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Sobolev inequalities with jointly concave weights on convex cones

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Abstract

Using optimal mass transport arguments, we prove weighted Sobolev inequalities of the form

$$\left(\int_E |u(x)|^q\,\omega(x)\,dx\right)^{1/q} \leqslant K_0\left(\int_E |\nabla u(x)|^p\,\sigma(x)\,dx\right)^{1/p}, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{WSI}$$

where $p \geqslant 1$ and $q > 0$ is the corresponding Sobolev critical exponent. Here $E \subseteq \mathbb{R}^n$ is an open convex cone, and $\omega, \sigma : E \to (0,\infty)$ are two homogeneous weights verifying a general concavity-type structural condition. The constant $K_0 = K_0(n,p,q,\omega,\sigma) > 0$ is given by an explicit formula. Under mild regularity assumptions on the weights, we also prove that K_0 is optimal in (WSI) if and only if ω and σ are equal up to a multiplicative factor. Several previously known results, including the cases for monomials and radial weights, are covered by our statement. Further examples and applications to partial differential equations are also provided.

Contents

1. *Introduction*

Driven by numerous applications to the calculus of variations and PDEs, there is a rich literature of weighted Sobolev inequalities, for example, Bakry, Gentil and Ledoux [**2**], Kufner [**16**], and Saloff-Coste [**27**]. Our purpose in this paper is to prove Sobolev inequalities for two weights of the form

$$\left(\int_E |u(x)|^q\,\omega(x)\,dx\right)^{1/q} \leqslant K_0\left(\int_E |\nabla u(x)|^p\,\sigma(x)\,dx\right)^{1/p} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n), \tag{WSI}$$

with $K_0 > 0$ independent on $u \in C_0^\infty(\mathbb{R}^n)$. Here $E \subseteq \mathbb{R}^n$ is an open convex cone, and $\omega, \sigma : E \to (0,\infty)$ are two homogeneous weights verifying some general concavity-type structural conditions to be described.

There are a few ways to prove inequalities of this type when the weights ω and σ are equal. One recent approach, based on the ABP method, is due to Cabré, Ros-Oton and Serra, see [**5**] for monomial weights, and [**6**] for homogeneous weights. A second method used is based on

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optimal transport and was initiated by Cordero-Erausquin, Nazaret and Villani in [11] to show the classical unweighted Sobolev inequalities. This second method has been further developed by Nguyen [25] to deal with the case of monomial weights $\omega = \sigma = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\alpha_i \geqslant 0$, $i = 1, \dots, n$. In addition, Ciraolo, Figalli and Roncoroni [10] recently considered the case of general α-homogeneous weights $\omega = \sigma$ with the property that $\sigma^{1/\alpha}$ is concave.

In this paper, we continue the aforementioned line of research for two different weights ω and σ satisfying a *joint structural concavity condition* and prove (WSI) under this assumption using optimal transport. In fact, the study of (WSI) is motivated by reaction–diffusion problems (see Cabré and Ros-Oton [3, 4]) and Sobolev inequalities on Heisenberg groups for axially symmetric functions (see Section 5.2). Furthermore, the cases considered in [10, 11, 25] turn out to be particular cases of our results which also contain the results of Castro [9] for possible different monomial weights, see Section 4.

We begin introducing notation and the general set up. Let $n \geqslant 2$, and let $E \subseteq \mathbb{R}^n$ be an open convex cone, that is, an open convex set such that $\lambda x \in E$ for all $\lambda > 0$ and $x \in E$; in particular, $0 \in \overline{E}$. Let $p \geqslant 1$ and $\omega, \sigma : E \to (0, \infty)$ be two locally integrable weights in $\overline{E}$, continuous in E, and satisfying the homogeneity conditions

$$\omega(\lambda x) = \lambda^\tau \omega(x), \qquad \sigma(\lambda x) = \lambda^\alpha \sigma(x) \ \text{ for all } \lambda > 0, x \in E, \tag{1.1}$$

where the parameters $\tau, \alpha \in \mathbb{R}$ verify

$$1 \leqslant p < \alpha + n \leqslant \tau + p + n, \tag{1.2}$$

and

$$\alpha \geqslant \left(1 - \frac{p}{n}\right)\tau. \tag{1.3}$$

Clearly, the local integrability of ω and σ implies $\tau + n > 0$ and $\alpha + n > 0$, respectively. Moreover, (1.2) implies $\alpha > -n + 1$. We remark that both integrals in (WSI) are considered only on E and the functions u involved need not vanish on ∂E. By scaling, (WSI) implies the dimensional balance condition

$$\frac{\tau + n}{q} = \frac{\alpha + n}{p} - 1. \tag{1.4}$$

The choice of the precise parameter range given by (1.2) and (1.3) is not arbitrary; indeed, these ranges are *necessary* for the validity of (WSI) as it is shown in Section 5.1. From (1.4) and (1.2), we immediately obtain that

$$q = \frac{p(\tau + n)}{\alpha + n - p} \geqslant p.$$

An important quantity, called fractional dimension n_a, is given by

$$\frac{1}{n_a} = \frac{1}{p} - \frac{1}{q}. \tag{1.5}$$

From (1.4), the inequality (1.3) is equivalent to

$$n_a \geqslant n.$$

It may happen that $n_a = +\infty$ which is equivalent to $p = q$, that is, to $\alpha = p + \tau$. As usual, denote $p' = \frac{p}{p-1}$ for $p > 1$, and $p' = +\infty$ when $p = 1$.

In addition to the homogeneity assumption (1.1) and necessary conditions (1.2)–(1.4), we assume that the weights $\omega, \sigma : E \to (0, \infty)$ are differentiable almost everywhere (a.e) in E and satisfy either one of the following joint structural concavity conditions.

C-0: If $n_a > n$, then there exists a constant $C_0 > 0$ such that

$$\left(\left(\frac{\sigma(y)}{\sigma(x)}\right)^{1/p}\left(\frac{\omega(x)}{\omega(y)}\right)^{1/q}\right)^{n_a/(n_a-n)} \leqslant C_0\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)}+\frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot y \tag{1.6}$$

for almost every (a.e.) $x \in E$ and for all $y \in E$.

C-1: If $n_a = n$, then $\sup_{x\in E}\frac{\omega(x)^{1/q}}{\sigma(x)^{1/p}} =: C_1 \in (0,\infty)$, and

$$0 \leqslant \left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)}+\frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot y \tag{1.7}$$

for a.e. $x \in E$ and for all $y \in E$.

We note that whenever $\omega = \sigma$ is a homogeneous weight of degree $\alpha > 0$ and $C_0 = \frac{1}{\alpha}$, relation (1.6) in C-0 turns to be equivalent to the concavity of $\sigma^{1/\alpha}$, see [**6**, Lemma 5.1]. Even more, Proposition 3.1 reveals an unexpected rigidity connection between condition C-0 and the concavity of the weights ω and σ in a limiting case.

Our main results are that under either one of these assumptions (WSI) holds. Our first main result is then as follows.

THEOREM 1.1. *Let $p > 1$, $E \subseteq \mathbb{R}^n$ be an open convex cone and weights $\omega, \sigma : E \to (0,\infty)$ satisfying relations (1.1)–(1.4), continuous in E and differentiable a.e. in E. Then we have:*

(i) *if condition C-0 holds for some $C_0 > 0$, then (WSI) holds with*

$$K_0 = \max\left\{C_0\left(1-\frac{n}{n_a}\right), \frac{1}{n_a}\right\} q\left(\frac{1}{p'}+\frac{1}{q}\right)$$

$$\times \inf_{\int_E v(y)\,dy=1, v\in C_0^\infty(\mathbb{R}^n), v\geqslant 0} \frac{\left(\int_E v(y)\,|y|^{p'}\,dy\right)^{\frac{1}{p'}}}{\int_E v(y)^{1-\frac{1}{n_a}}\omega(y)^{-\frac{1}{q}}\,\sigma(y)^{\frac{1}{p}}\,dy};$$

(ii) *if condition C-1 holds for some $C_1 > 0$, then (WSI) holds with*

$$K_0 = \frac{C_1}{n} q\left(\frac{1}{p'}+\frac{1}{q}\right) \inf_{\int_E v(y)\,dy=1, v\in C_0^\infty(\mathbb{R}^n), v\geqslant 0} \frac{\left(\int_E v(y)\,|y|^{p'}\,dy\right)^{\frac{1}{p'}}}{\int_E v(y)^{1-\frac{1}{n}}\,dy}.$$

The proof of this theorem is based on optimal transport arguments *à la* Cordero-Erausquin, Nazaret and Villani [**11**]. The statement of the theorem is general enough to cover several well-known results and flexible enough to apply to new cases as well. A well-known Sobolev inequality for radial weights of the form $\omega(x) = |x|^\tau$ and $\sigma(x) = |x|^\alpha$ (see Caffarelli, Kohn and Nirenberg [**7**]) follows as a corollary of this theorem. Considering equal weights $\omega = \sigma$ in Theorem 1.1(i) we recover the isotropic weighted Sobolev inequality in [**10**, Appendix A] and [**25**] when $\omega = \sigma = w$ is a monomial weight. When ω and σ are monomial weights not necessarily equal, Theorem 1.1 contains also the main result of Castro [**9**], providing in addition an explicit Sobolev constant in (WSI). Moreover, our setting allows that some parameters $\tau_i \in \mathbb{R}$ in the monomial $\omega(x_1,\dots,x_n) = x_1^{\tau_1}\cdots x_n^{\tau_n}$ can take negative values, which is an unexpected phenomenon that does not appear in the papers [**5, 10, 25**].

