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & -1 \end{pmatrix}.$$

The matrix $R_{m,q}(f, g)$ can be obtained from $R_{0,q}(f, g)$ by removing its first and last m columns and its first m rows together with the $(n+1)$ -th, $(n+2)$ -th, \dots , $(n+m)$ -th rows.

Let $\tilde{f}(x) = \sum_{i=0}^{k-1} a_i x^{q^i}$ and $g(x) = x^{q^n} - x$. If we substitute $a_k = 0$ in $R_{m,q}(f, g)$, then its determinant equals either $|R_{m,q}(\tilde{f}, g)|$ or $-|R_{m,q}(\tilde{f}, g)|$. This argument can be iterated and hence one can use Corollary 2.4 even if the q -degree of f is not known, by considering the $(2n-1-2m) \times (2n-1-2m)$ m -th scalar q -subresultants of $\sum_{i=0}^{n-1} a_i x^{q^i}$ and $g(x)$.

3 A connection with Dickson matrices

In this section we prove Theorem 1.3 but before that we need some preparation.

Result 3.1 (Schur's determinant identity, [3]). *Consider the square matrix*

$$M := \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where W is also square and invertible. Then $\det(M) = \det(W) \det(X - YW^{-1}Z)$.

Corollary 3.2. *Consider the square matrices*

$$M := \begin{pmatrix} A & B & C \\ I_l & O & -I_l \end{pmatrix},$$

$$N := \begin{pmatrix} B & A + C \end{pmatrix},$$

where A and C are $k \times l$ matrices, B is $k \times (k-l)$, I_l denotes the $l \times l$ identity matrix and O is the $l \times (k-l)$ zero matrix. Then $\det(M) = (-1)^{l(k-l+1)} \det(N)$.

Proof. Result 3.1 with $X = \begin{pmatrix} A & B \end{pmatrix}$, $Y = C$, $Z = \begin{pmatrix} I_l & O \end{pmatrix}$ and $W = -I_l$ gives

$$\det(M) = \det(-I_l) \det\left(\begin{pmatrix} A & B \end{pmatrix} + C \begin{pmatrix} I_l & O \end{pmatrix}\right) = (-1)^l \det\begin{pmatrix} A + C & B \end{pmatrix}.$$

The result follows since N can be obtained from $\begin{pmatrix} A + C & B \end{pmatrix}$ by $l(k-l)$ column changes. \square

Let us introduce the abbreviation

$$R_m(f) := R_{m,q}(f, g),$$

where $g(x) = x^{q^n} - x$ and $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ for some $a_i \in \mathbb{F}_{q^n}$.

Lemma 3.3. $|D_m(f)| = |R_m(f)|$.

Proof. Note that $D_{n-1}(f) = R_{n-1}(f) = (a_{n-1})$, so we may assume $m < n - 1$. Let T_k denote the $k \times k$ anti-diagonal matrix whose non-zero entries equal to one and let I_k denote the $k \times k$ identity matrix. By O we will always denote a zero matrix whose dimension will be clear from the context. We distinguish two cases.

If $m \geq (n - 1)/2$, then $2n - 1 - 2m \leq n$ and hence $R_m(f)$ has the form:

$$\begin{pmatrix} A & B \\ I_{n-1-m} & O \end{pmatrix},$$

where $B = T_{n-m}D_m(f)T_{n-m}$. We have

$$\left| \begin{pmatrix} A & B \\ I_{n-1-m} & O \end{pmatrix} \right| = (-1)^{(n-m-1)(n-m)} \left| \begin{pmatrix} B & A \\ O & I_{n-1-m} \end{pmatrix} \right|,$$

and hence by Result 3.1

$$|R_m(f)| = |B| = |D_m(f)|.$$

If $m < (n - 1)/2$, then first consider the last m rows of $R_m(f)$: for $k \in \{0, 1, \dots, m - 1\}$ the $(2n - 2m - 1 - k)$ -th row of $R_m(f)$ contains only one non-zero entry, namely, a 1 at position $n - 1 - m - k$. Then it is easy to see by row expansion applied to the last m rows that:

$$(-1)^{(n-1)m} |R_m(f)| = \left| \begin{pmatrix} A & B & C \\ I_{n-2m-1} & O & -I_{n-2m-1} \end{pmatrix} \right|,$$

where A and C are $(n - m) \times (n - 2m - 1)$ matrices and

$$\begin{pmatrix} B & A + C \end{pmatrix} = T_{n-m}D_m(f)T_{n-m}.$$

According to Corollary 3.2,

$$(-1)^{(n-1)m} |R_m(f)| = (-1)^{(n-2m-1)(m+2)} |T_{n-m}D_m(f)T_{n-m}|,$$

which proves the assertion. □

Lemma 3.3 immediately yields Theorem 1.3.

For some s with $\gcd(s, n) = 1$ put $\sigma := q^s$. The set of σ -polynomials over \mathbb{F}_{q^n} is isomorphic to the skew-polynomial ring $\mathbb{F}_{q^n}[t, \sigma]$ where $t\alpha = \alpha^\sigma t$ for

all $\alpha \in \mathbb{F}_{q^n}$. Analogies for some of the results of Section 2 should hold in these non-commutative polynomial rings as well. Next we show a generalization of Theorem 1.3 for σ -polynomials.

Consider the σ -polynomial $f(x) := \sum_{i=0}^{n-1} a_i x^{\sigma^i} \in \mathbb{F}_{q^n}[x]$, which is also a q -polynomial. As before, by $\ker f$ we will denote $\gcd(f(x), x^{q^n} - x)$ and similarly to $D(f)$ we define

$$D_\sigma(f) := \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1}^\sigma & a_0^\sigma & \dots & a_{n-2}^\sigma \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{\sigma^{n-1}} & a_2^{\sigma^{n-1}} & \dots & a_0^{\sigma^{n-1}} \end{pmatrix}.$$

We will denote by $D_{m,\sigma}(f)$ the $(n-m) \times (n-m)$ matrix obtained from $D_\sigma(f)$ after removing its first m columns and last m rows. Because of the applications it might be useful to have conditions on other minors of $D_\sigma(f)$. In the next corollary we show some results also in this direction.

Corollary 3.4. *If $f(x) = \sum_{i=0}^{n-1} a_i x^{\sigma^i} \in \mathbb{F}_{q^n}[x]$ with $\gcd(s, n) = 1$, then $\dim_q(\ker f) = \mu$ if and only if*

$$|D_{0,\sigma}(f)| = |D_{1,\sigma}(f)| = \dots = |D_{\mu-1,\sigma}(f)| = 0 \quad (8)$$

and $|D_{\mu,\sigma}(f)| \neq 0$.

Index the rows and columns of $D_\sigma(f)$ from 0 to $n-1$. For $0 \leq m \leq \dim_q(\ker f)$ if $J, K \subseteq \{0, 1, \dots, n-1\}$ are two sets of m consecutive integers modulo n then let $M_{J,K}(f)$ denote the $(n-m) \times (n-m)$ matrix obtained from $D_\sigma(f)$ after removing its rows and columns with indices in J and K , respectively. Then

$$|M_{J,K}(f)| = 0 \Leftrightarrow |D_{m,\sigma}(f)| = 0.$$

Proof. Consider f as a q -polynomial with $\dim_q(\ker f) = \mu$. This happens if and only if $D(f)$ has rank μ . Recall that rows and columns of $D(f)$ are indexed from 0 to $n-1$ and let P denote the permutation matrix for which the i -th row of PA is the si -th row of A (considered modulo n). Then $PAP^{-1} = D_\sigma(f)$ and hence the rank of $D_\sigma(f)$ is the same as the rank of $D(f)$ (cf. also [8, Remark 2.3]). Note that $D_\sigma(f)$ is the Dickson matrix of a σ -polynomial considered as an \mathbb{F}_σ -linear transformation of \mathbb{F}_{σ^n} with kernel a μ -dimensional \mathbb{F}_σ -subspace of \mathbb{F}_{σ^n} . By Theorem 1.3 this happens if and only if the conditions on $|D_{m,\sigma}(f)|$ holds for $0 \leq m \leq \mu$.