When $p = 1$, with a proof similar to that of Theorem 1.1, we obtain isoperimetric-type inequalities for two weights. In this case, we have $\frac{1}{n_a}+\frac{1}{q} = 1$ and $\frac{1}{p'} = 0$, and both conditions C-0 and C-1 are understood with these values; see (2.2) and the end of the proof of Lemma 2.1. For further use, let $B := \{x \in \mathbb{R}^n : |x| \leqslant 1\}$. Our second main result is then the following.

THEOREM 1.2. *Let $p = 1$, $E \subseteq \mathbb{R}^n$ be an open convex cone and weights $\omega, \sigma : E \to (0, \infty)$ satisfying relations (1.1)–(1.4), continuous in E and differentiable a.e. in E. Then we have:*

(i) *if condition C-0 holds for some $C_0 > 0$, then (WSI) holds with*

$$K_0 = \max\left\{C_0\left(1 - \frac{n}{n_a}\right), \frac{1}{n_a}\right\}\frac{\left(\int_{B\cap E}\omega(y)\,dy\right)^{1-\frac{1}{n_a}}}{\int_{B\cap E}\sigma(y)\,dy};$$

(ii) *if condition C-1 holds for some $C_1 > 0$, then (WSI) holds with*

$$K_0 = \frac{C_1}{n}\frac{\left(\int_{B\cap E}\omega(y)\,dy\right)^{1-\frac{1}{n}}}{\int_{B\cap E}\omega(y)^{1-\frac{1}{n}}\,dy}.$$

Moreover, inequality (WSI) extends to functions with σ-bounded variation on E.

This statement covers the main results in [6] on weighted isoperimetric inequalities when $\omega = \sigma$. To be more precise, let us introduce a few definitions to conclude from Theorem 1.2 isoperimetric inequalities. A function $f : \mathbb{R}^n \to \mathbb{R}$ has σ-bounded variation on E if

$$V_\sigma(f, E) = \sup\left\{\int_E f(x)\mathrm{div}(\sigma(x)X(x))dx : X \in C_0^1(E, \mathbb{R}^n), |X(x)| \leqslant 1, \forall x \in E\right\} < +\infty.$$

Let $BV_\sigma(\mathbb{R}^n)$ be the set of these functions. It is clear that $\dot{W}^{1,1}_\sigma(\mathbb{R}^n) \subset BV_\sigma(\mathbb{R}^n)$ and for every $u \in \dot{W}^{1,1}_\sigma(\mathbb{R}^n)$, we have

$$\int_E |\nabla u(x)|\sigma(x)dx = V_\sigma(u, E).$$

Here for each $p \geqslant 1$, $\dot{W}^{1,p}_\sigma(\mathbb{R}^n)$ denotes the set of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that the level sets $\{x \in E : |u(x)| > s\}$, $s > 0$, have finite σ-measure and $|\nabla u||_E \in L^p_\sigma(E)$, the space of functions that are pth integrable with respect to σ in E.

A measurable set $\Omega \subset \mathbb{R}^n$ has σ-bounded variation on E if $\mathbb{1}_\Omega \in BV_\sigma(\mathbb{R}^n)$, and its weighted perimeter with respect to the convex cone E is given by

$$P_\sigma(\Omega, E) = V_\sigma(\mathbb{1}_\Omega, E).$$

The conclusions of Theorem 1.2 can be then reformulated in terms of weighted isoperimetric inequalities, that is, for any set $\Omega \subset \mathbb{R}^n$ having σ-bounded variation on E, one has

$$K_0^{-1}\left(\int_{\Omega\cap E}\omega(x)dx\right)^{1-\frac{1}{n_a}} \leqslant P_\sigma(\Omega, E), \tag{1.8}$$

where $K_0 > 0$ is the constant given by Theorem 1.2. When $\omega = \sigma$, (1.8) is the sharp weighted isoperimetric inequality of [6] and [25] in the monomial case. Moreover, for different monomial weights we recover from (1.8) the results of Abreu and Fernandes [1].

The next question considered is to describe the *equality* cases in Theorems 1.1 and 1.2. As expected, the candidates for extremal functions belong to $\dot{W}^{1,p}_\sigma(\mathbb{R}^n)$ rather than to $C_0^\infty(\mathbb{R}^n)$. Therefore, we may assume that (WSI) is extended to functions in $\dot{W}^{1,p}_\sigma(\mathbb{R}^n)$. The equality cases in Theorems 1.1 and 1.2 are described in the following result.

THEOREM 1.3. *Let $p \geqslant 1$, $E \subseteq \mathbb{R}^n$ be an open convex cone and weights $\omega, \sigma : E \to (0, \infty)$ satisfying relations (1.1)–(1.4), continuous in E, differentiable a.e. in E, and one of them locally Lipschitz in E. Then we have:*

(i) *if condition C-0 holds for some $C_0 > 0$ and $n_a < +\infty$, then there exist nonzero extremal functions in (WSI) (with the constant K_0 in Theorem 1.1(i) or Theorem 1.2(i)) if* *and only if ω and σ are equal up to a multiplicative factor, $\sigma^{\frac{1}{\alpha}}$ is concave and $C_0 = \frac{1}{n_a - n}$;*

(ii) *if condition C-0 holds and $n_a = +\infty$, there are no extremal functions in (WSI);*

(iii) *if condition C-1 holds for some $C_1 > 0$, then there exist nonzero extremal functions in (WSI) (with the constant K_0 in Theorem 1.1(ii) or Theorem 1.2(ii)) if and only if both weights are constant, that is, $\omega \equiv c_\omega > 0$ and $\sigma \equiv c_\sigma > 0$ with $c_\omega^{\frac{1}{q}} = C_1 c_\sigma^{\frac{1}{p}}$.*

Theorem 1.3 follows by a careful analysis of the equality cases in the proof of Theorems 1.1 and 1.2. Besides the regularity properties of the optimal transport map — similar to those in [**11**] (see also [**25**] when the weights are two equal monomials) — the main novelty in our argument is a rigidity phenomenon showing up from conditions C-0 and C-1 which implies that the weights ω and σ are *equal* up to a multiplicative factor. For a technical reason, in order to establish Theorem 1.3, our argument requires further regularity on the weights with respect to Theorems 1.1 and 1.2, that is, one of them is assumed to be locally Lipschitz. On one hand, Theorem 1.3 shows in a certain sense the limits of our approach. In particular, no characterization can be provided for the equality cases in axially symmetric Sobolev inequalities on the Heisenberg group $\mathbb{H}^1$, since in that case $\omega/\sigma \neq$ constant (see Section 5.2). On the other hand, Theorem 1.3 shows that the results from [**6, 10, 25**] are optimal in the sense that the only reasonable scenario to obtain sharp (WSI) inequalities with the constants given above is when the two weights are constant multiples of each other. The difference between the cases $p > 1$ and $p = 1$ in Theorem 1.3(i) and (iii) appears in the shape of the extremal functions. In the former case, it is Talenti-type radial function (independently on the weight), while in the latter case it is the indicator function of $B \cap E$.

We complete this introduction summarizing the organization of the paper. In Section 2, we prove Theorems 1.1 and 1.2. Section 3 begins with a discussion concerning a concavity rigidity arising from condition C-0, and then we provide the proof of Theorem 1.3. In Section 4, we give various examples and applications of our results. In particular, examples of pairs of weights (ω, σ) satisfying conditions C-0 and C-1 are given in Section 4.1 showing that several known results are simple corollaries of Theorems 1.1 and 1.2. In Section 4.2, we provide some applications by estimating the spectral gap in a weighted eigenvalue problem and discuss the existence of nontrivial weak solution for a weighted PDE. Finally, in Section 5.1, we show that (1.2)–(1.4) are necessary conditions for the validity of (WSI), and next in Section 5.2 we establish the relation between (WSI) and Sobolev inequalities in the Heisenberg group. We finish the paper with final comments and open questions.

2. *Proof of Theorems 1.1 and 1.2*

We start this section with some preliminary remarks on conditions C-0 and C-1. Let us note that, from Euler's theorem for homogeneous functions, one has $\nabla\omega(x) \cdot x = \tau\omega(x)$ and $\nabla\sigma(x) \cdot x = \alpha\sigma(x)$ for a.e. $x \in E$. Picking $y = x \in E$ in C-0 yields $1 \leqslant C_0(\frac{\tau}{p'} + \frac{\alpha}{p})$, implying that if C-0 holds, then at least one of the parameters τ or α must be strictly positive. Clearly, C-1 holds for constant weights.

REMARK 2.1. (i) Using (1.4) and (1.5), condition C-0 can be written in terms of α and τ as follows.

$$\left(\left(\frac{\sigma(y)}{\sigma(x)} \right)^{\tau+n} \left(\frac{\omega(x)}{\omega(y)} \right)^{\alpha+n-p} \right)^{\frac{1}{n(\alpha-\tau)+p\tau}} \leqslant C_0 \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot y, \tag{2.1}$$

for a.e. $x \in E$ and all $y \in E$.

(ii) When $n_a = +\infty$ (that is, $p = q$, which is equivalent to $\alpha = p + \tau$), from (i), it is easy to see that condition C-0 takes the form

$$\left(\frac{\sigma(y)}{\sigma(x)}\frac{\omega(x)}{\omega(y)}\right)^{1/p} \leqslant C_0\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot y \quad \text{for a.e. } x \in E \text{ and all } y \in E. \tag{2.2}$$

(iii) When $n_a \to n$ in condition C-0, the only reasonable relation we obtain is precisely (1.7) in condition C-1. Indeed, if we fix $x, y \in E$ such that $\frac{\omega(x)^{1/q}}{\sigma(x)^{1/p}} < \frac{\omega(y)^{1/q}}{\sigma(y)^{1/p}}$, then the left-hand side of (1.6) tends to 0 whenever $n_a \to n$.

(iv) When $n_a = n$, (1.4) implies $\frac{\tau}{q} = \frac{\alpha}{p}$, and so by (1.1) the function $\frac{\omega^{1/q}}{\sigma^{1/p}}$ is homogeneous of degree zero. Thus, the constant C_1 in condition C-1 equals

$$C_1 := \sup_{x\in E\cap\mathbb{S}^{n-1}} \frac{\omega(x)^{1/q}}{\sigma(x)^{1/p}} < \infty.$$

In spite of the fact that $\frac{\omega^{1/q}}{\sigma^{1/p}}$ is homogeneous of degree zero, the last condition is not automatically satisfied; indeed, the function $(x_1, x_2) \mapsto \frac{x_1}{x_2}$ is 0-homogeneous in $E = (0,\infty)^2$ but it certainly blows up when $x_2 \to 0^+$.

2.1. *Weighted divergence type inequalities*

The proof of Theorems 1.1 and 1.2 are based on a pointwise divergence type inequality stated in the following lemma. Let us recall that if $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $D^2_A\phi$ denotes its Hessian in the sense of Alexandrov, that is, the absolutely continuous part of the distributional Hessian of ϕ, see, for example, Villani [**31**]. In the same sense, let $\Delta_A\phi = \mathrm{tr}D^2_A\phi$ be the Laplacian and for $f \in C^1(\mathbb{R}^n)$, let $\mathrm{div}_A(f\nabla\phi) = \nabla f\cdot\nabla\phi + f\Delta_A\phi$.

LEMMA 2.1. *Let $\omega, \sigma : E \to (0,\infty)$ be weights satisfying (1.1)–(1.4), continuous in E and differentiable a.e. in E. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $\nabla\phi(E) \subseteq E$.*

Then we have:

(i) *if C-0 holds with $C_0 > 0$, then for a.e. $x \in E$ one has*

$$\omega(x)^{1-\frac{1}{n_a}}\,\omega(\nabla\phi(x))^{-1/q}\,\sigma(\nabla\phi(x))^{1/p}\left(\det D^2_A\phi(x)\right)^{1/n_a} \leqslant \tilde{C}_0\,\mathrm{div}_A\left(\omega(x)^{1/p'}\sigma(x)^{1/p}\,\nabla\phi\right),$$

with

$$\tilde{C}_0 = \max\left\{C_0\left(1-\frac{n}{n_a}\right), \frac{1}{n_a}\right\}; \tag{2.1}$$

(ii) *if C-1 holds with $C_1 > 0$, then*

$$\omega(x)^{1-\frac{1}{n_a}}\left(\det D^2_A\phi(x)\right)^{1/n_a} \leqslant \frac{C_1}{n_a}\,\mathrm{div}_A\left(\omega(x)^{1/p'}\sigma(x)^{1/p}\,\nabla\phi\right) \text{ for a.e. } x \in E.$$

Proof. Let us begin proving (i). We divide the proof into several cases.

Case 1: $p > 1$ and $n_a < +\infty$. Since $\nabla\phi(E) \subseteq E$, $\omega(\nabla\phi(x))$ and $\sigma(\nabla\phi(x))$ are well defined for a.e. $x \in E$. Therefore, for a.e. $x \in E$, we have

$$\begin{aligned}
&\omega(x)^{1-\frac{1}{n_a}}\,\omega(\nabla\phi(x))^{-1/q}\,\sigma(\nabla\phi(x))^{1/p}\left(\det D^2_A\phi(x)\right)^{1/n_a}\\
&\leqslant \omega(x)^{1-\frac{1}{n_a}}\,\omega(\nabla\phi(x))^{-1/q}\,\sigma(\nabla\phi(x))^{1/p}\left(\frac{\Delta_A\phi(x)}{n}\right)^{n/n_a} \qquad \text{(from the AM-GM inequality)}\\
&= \omega(x)^{1-\frac{1}{n_a}}\left(\frac{\omega(\nabla\phi(x))^{-1/q}\,\sigma(\nabla\phi(x))^{1/p}}{\omega(x)^{-n/qn_a}\,\sigma(x)^{n/pn_a}}\right)\left(\frac{\Delta_A\phi(x)}{n}\,\omega(x)^{-1/q}\,\sigma(x)^{1/p}\right)^{n/n_a}
\end{aligned}$$

$$= \omega(x)^{1-\frac{1}{n_a}} \left(\left(\frac{\omega(\nabla\phi(x))^{-1/q}\, \sigma(\nabla\phi(x))^{1/p}}{\omega(x)^{-n/qn_a}\, \sigma(x)^{n/pn_a}} \right)^{n_a/(n_a-n)} \right)^{1-\frac{n}{n_a}}$$

$$\times \left(\frac{\Delta_A \phi(x)}{n}\, \omega(x)^{-1/q}\, \sigma(x)^{1/p} \right)^{n/n_a}$$

$$\leqslant \omega(x)^{1-\frac{1}{n_a}} \left(\left(1-\frac{n}{n_a}\right) \left(\frac{\omega(\nabla\phi(x))^{-1/q}\, \sigma(\nabla\phi(x))^{1/p}}{\omega(x)^{-n/qn_a}\, \sigma(x)^{n/pn_a}} \right)^{n_a/(n_a-n)} \right.$$

$$\left. + \frac{1}{n_a}\, \omega(x)^{-1/q}\, \sigma(x)^{1/p}\, \Delta_A \phi(x) \right)$$

$$\leqslant \omega(x)^{1-\frac{1}{n_a}}$$

$$\times \left(1-\frac{n}{n_a}\right) \left(\frac{\left(C_0 \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot \nabla\phi(x) \right)^{(n_a-n)/n_a} \omega(x)^{-1/q} \sigma(x)^{1/p}}{\omega(x)^{-n/qn_a}\, \sigma(x)^{n/pn_a}} \right)^{n_a/(n_a-n)}$$

$$+ \frac{1}{n_a} \omega(x)^{1-\frac{1}{n_a}}\, \omega(x)^{-1/q}\, \sigma(x)^{1/p}\, \Delta_A \phi(x) \qquad \text{(from C-0)}$$

$$= \omega(x)^{1-\frac{1}{n_a}} \left(1-\frac{n}{n_a}\right) \frac{C_0 \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot \nabla\phi(x)}{\omega(x)^{1/q}\, \sigma(x)^{-1/p}}$$

$$+ \frac{1}{n_a} \omega(x)^{1-\frac{1}{n_a}}\, \omega(x)^{-1/q}\, \sigma(x)^{1/p}\, \Delta_A \phi(x)$$

$$= \omega(x)^{1/p'}\, \sigma(x)^{1/p} \left(1-\frac{n}{n_a}\right) C_0 \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot \nabla\phi(x)$$

$$+ \frac{1}{n_a} \omega(x)^{1/p'}\, \sigma(x)^{1/p}\, \Delta_A \phi(x)$$

$$\leqslant \max\left\{ C_0 \left(1-\frac{n}{n_a}\right), \frac{1}{n_a} \right\} \left(\omega(x)^{1/p'}\, \sigma(x)^{1/p} \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \right.$$

$$\left. \cdot \nabla\phi(x) + \omega(x)^{1/p'}\, \sigma(x)^{1/p}\, \Delta_A \phi(x) \right)$$

$$= \max\left\{ C_0 \left(1-\frac{n}{n_a}\right), \frac{1}{n_a} \right\} \operatorname{div}_A\Big(\omega(x)^{1/p'} \sigma(x)^{1/p}\, \nabla\phi \Big),$$

which proves (i) whenever $p > 1$. In the above estimates, we used that both terms $\Delta_A \phi(x)$ and $(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)}) \cdot \nabla\phi(x)$ are nonnegative. *Case* 2: $p = 1$ and $n_a < +\infty$. Then $\frac{1}{n_a} + \frac{1}{q} = 1$ and $\frac{1}{p'} = \frac{p-1}{p} = 0$; accordingly, condition C-0 takes the form

$$\left(\frac{\sigma(y)}{\sigma(x)} \left(\frac{\omega(x)}{\omega(y)} \right)^{(n_a-1)/n_a} \right)^{n_a/(n_a-n)} \leqslant C_0 \frac{\nabla\sigma(x)}{\sigma(x)} \cdot y \quad \text{for all } x, y \in E. \tag{2.2}$$