For the second part take $0 \leq m \leq \dim_q(\ker f)$. Note that for any σ -polynomial $g(x) = \sum_{i=0}^{n-1} b_i x^{\sigma^i} \in \mathbb{F}_{q^n}[x]$ and for any non-negative integer t the rank of $g(x)$ is the same as

1. the rank of $g(x)^{\sigma^t}$ considered modulo $x^{q^n} - x$,
2. the rank of $\hat{g}(x) := \sum_{i=0}^{n-1} b_{n-i}^{\sigma^i} x^{\sigma^i}$ (since $D_{\sigma}(g)^T = D_{\sigma}(\hat{g})$, where by T we denote matrix transposition).

Suppose $J = \{j, j+1, \dots, j+m-1\}$ and $K = \{k, k+1, \dots, k+m-1\}$ considered modulo n . Then $f_1(x) := f(x)^{\sigma^{n-k-m}}$ modulo $x^{q^n} - x$ has the same rank as $f(x)$ and $|M_{J,K'}(f_1)| = |M_{J,K}(f)|^{\sigma^{n-k-m}}$ where $K' = \{n-m, m+1, \dots, n-1\}$. Then $\hat{f}_1(x)$ has the same rank as $f_1(x)$ and $|M_{K',J}(\hat{f}_1)| = |M_{J,K'}(f_1)|$. Finally, $f_2(x) := \hat{f}_1(x)^{\sigma^{n-j}}$ modulo $x^{q^n} - x$ has the same rank as $\hat{f}_1(x)$ and $|M_{K',J'}(f_2)| = |M_{K',J}(\hat{f}_1)|^{\sigma^{n-j}}$ where $J' = \{0, 1, \dots, m-1\}$. By definition $M_{K',J'}(f_2) = D_{m,\sigma}(f_2)$, and hence

$$|D_{m,\sigma}(f_2)| = 0 \Leftrightarrow |M_{K',J}(\hat{f}_1)| = 0 \Leftrightarrow |M_{J,K'}(f_1)| = 0 \Leftrightarrow |M_{J,K}(f)| = 0.$$

Recall $0 \leq m \leq \dim_q(\ker f)$. Since f_2 and f has the same rank, it follows from the first part of the assertion that $|D_{m,\sigma}(f_2)| = 0 \Leftrightarrow |D_{m,\sigma}(f)| = 0$ and this finishes the proof. \square

3.1 Applications

A q -polynomial $f(x) \in \mathbb{F}_{q^n}[x]$ is called *scattered* if $\{f(x)/x : x \in \mathbb{F}_{q^n} \setminus \{0\}\}$ (the *set of directions determined by the graph of f*) has maximum size, that is $(q^n - 1)/(q - 1)$. Put $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$, which is an n -dimensional \mathbb{F}_q -subspace of $\mathbb{F}_{q^n}^2$. The *linear set* of $\text{PG}(1, q^n)$ defined by f is the set of projective points $L_f := \{\langle (x, f(x)) \rangle_{q^n} : x \in \mathbb{F}_{q^n} \setminus \{0\}\}$. The *weight* of a point $\langle (a, b) \rangle_{\mathbb{F}_{q^n}} \in \text{PG}(1, q^n)$ w.r.t. the \mathbb{F}_q -subspace U_f is $\dim_q \langle (a, b) \rangle_{\mathbb{F}_{q^n}} \cap U_f$. The polynomial f is scattered if and only if the points of L_f have weight 1. In this case L_f and U_f are called *maximum scattered*. This happens if and only if the \mathbb{F}_q -linear transformations of \mathbb{F}_{q^n} in the \mathbb{F}_{q^n} -subspace $M := \langle x, f(x) \rangle_{\mathbb{F}_{q^n}}$ have rank at least $n - 1$. Equivalently, M is equivalent to an \mathbb{F}_{q^n} -linear maximum rank distance (MRD for short) code of $\mathbb{F}_q^{n \times n}$ with minimum distance $n - 1$. For more details about these objects and the relations among them we refer to [16, Section 13.3.6] and the references therein.