A similar argument as before gives

$$\omega(x)^{1-\frac{1}{n_a}}\,\omega(\nabla\phi(x))^{-1/q}\,\sigma(\nabla\phi(x))\left(\det D_A^2\phi(x)\right)^{1/n_a}\leqslant \tilde{C}_0\,\mathrm{div}_A(\sigma(x)\,\nabla\phi(x))\text{ for a.e. }x\in E,$$

which is the desired inequality.

Case 3: $p>1$ and $n_a=+\infty$. Since $n_a=+\infty$, we have $q=p$. Thus, by (2.2) and $\Delta_A\phi(x)\geqslant 0$ for a.e. $x\in E$, it turns out that

$$\begin{aligned}\omega(x)\,\omega(\nabla\phi(x))^{-1/p}\,\sigma(\nabla\phi(x))^{1/p}&\leqslant C_0\omega(x)^{\frac{1}{p'}}\,\sigma(x)^{1/p}\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)}+\frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot\nabla\phi(x)\\&\leqslant C_0\,\mathrm{div}_A\Big(\omega(x)^{1/p'}\sigma(x)^{1/p}\,\nabla\phi(x)\Big),\end{aligned}$$

which is the required inequality with $\tilde{C}_0=C_0$.

Case 4: $p=1$ and $n_a=+\infty$. Since in this case $p=q=1$, condition C-0 reduces to

$$\frac{\sigma(y)}{\sigma(x)}\frac{\omega(x)}{\omega(y)}\leqslant C_0\frac{\nabla\sigma(x)}{\sigma(x)}\cdot y\quad\text{for a.e. }x\in E\text{ and all }y\in E.\tag{2.3}$$

Therefore, by (2.3) and $\Delta_A\phi(x)\geqslant 0$ for a.e. $x\in E$, one has

$$\omega(x)\,\omega(\nabla\phi(x))^{-1}\,\sigma(\nabla\phi(x))\leqslant C_0\,\nabla\sigma(x)\cdot\nabla\phi(x)\leqslant C_0\,\mathrm{div}_A(\sigma(x)\,\nabla\phi(x))\text{ for a.e. }x\in E,$$

concluding the proof of (i).

To show(ii), we divide the proof into two parts.

Case 1: $p>1$ and $n_a=n$. Since $n_a=n$, one has $\frac{1}{p}-\frac{1}{q}=\frac{1}{n}$. Moreover, by the definition of $C_1>0$ in condition C-1, it follows that

$$\omega(x)^{1-\frac{1}{n}}\leqslant C_1\omega(x)^{1/p'}\sigma(x)^{1/p},\quad x\in E.\tag{2.4}$$

Then for a.e. $x\in E$, one has

$$\begin{aligned}\omega(x)^{1-\frac{1}{n_a}}\left(\det D_A^2\phi(x)\right)^{1/n_a}&=\omega(x)^{1-\frac{1}{n}}\left(\det D_A^2\phi(x)\right)^{1/n}\\&\leqslant\omega(x)^{1-\frac{1}{n}}\frac{\Delta_A\phi(x)}{n}\qquad\text{(from the AM-GM inequality)}\\&\leqslant\frac{C_1}{n}\omega(x)^{1/p'}\sigma(x)^{1/p}\Delta_A\phi(x)\\&\leqslant\frac{C_1}{n}\left(\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)}+\frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot\nabla\phi(x)+\omega(x)^{1/p'}\sigma(x)^{1/p}\Delta_A\phi(x)\right)\\&\qquad\text{(from }\nabla\phi(E)\subseteq E\text{ and }C-1)\\&=\frac{C_1}{n}\mathrm{div}_A\Big(\omega(x)^{1/p'}\sigma(x)^{1/p}\,\nabla\phi(x)\Big),\end{aligned}$$

which concludes the proof whenever $p>1$.

Case 2: $p=1$ and $n_a=n$. Since $p=1$, one has $\frac{1}{p'}=0$, and condition C-1 reads as $\sup_{x\in E}\frac{\omega(x)^{1/q}}{\sigma(x)}=C_1\in(0,\infty)$ and $0\leqslant\nabla\sigma(x)\cdot y$ for all $x,y\in E$. In particular, since $\frac{1}{q}=1-\frac{1}{n}$, then $\omega(x)^{1-\frac{1}{n}}\leqslant C_1\sigma(x)$ for every $x\in E$. A similar argument as in the previous case provides the inequality

$$\omega(x)^{1-\frac{1}{n_a}}\left(\det D_A^2\phi(x)\right)^{1/n_a}\leqslant\frac{C_1}{n}\mathrm{div}_A(\sigma(x)\,\nabla\phi(x))\quad\text{for a.e. }x\in E,$$

which concludes the proof of the lemma. □

2.2. *Proof of Theorem 1.1*

From Lemma 2.1, we can now give the proof of the desired weighted Sobolev inequalities on convex cones.

Let $u \in C_0^\infty(\mathbb{R}^n)$ be fixed. If $\mathcal{L}^n(\operatorname{supp}(u) \cap E) = 0$, we have nothing to prove; hereafter, $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure. Thus, we may assume that $\mathcal{L}^n(\operatorname{supp}(u) \cap E) > 0$ and to simplify the notation, let $U = \operatorname{supp}(u)$. We may assume that u is nonnegative and by scaling

$$\int_E u(x)^q \, \omega(x) \, dx = 1.$$

We also fix $v \in C_0^\infty(\mathbb{R}^n)$ a nonnegative function satisfying

$$\int_E v(y) \, dy = 1.$$

Consider the probability measures in E, $\mu = u^q \, \omega \, dx$ and $\nu = v \, dy$, and let T be the optimal map with respect to the quadratic cost such that $T_\sharp \mu = \nu$. By Brenier's theorem, there is ϕ convex in $\mathbb{R}^n$ such that $T = \nabla \phi$ and $\nabla\phi(E) \subseteq \operatorname{supp}\nu \subseteq E$. This is equivalent to the following Monge–Ampère equation

$$u^q(x) \, \omega(x) = v(\nabla\phi(x)) \, \det D_A^2 \phi(x) \quad \text{for a.e. } x \in U \cap E. \tag{2.5}$$

Proof of (i). Raising (2.5) to the power $1 - \frac{1}{n_a}$ and rewriting the resulting equation yields

$$v^{1-\frac{1}{n_a}}(\nabla\phi(x)) \, h(\nabla\phi(x)) \, \det D_A^2\phi(x) = u^{q\left(1-\frac{1}{n_a}\right)}(x) \, \omega^{1-\frac{1}{n_a}}(x) \, h(\nabla\phi(x)) [\det D_A^2\phi(x)]^{\frac{1}{n_a}}, \tag{2.6}$$

where $h(x) = \omega(x)^{-1/q} \, \sigma(x)^{1/p}$. Integrating this identity over $U \cap E$, changing variables on the left-hand side, and using Lemma 2.1(i) on the right-hand side, yields

$$\int_E v(y)^{1-\frac{1}{n_a}} \, h(y) \, dy \leqslant \tilde{C}_0 \int_{U\cap E} u(x)^{q\left(1-\frac{1}{n_a}\right)} \operatorname{div}_A\left(\sigma(x)^{1/p} \omega(x)^{1/p'} \, \nabla\phi\right) dx := \tilde{C}_0 \, I.$$

Since $\Delta_A \phi \leqslant \Delta_{\mathcal{D}'}\phi$, where $\Delta_{\mathcal{D}'}$ is the distributional Laplacian, integrating by parts, one gets

$$\begin{aligned} I &\leqslant \int_{U\cap E} u^{q\left(1-\frac{1}{n_a}\right)}(x) \operatorname{div}_{\mathcal{D}'}(\omega(x)^{\frac{1}{p'}} \sigma(x)^{\frac{1}{p}} \nabla\phi(x)) dx \\ &= \int_{\partial(U\cap E)} u^{q\left(1-\frac{1}{n_a}\right)}(x) \, \omega(x)^{\frac{1}{p'}} \sigma(x)^{\frac{1}{p}} \nabla\phi(x) \cdot \mathbf{n}(x) \, ds(x) \\ &\quad - q\left(1 - \frac{1}{n_a}\right) \int_{U\cap E} u^{q\left(1-\frac{1}{n_a}\right)-1}(x) \, \omega(x)^{\frac{1}{p'}} \sigma(x)^{\frac{1}{p}} \nabla\phi(x) \cdot \nabla u(x) \, dx, \end{aligned} \tag{2.7}$$

where $\mathbf{n}(x)$ is the outer normal vector at $x \in \partial(U \cap E)$. Since E is a convex cone, $y \cdot \mathbf{n}(x) \leqslant 0$ for each $y \in \bar{E}$ and $x \in \partial E$. In particular, $\nabla\phi(x) \cdot \mathbf{n}(x) \leqslant 0$ for each $x \in \partial E$, since $\nabla\phi(\overline{E}) \subseteq \overline{E}$. On the other hand, $\partial(U \cap E) \subset \partial U \cup \partial E$. So we obtain that the integrand in the boundary integral is nonpositive for $x \in \partial E$ and is zero for $x \in \partial U$ since $q(1 - \frac{1}{n_a}) > 0$. Therefore, the boundary integral in (2.7) can be dropped and by Hölder's inequality it follows that

$$I \leqslant q\left(1 - \frac{1}{n_a}\right) \left(\int_E u^q(x) \omega(x) |\nabla\phi(x)|^{p'} dx\right)^{\frac{1}{p'}} \left(\int_E |\nabla u(x)|^p \sigma(x) dx\right)^{\frac{1}{p}},$$

since $(q(1 - \frac{1}{n_a}) - 1)p' = q$. Using once again the Monge–Ampère equation (2.5) yields

$$\int_E u(x)^q \, \omega(x) \, |\nabla\phi(x)|^{p'} \, dx = \int_E v(\nabla\phi(x)) \, |\nabla\phi(x)|^{p'} \, \det D_A^2\phi(x) \, dx = \int_E v(y) \, |y|^{p'} \, dy.$$

Therefore, the above estimates give

$$\int_E v(y)^{1-\frac{1}{n_a}}\, h(y)\, dy \leqslant \tilde{C}_0\, q\left(1-\frac{1}{n_a}\right)\left(\int_E v(y)\,|y|^{p'}\, dy\right)^{\frac{1}{p'}}\left(\int_E |\nabla u(x)|^p \sigma(x)\, dx\right)^{\frac{1}{p}},$$

which completes the proof of (i).

Proof of (ii). Since C-1 holds, one has $n_a = n$. From (2.5), we have

$$v^{1-\frac{1}{n_a}}(\nabla\phi(x))\,\det D_A^2\phi(x) = u^{q\left(1-\frac{1}{n_a}\right)}(x)\,\omega^{1-\frac{1}{n_a}}(x)[\det D_A^2\phi(x)]^{\frac{1}{n_a}},\ x\in E.$$

Integrating the last equation and using Lemma 2.1(ii) gives

$$\int_E v(y)^{1-\frac{1}{n_a}}\, dy \leqslant \frac{C_1}{n_a}\int_{U\cap E} u(x)^{q\left(1-\frac{1}{n_a}\right)}\,\mathrm{div}_A\Big(\sigma(x)^{1/p}\omega(x)^{1/p'}\,\nabla\phi\Big)\, dx. \tag{2.8}$$

We now proceed as in case (i), obtaining that

$$\int_E v(y)^{1-\frac{1}{n_a}}\, dy \leqslant \frac{C_1}{n_a} q\left(1-\frac{1}{n_a}\right)\left(\int_E v(y)\,|y|^{p'}\, dy\right)^{\frac{1}{p'}}\left(\int_E |\nabla u(x)|^p\sigma(x)\right)^{\frac{1}{p}},$$

which completes the proof of the theorem. □

2.3. *Proof of Theorem 1.2*

Let us start with an arbitrarily fixed nonnegative function $u\in C_0^\infty(\mathbb{R}^n)$ with the property $\int_E u(x)^{\frac{n_a}{n_a-1}}\omega(x)dx = 1$, and $v(y) := \frac{\omega(y)}{\int_{B\cap E}\omega(y)dy}\mathbb{1}_{B\cap E}(y)$. Let us consider the optimal transport map $T=\nabla\phi$ such that $T_\sharp\mu = \nu$ for $\mu = u^{\frac{n_a}{n_a-1}}\omega dx$ and $\nu = v dx$. We may repeat the arguments from Theorem 1.1 with suitable modifications.

Proof of (i). If C-0 holds, then since $\nabla\phi(x)\in \mathrm{supp} v = B\cap E$ for a.e. $x\in U\cap E$, we can use Lemma 2.1/(i) for $p=1$. In this case, we note that $1-\frac{1}{n_a}=\frac{1}{q}$. The divergence theorem and $\nabla\phi(x)\in B\cap E$ for a.e. $x\in U\cap E$ imply

$$\begin{aligned}
\int_{B\cap E} v(y)^{1-\frac{1}{n_a}}\,\omega(y)^{-\frac{1}{q}}\,\sigma(y)\, dy &\leqslant \tilde{C}_0\int_{U\cap E} u(x)^{q\left(1-\frac{1}{n_a}\right)}\,\mathrm{div}_A(\sigma(x)\,\nabla\phi)\, dx\\
&= \tilde{C}_0\int_{U\cap E} u(x)\,\mathrm{div}_A(\sigma(x)\,\nabla\phi)\, dx\\
&\leqslant \tilde{C}_0\Bigg(\int_{\partial(U\cap E)} u(x)\sigma(x)\nabla\phi(x)\cdot\mathbf{n}(x)\, ds(x)\\
&\qquad -\int_{U\cap E}\sigma(x)\nabla u(x)\cdot\nabla\phi(x)\, dx\Bigg)\\
&\leqslant \tilde{C}_0\int_{U\cap E}\sigma(x)|\nabla u(x)||\nabla\phi(x)|\, dx\\
&\leqslant \tilde{C}_0\int_E |\nabla u(x)|\sigma(x)\, dx.
\end{aligned}$$

Using again the relation $1-\frac{1}{n_a}=\frac{1}{q}$, we obtain

$$\int_{B\cap E} v(y)^{1-\frac{1}{n_a}}\,\omega(y)^{-\frac{1}{q}}\,\sigma(y)\, dy = \frac{\int_{B\cap E}\sigma(y)\, dy}{\left(\int_{B\cap E}\omega(y)\, dy\right)^{1-\frac{1}{n_a}}}.$$

Proof of (ii). Suppose, that condition C-1 holds for some $C_1 > 0$. In this case, instead of (2.8), we use Lemma 2.1/(ii) for $p = 1$. We conclude

$$\int_{B\cap E} v(y)^{1-\frac{1}{n_a}}\, dy \leqslant \frac{C_1}{n_a} \int_{U\cap E} u(x)^{q\left(1-\frac{1}{n_a}\right)} \operatorname{div}_A(\sigma(x)\,\nabla\phi)\, dx = \frac{C_1}{n_a} \int_{U\cap E} u(x) \operatorname{div}_A(\sigma(x)\,\nabla\phi)\, dx.$$

Proceeding as before yields

$$\int_E v(y)^{1-\frac{1}{n_a}}\, dy \leqslant \frac{C_1}{n_a} \int_E |\nabla u(x)|\sigma(x) dx,$$

which concludes the proof.

Clearly, both (i) and (ii) can be extended to functions with σ-bounded variation on E. ☐

Remark 2.2. Theorems 1.1 and 1.2 can be formulated in the *anisotropic* setting as well, by considering any norm instead of the usual Euclidean one. The only technical difference is the use of Hölder's inequality for the norm and its polar transform, see, for example, [**10, 11**]. When $\omega = \sigma = 1$, the weights are homogeneous of degree zero and one has $n_a = n$. Choosing $E = \mathbb{R}^n$, condition C-1 trivially holds with constant $C_1 = 1$. Thus Theorems 1.1(ii) and 1.2(ii) yield the well-known sharp Sobolev inequality ($p > 1$) and sharp isoperimetric inequality ($p = 1$), respectively, in Del Pino and Dolbeault [**12, 13**] and Cordero-Erausquin, Nazaret and Villani [**11**, Theorems 2 and 3].