Corollary 3.5. *Consider the q -polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ and with y as a variable consider the matrix*

$$H := \begin{pmatrix} y & a_1 & \dots & a_{n-1} \\ a_{n-1}^q & y^q & \dots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \dots & y^{q^{n-1}} \end{pmatrix}.$$

The determinant of the $(n - m) \times (n - m)$ matrix obtained from H after removing its first m columns and last m rows is a polynomial $H_m(y) \in \mathbb{F}_{q^n}[y]$. Then the following holds:

1. The roots of $H_0(y)$ are in \mathbb{F}_{q^n} ,
2. the number of points of weight μ of L_f w.r.t. U_f is the same as the number of common roots of $H_0(y), H_1(y), \dots, H_{\mu-1}(y)$ which are not roots of $H_\mu(y)$,
3. in particular $f(x)$ is scattered if and only if $H_0(y)$ and $H_1(y)$ have no common roots.

Proof. Let y_0 be a root of $H_0(y)$. Note that Lemma 3.3 does not require the coefficients of f to be in \mathbb{F}_{q^n} , thus also for $y_0 \in \overline{\mathbb{F}}_q$ we have $0 = H_0(y_0) = |R_{0,q}(y_0x + \sum_{i=1}^{n-1} a_i x^{q^i}, x^{q^n} - x)|$ and hence by Theorem 2.1 there exists $x_0 \in \mathbb{F}_{q^n} \setminus \{0\}$ such that $y_0 = -\sum_{i=1}^{n-1} a_i x_0^{q^i-1}$. Here the right-hand side is in \mathbb{F}_{q^n} and hence $y_0 \in \mathbb{F}_{q^n}$.

By Theorem 1.3 $H_0(y_0) = H_1(y_0) = \dots = H_{\mu-1}(y_0) = 0$ and $H_\mu(y_0) \neq 0$ hold if and only if the q -polynomial $(y_0 - a_0)x + f(x) \in \mathbb{F}_{q^n}[x]$ has nullity μ , equivalently, the point $\langle (1, a_0 - y_0) \rangle_{q^n}$ has weight μ .

The last part follows from the fact that f is scattered if and only if L_f does not have points of weight larger than 1. \square

In [2] Part 3. of Corollary 3.5 is used to derive sufficient and necessary conditions for $f(x) = bx^q + x^{q^4} \in \mathbb{F}_{q^6}[x]$ to be a scattered polynomial and to prove [6, Conjecture 7.5] regarding the number of scattered polynomials of this form.

In [4] the authors study MRD-codes with maximum idealisers, or equivalently, the problem of finding sets of distinct integers $\{t_0, t_1, \dots, t_k\}$ such that every \mathbb{F}_q -linear transformation of \mathbb{F}_{q^n} in the \mathbb{F}_{q^n} -subspace $\langle x^{q^{t_0}}, x^{q^{t_1}}, \dots, x^{q^{t_k}} \rangle_{\mathbb{F}_{q^n}}$ has rank at least $n - k$. In [4, Corollary 3.6] it is stated that in $M := \langle x, x^q, x^{q^2}, x^{q^4} \rangle_{\mathbb{F}_{q^9}}$ one can find an \mathbb{F}_q -linear transformation of \mathbb{F}_{q^9} with rank at most 5 and hence the set of integers $\{0, 1, 2, 4\}$ does not satisfy the above mentioned condition. In [4] this was proved by calculating sixteen 6×6 submatrices of $D(f)$, where $f(x) = -x + (1 + c^{-q})x^q + cx^{q^2} - x^{q^4}$ and $c \in \mathbb{F}_{q^9}$ satisfies certain conditions, and by proving that each of them has zero determinant. According to Theorem 1.3 the same result follows also by calculating only $|D_0(f)|$, $|D_1(f)|$, $|D_2(f)|$, $|D_3(f)|$ and by proving that all of them are zero.

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