3. *Discussion of the equality cases: proof of Theorem* 1.3

In this section, we are going to prove Theorem 1.3, that is, to identify the equality cases in Theorems 1.1 and 1.2. As we already pointed out after the statement of Theorem 1.2, we may extend (WSI) from $C_0^\infty(\mathbb{R}^n)$ to $\dot{W}^{1,p}_\sigma(\mathbb{R}^n)$, that is larger space in order to search for a suitable candidate as an extremal function. To do this extension, a careful approximation argument is needed which is similar to the one carried out in [**11**, Lemma 7] for the unweighted case, and that was adapted to equal monomial weights in [**25**]. In fact, the idea to do this is to extend the integration by parts formula (2.7) to functions u in $\dot{W}^{1,p}_\sigma(\mathbb{R}^n)$, a technical issue discussed in detail in [**11, 25**]. Since the same technique can be adapted also to our setting, we thus omit the details.

In order to prove Theorem 1.3, we shall need some preliminary results. First, we have the following characterization of concavity.

Lemma 3.1. *Let $E \subseteq \mathbb{R}^n$ be an open convex set and $h : E \to \mathbb{R}$ be a continuous function which is a.e. differentiable in E. Then the following statements are equivalent.*

(a) *h is concave in E.*
(b) *For a.e. $x \in E$ and all $y \in E$, one has $h(y) - h(x) \leqslant \nabla h(x) \cdot (y - x)$.*

Proof. Although standard, we provide the proof since we did not find it in the literature. The implication '(a)⇒(b)' is trivial. For '(b)⇒(a)', let $E_0 \subset E$ be the set where h is differentiable; clearly, $\mathcal{L}^n(E \setminus E_0) = 0$. Let $x_0, y_0 \in E$, $0 < t < 1$, and $z_0 = (1-t)x_0 + ty_0$. If $z_0 \in E_0$, then by our assumption, we have that $h(x_0) - h(z_0) \leqslant \nabla h(z_0) \cdot (x_0 - z_0)$ and $h(y_0) - h(z_0) \leqslant \nabla h(z_0) \cdot (y_0 - z_0)$. Multiplying the first inequality by $(1-t)$, the second by t, and adding them up yields $(1-t)\, h(x_0) + t\, h(y_0) - h(z_0) \leqslant 0$. On the other hand, if $z_0 \notin E_0$, pick a sequence $z_k \in E_0$ such that $z_k \to z_0$. Since E is open, we can pick sequences $x_k, y_k \in E$ such that $x_k \to x_0$, $y_k \to y_0$, with $z_k = (1-t)x_k + ty_k$. In particular, we have that $h(x_k) - h(z_k) \leqslant \nabla h(z_k) \cdot (x_k - z_k)$ and $h(y_k) - h(z_k) \leqslant \nabla h(z_k) \cdot (y_k - z_k)$. Multiplying the latter inequality by t and the former by $(1-t)$ yields $(1-t)\,h(x_k)+t\,h(y_k)-h(z_k)\leqslant 0$. Since h is continuous, letting $k\to\infty$ we obtain the concavity of h. $\square$

We are ready to prove a rigidity result based on the validity of condition C-0.

Proposition 3.1. *Let $E\subseteq\mathbb{R}^n$ be an open convex cone and weights $\omega,\sigma:E\to(0,\infty)$ satisfying relation (1.1) with $\alpha>0$, $\tau\in\mathbb{R}$, continuous in E, differentiable a.e. in E. Assume in addition that at least one of the weights ω or σ is locally Lipschitz in E. If $n_a<+\infty$, we have:*

- (i) *if condition C-0 holds with $C_0>0$ and $\tau\leqslant\alpha$, then $C_0\geqslant\frac{1}{n_a-n}$;*
- (ii) *the following statements are equivalent.*
  - (a) *Condition C-0 holds for $C_0=\frac{1}{n_a-n}$ and $\tau\leqslant\alpha$.*
  - (b) *$\omega=c\sigma$ for some $c>0$ (thus $\alpha=\tau$) and $\sigma^{1/\alpha}$ is concave in E.*

Proof. (i) From Euler's theorem for homogeneous functions, $\nabla\omega(x)\cdot x=\tau\omega(x)$ and $\nabla\sigma(x)\cdot x=\alpha\sigma(x)$ for all a.e. $x\in E$. Picking $y=x\in E$ in C-0 yields $1\leqslant C_0(\frac{\tau}{p'}+\frac{\alpha}{p})$. Using (1.4) and (1.5), we get that $n_a=\frac{p\,(\tau+n)}{\tau-\alpha+p}$, and

$$n_a-n=\frac{p\,\tau+n(\alpha-\tau)}{\tau-\alpha+p}\geqslant\tau+\frac{n}{p}(\alpha-\tau)\geqslant\tau+\frac{1}{p}(\alpha-\tau)=\frac{\tau}{p'}+\frac{\alpha}{p},$$

where in the last estimate we used the assumption $\tau\leqslant\alpha$. The lower estimate for C_0 then follows.

(ii) '(b)⇒(a)' On one hand, by Lemma 3.1, we note that the concavity of $\sigma^{1/\alpha}$ in E implies

$$\sigma(y)^{1/\alpha}-\sigma(x)^{1/\alpha}\leqslant\nabla\sigma^{1/\alpha}(x)\cdot(y-x)\quad\text{for a.e. } x\in E \text{ and all } y\in E.$$

By the 1-homogeneity of $\sigma^{1/\alpha}$ and Euler's theorem, it turns out that $\sigma(x)^{1/\alpha}=\nabla\sigma^{1/\alpha}(x)\cdot x$ for a.e. $x\in E$, thus the last inequality is equivalent to

$$\sigma(y)^{1/\alpha}\leqslant\nabla\sigma^{1/\alpha}(x)\cdot y=\frac{1}{\alpha}\sigma(x)^{1/\alpha-1}\nabla\sigma(x)\cdot y\quad\text{for a.e. } x\in E \text{ and all } y\in E. \tag{3.1}$$

On the other hand, since by assumption $\omega=c\sigma$ (for some $c>0$), one has $\tau=\alpha$ and $n_a=n+\alpha$. Now using (2.1) we see that condition C-0 means

$$\sigma(x)\left(\frac{\sigma(y)}{\sigma(x)}\right)^{1/\alpha}\leqslant C_0\nabla\sigma(x)\cdot y\quad\text{for a.e. } x\in E \text{ and all } y\in E.$$

On account of (3.1), condition C-0 holds for $C_0=\frac{1}{\alpha}=\frac{1}{n_a-n}$.
'(a)⇒(b)' This is the trickiest part of the proof and at the same time is the most important result to use later in the description of equality in (WSI).

Since by assumption, condition C-0 holds with $C_0=\frac{1}{n_a-n}$, it follows from (2.1) that

$$\begin{aligned}&\left(\left(\frac{\sigma(y)}{\sigma(x)}\right)^{\tau+n}\left(\frac{\omega(x)}{\omega(y)}\right)^{\alpha+n-p}\right)^{\frac{1}{n(\alpha-\tau)+p\tau}}\\&\qquad\leqslant\frac{1}{n_a-n}\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)}+\frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot y\quad\text{for a.e. } x\in E \text{ and all } y\in E.\end{aligned} \tag{3.2}$$

Choosing $y=x$ in (3.2) yields

$$1\leqslant\frac{1}{n_a-n}\left(\frac{\tau}{p'}+\frac{\alpha}{p}\right). \tag{3.3}$$

Let us recall from the proof of Part (i) that

$$n_a - n = \frac{p\tau + n(\alpha-\tau)}{\tau-\alpha+p}.$$

This inserted into (3.3) yields

$$\frac{p\tau + n(\alpha-\tau)}{\tau-\alpha+p} \leqslant \tau + \frac{\alpha-\tau}{p},$$

which is equivalent to

$$\left(\frac{n+\tau}{\alpha-\tau+p} - \frac{1}{p}\right)(\alpha-\tau) \leqslant 0.$$

Once again from the expression of n_a, the last inequality is equivalent to $(\alpha-\tau)(n_a-1)/p \leqslant 0$. Since $n_a > n \geqslant 2$, this implies that $\alpha \leqslant \tau$, and since by assumption $\tau \leqslant \alpha$, we conclude that $\alpha = \tau$. In particular, we have that $n_a = n+\alpha$ and (3.2) reduces to

$$\begin{aligned}\left(\left(\frac{\sigma(y)}{\sigma(x)}\right)^{\alpha+n}\left(\frac{\omega(x)}{\omega(y)}\right)^{\alpha+n-p}\right)^{\frac{1}{p\alpha}}& \\ \leqslant \frac{1}{\alpha}\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot y &\quad \text{for a.e. } x\in E \text{ and all } y \in E. \end{aligned}\tag{3.4}$$

Let us define the function $f : E \to (0,\infty)$ given by $f(x) = \frac{\omega(x)}{\sigma(x)}$, $x \in E$. Our task is to prove that f is constant on E. To do this, we first rewrite (3.4) in terms of f and σ to eliminate ω. In this way, we obtain

$$\left[\left(\frac{f(x)}{f(y)}\right)^{\alpha+n-p}\left(\frac{\sigma(y)}{\sigma(x)}\right)^{p}\right]^{\frac{1}{p\alpha}} \leqslant \frac{1}{\alpha}\left[\frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)} + \frac{1}{p'}\frac{f(x)\nabla\sigma(x)+\nabla f(x)\sigma(x)}{f(x)\cdot\sigma(x)}\right]\cdot y, \tag{3.5}$$

for a.e $x \in E$ and for all $y \in E$. Motivated by this inequality, we define for a.e. $x \in E$ the function $g_x : E \to \mathbb{R}$ given by

$$g_x(y) = \frac{1}{\alpha}\left(\frac{1}{p'}\frac{\nabla f(x)}{f(x)} + \frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot y - \left(\frac{\sigma(y)}{\sigma(x)}\right)^{\frac{1}{\alpha}}\left(\frac{f(x)}{f(y)}\right)^{\frac{n+\alpha}{\alpha q}}, \quad y \in E.$$

Clearly g_x is continuous in E, and since $\alpha = \tau$ and (1.4), (3.5) is equivalent to $g_x(y) \geqslant 0$ for a.e $x \in E$ and all $y \in E$. Furthermore, since f is homogeneous of degree zero and differentiable a.e., one has that $\nabla f(x)\cdot x = 0$, and thus $g_x(x) = 0$ for a.e. $x \in E$. In particular, for a.e. $x \in E$, the function $y \mapsto g_x(y)$ has a global minimum on E at $y = x$ and since $y \mapsto g_x(y)$ is differentiable at $y = x$, we obtain $\nabla g_x(y)|_{y=x} = 0$. This means that for a.e. $x \in E$, one has

$$\frac{1}{\alpha}\left(\frac{1}{p'}\frac{\nabla f(x)}{f(x)} + \frac{\nabla\sigma(x)}{\sigma(x)}\right) - \frac{1}{\alpha}\frac{\nabla\sigma(x)}{\sigma(x)} + \frac{n+\alpha}{\alpha q}\frac{\nabla f(x)}{f(x)} = 0,$$

which is equivalent to

$$\left(\frac{1}{\alpha p'} + \frac{n+\alpha}{\alpha q}\right)\frac{\nabla f(x)}{f(x)} = 0 \;\text{ for a.e. } x \in E.$$

Since $\frac{1}{p'} + \frac{n+\alpha}{q} > 0$, it follows that

$$\nabla f(x) = 0 \quad \text{for a.e. } x \in E. \tag{3.6}$$

We are going to prove that f is locally Lipschitz in E; once we do that, by (3.6) we may conclude that f is constant. To see this, let $h : E \to (0, \infty)$ be the continuous, a.e. differentiable function given by

$$h(x) = \left(\frac{\sigma(x)^{\alpha+n}}{\omega(x)^{\alpha+n-p}} \right)^{\frac{1}{p\alpha}}, \quad x \in E. \tag{3.7}$$

From (3.6), it follows that $\frac{\nabla\omega(x)}{\omega(x)} = \frac{\nabla\sigma(x)}{\sigma(x)}$ for a.e. $x \in E$. A simple computation then shows that $\nabla h(x) = \frac{1}{\alpha} h(x) \frac{\nabla\omega(x)}{\omega(x)}$ for a.e. $x \in E$. Therefore, relation (3.4) reduces to

$$h(y) \leqslant \nabla h(x) \cdot y \quad \text{for a.e. } x \in E \text{ and all } y \in E. \tag{3.8}$$

Since h is homogeneous of degree one, it follows by (3.8) that

$$h(y) - h(x) \leqslant \nabla h(x) \cdot (y - x) \quad \text{for a.e. } x \in E \text{ and all } y \in E. \tag{3.9}$$

Now, Lemma 3.1 implies that h is concave in E, thus locally Lipschitz in E. By assumption, one of the weights is locally Lipschitz, thus the other one is locally Lipschitz too. In particular, f is also locally Lipschitz in E, and so from (3.6) we conclude the proof that f is constant. Hence $\omega = c\sigma$ in E for some $c > 0$, and so $h(x) = c^{\frac{p-\alpha-n}{p\alpha}} \sigma(x)^{\frac{1}{\alpha}}$ for every $x \in E$. Therefore $\sigma^{\frac{1}{\alpha}}$ is concave in E concluding the proof. ☐

Proof of Theorem 1.3. Let us assume that equality holds in (WSI) for some $u \in \dot{W}^{1,p}_{\sigma}(\mathbb{R}^n) \setminus \{0\}$, and without loss of generality, assume that u is nonnegative with

$$\int_E u(x)^q \, \omega(x) \, dx = 1.$$

A similar argument as in [**11**, Proposition 6] implies that $\Delta_{\mathcal{D}'}\phi$ is absolutely continuous on E_0, where E_0 denotes the interior of the set $\{x \in \mathbb{R}^n : \phi(x) < +\infty\}$. We note that $U \cap E = \mathrm{supp}(u) \cap E \subset \overline{E_0}$.

To prove Theorem 1.3, we discuss separately the equality cases for $p > 1$ (see Theorem 1.1) and $p = 1$ (see Theorem 1.2), respectively. ☐

3.1. *Case* $p > 1$

We split the proof into several cases.

Case 1: condition C-0 holds, $p > 1$ and $n_a < +\infty$.

Since u gives equality in (WSI), we must have equality in each step in the proof of Lemma 2.1(i), Case 1. In particular, we have equality in the AM-GM inequality $\det D^2_A\phi(x) \leqslant (\frac{\Delta_A\phi(x)}{n})^n$ for μ-a.e. $x \in E$ (recall that $\mu = u^q \, \omega \, dx$), thus identifying $D^2_A\phi$ with $D^2_{\mathcal{D}'}\phi$, it turns out that $D^2_A\phi(x) = \lambda I_n$ for a.e. $x \in E$, where $\lambda > 0$ and I_n is the $n \times n$-identity matrix. Therefore, for some $x_0 \in \mathbb{R}^n$, one has

$$\nabla\phi(x) = \lambda x + x_0 \quad \text{for a.e.} \quad x \in E \cap E_0. \tag{3.10}$$

Since $\nabla\phi(\overline{E}) \subseteq \overline{E}$ and $0 \in \overline{E}$, we necessarily have that $x_0 \in \overline{E}$.

The equality in the second AM-GM inequality in the proof of Lemma 2.1(i) yields

$$\left(\frac{\omega(\nabla\phi(x))^{-1/q} \, \sigma(\nabla\phi(x))^{1/p}}{\omega(x)^{-n/qn_a} \, \sigma(x)^{n/pn_a}} \right)^{n_a/(n_a-n)} = \frac{\Delta_A\phi(x)}{n} \omega(x)^{-1/q} \, \sigma(x)^{1/p} \quad \text{for a.e.} \quad x \in E \cap E_0.$$

By rearranging the last equation, combined with $\Delta_A\phi(x) = \lambda n$ for a.e. $x \in E \cap E_0$ and (3.10), it follows that

$$\omega(\lambda x + x_0)^{-1/q} \, \sigma(\lambda x + x_0)^{1/p} = \lambda^{(n_a-n)/n_a} \omega(x)^{-1/q} \, \sigma(x)^{1/p} \quad \text{for a.e.} \quad x \in E \cap E_0. \tag{3.11}$$

When we apply condition C-0 in the proof of Lemma 2.1(i), the equality means that for a.e. $x \in E \cap E_0$, we have

$$\omega(\nabla\phi(x))^{-1/q}\,\sigma(\nabla\phi(x))^{1/p} = \left(C_0\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot\nabla\phi(x)\right)^{(n_a-n)/n_a}\omega(x)^{-1/q}\sigma(x)^{1/p}.$$

Thus, by (3.10) and (3.11), it turns out that

$$\lambda = C_0\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot(\lambda x + x_0) \quad \text{for a.e. } \ x \in E \cap E_0.$$

By (1.1), the last relation is equivalent to

$$1 = C_0\left(\frac{\tau}{p'} + \frac{\alpha}{p} + I_0(x)\right) \quad \text{for a.e. } \ x \in E \cap E_0, \tag{3.12}$$

where

$$I_0(x) = \lambda^{-1}\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot x_0.$$

By using condition C-0 for $y := y_k$, where $\{y_k\}_k \subset E$ is a sequence converging to $x_0 \in \overline{E}$, we immediately obtain that $I_0(x) \geqslant 0$ for a.e. $x \in E$. Therefore, by (3.12) we have that

$$1 \geqslant C_0\left(\frac{\tau}{p'} + \frac{\alpha}{p}\right). \tag{3.13}$$

Finally, in the last estimate of the proof of Lemma 2.1/(i), the equality requires

$$C_0\left(1 - \frac{n}{n_a}\right) = \frac{1}{n_a},$$

that is,

$$C_0 = \frac{1}{n_a - n}. \tag{3.14}$$

This means that we have $\frac{\tau}{p'} + \frac{\alpha}{p} \leqslant n_a - n$ that is precisely the reverse inequality to (3.3). A similar reasoning as before using (3.13) together with (3.14) imply now the reverse conclusion, that is $\tau \leqslant \alpha$.

We also note that $\alpha > 0$. Indeed, if we assume that $\alpha \leqslant 0$, we would have $\tau \leqslant \alpha \leqslant 0$ and by picking $y = x \in E$ in C-0, it follows $1 \leqslant C_0(\frac{\tau}{p'} + \frac{\alpha}{p}) \leqslant 0$; a contradiction.

Summing up, from the above arguments one concludes that condition C-0 holds with $C_0 = \frac{1}{n_a - n}$ and $\tau \leqslant \alpha$ with $\alpha > 0$. But now from Proposition 3.1(ii) it follows that there exists $c > 0$ such that $\omega(x) = c\sigma(x)$ for every $x \in E$ (thus $\alpha = \tau$ and $n_a = n + \alpha$) and $\sigma^{1/\alpha}$ is concave in E.

Now, we are precisely in the setting of [**10**, Theorem A.1]. In particular, by the equality in the Hölder inequality, it follows that the extremal function satisfies $|\nabla u(x)|^p\sigma(x) = c_0 u(x)^q|x + x_0|^{p'}\omega(x)$ for some $c_0 > 0$ and every $x \in E$. Thus, $|\nabla u(x)|^p = c_0 c u(x)^q|x + x_0|^{p'}$, obtaining that the extremal function in (WSI) is $u(x) := u_\gamma(x) = (\gamma + |x + x_0|^{p'})^{-\frac{n+\alpha-p}{p}}$, $\gamma > 0$. We note that (3.12) reduces to $I_0(x) = 0$ for a.e $x \in E$, thus $x_0 \in \overline{E}$ verifies $\nabla\omega(x)\cdot x_0 = \nabla\sigma(x)\cdot x_0 = 0$ for a.e. $x \in E$. In this way, (WSI) takes the more familiar form (with only one weight)

$$\left(\int_E |u(x)|^q\,\sigma(x)\,dx\right)^{1/q} \leqslant \tilde{K}_0\left(\int_E |\nabla u(x)|^p\,\sigma(x)\,dx\right)^{1/p} \quad \text{for all } u \in \dot{W}^{1,p}_\sigma(\mathbb{R}^n), \tag{3.15}$$

where

$$\tilde{K}_0 = \frac{p(n_a-1)}{n_a(n_a-p)} \frac{\left(\int_E u_\gamma(y)^q\,|y|^{p'}\sigma(y)\,dy\right)^{\frac{1}{p'}} \left(\int_E u_\gamma(y)^q\sigma(y)\,dy\right)^{\frac{1}{q}}}{\int_E u_\gamma(y)^{q(1-\frac{1}{n_a})}\sigma(y)\,dy} \tag{3.16}$$

is the best constant in (3.15) (not depending on $\gamma > 0$).

Case 2: condition C-0 holds, $p > 1$ and $n_a = +\infty$.

In order to have equality in (WSI), we must have equality in the proof of Lemma 2.1(i), Case 3. In particular, we have $\Delta_A\phi(x) = 0$ for a.e. $x \in E$, which leads us to the degenerate case $\nabla\phi(x) = x_0$ for a.e. $x \in E$, for some $x_0 \in E$, which is not compatible with the Monge–Ampère equation (2.5). Therefore, no equality can be obtained in (WSI).

Case 3: condition C-1 holds and $p > 1$.

Equality in (WSI) requires equality in each estimate in the proof of Lemma 2.1(ii), Case 1. First, as before, the equality in the AM-GM inequality $\det D_A^2\phi(x) \leqslant (\frac{\Delta_A\phi(x)}{n})^n$ for μ-a.e. $x \in E$ yields

$$\nabla\phi(x) = \lambda x + x_0 \quad \text{for a.e.} \quad x \in E \cap E_0 \tag{3.17}$$

for some $\lambda > 0$ and $x_0 \in \overline{E}$. The equality in the second estimate, where (2.4) is applied, together with the continuity of the weights σ and ω implies

$$\omega(x)^{\frac{1}{q}} = C_1\sigma(x)^{1/p} \quad \text{for all } x \in E, \tag{3.18}$$

where $C_1 > 0$ is the constant in condition C-1. Furthermore, the equality when we apply condition C-1 requires

$$\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot(\lambda x + x_0) = 0 \quad \text{for a.e.} \quad x \in E \cap E_0.$$

A similar argument as before shows that the latter relation can be transformed equivalently into

$$\frac{\tau}{p'} + \frac{\alpha}{p} + I_0(x) = 0 \quad \text{for a.e.} \quad x \in E \cap E_0, \tag{3.19}$$

where

$$I_0(x) = \lambda^{-1}\left(\frac{1}{p'}\frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p}\frac{\nabla\sigma(x)}{\sigma(x)}\right)\cdot x_0.$$

By condition C-1, it is clear that $\frac{\tau}{p'} + \frac{\alpha}{p} \geqslant 0$ (taking $y = x$) and $I_0(x) \geqslant 0$ for a.e. $x \in E$ (taking $y := y_k$ where $\{y_k\}_k \subset E$ converges to x_0). Therefore, by (3.19), we have that $\frac{\tau}{p'} + \frac{\alpha}{p} = 0$ and $I_0(x) = 0$ for a.e. $x \in E \cap E_0$. Since $n_a = n$, it follows that $\frac{\tau}{q} = \frac{\alpha}{p}$; this relation combined with $\frac{\tau}{p'} + \frac{\alpha}{p} = 0$ gives that $\tau = \alpha = 0$.

Due to (3.18), condition C-1 implies

$$\nabla\omega(x)\cdot y \geqslant 0, \quad \nabla\sigma(x)\cdot y \geqslant 0 \quad \text{for a.e } x \in E \text{ and all } y \in E. \tag{3.20}$$

Let $x \in E$ be any differentiability point of ω and fix $\rho > 0$ small enough such that $x + B_\rho \subset E$. Applying (3.20) for $y := x + z$ with arbitrarily $z \in B_\delta$ and using the fact that $\nabla\omega(x)\cdot x = 0$ (since $\tau = 0$), it follows that $\nabla\omega(x)\cdot z \geqslant 0$ for every $z \in B_\delta$. This holds in fact for every $z \in \mathbb{R}^n$, which implies $\nabla\omega(x) = 0$. Since ω is locally Lipschitz (thanks to our assumption an (3.18)), the latter relation implies ω is a constant, $\omega \equiv c_\omega > 0$; in a similar way, $\sigma \equiv c_\sigma > 0$. By (3.18), one has $c_\omega^{\frac{1}{q}} = C_1 c_\sigma^{\frac{1}{p}}$. We also note that x_0 can be arbitrarily fixed in $\overline{E}$.

A similar argument as in Case 1 shows that when we use Hölder inequality in the proof of Theorem 1.1(ii), the equality case implies that the extremal function verifies $|\nabla u(x)|^p = c_1 u(x)^q|x + x_0|^{p'}$ for some $c_1 > 0$ and every $x \in E$. The rest is the same as in (3.15) and (3.16), where we may choose without loss of generality $\sigma = 1$; in fact, (3.15) is a Talenti-type sharp Sobolev inequality on convex cones.

3.2. *Case* $p = 1$

We now turn our attention to analyze the equality cases in Theorem 1.2. Since the proof is similar to the case $p > 1$, we outline only the differences.

Case 1: condition C-0 holds, $p = 1$ and $n_a < +\infty$.

We follow the proof of Lemma 2.1(i), Case 2. First, for some $\lambda > 0$ and $x_0 \in \overline{E}$, one has that $\nabla\phi(x) = \lambda x + x_0$ for a.e. $x \in E \cap E_0$. Similarly to (3.11), one necessarily has that

$$\omega(\lambda x + x_0)^{-1/q}\, \sigma(\lambda x + x_0) = \lambda^{(n_a - n)/n_a} \omega(x)^{-1/q}\, \sigma(x) \ \text{ for a.e. } \ x \in E \cap E_0.$$

Furthermore, it follows that

$$\lambda = C_0 \frac{\nabla\sigma(x)}{\sigma(x)} \cdot (\lambda x + x_0) \ \text{ for a.e. } \ x \in E \cap E_0,$$

which can be written as

$$1 = C_0(\alpha + I_0(x)) \ \text{ for a.e. } \ x \in E \cap E_0,$$

where $I_0(x) = \lambda^{-1} \frac{\nabla\sigma(x)}{\sigma(x)} \cdot x_0$. Since $I_0(x) \geqslant 0$ for a.e. $x \in E$ (due to condition C-0 for $p = 1$), it follows that $C_0\alpha \leqslant 1$. Clearly, condition C-0 for $p = 1$ and $y = x$ gives that $1 \leqslant C_0\alpha$. Thus $C_0\alpha = 1$. On the other hand, we must also have $C_0(1 - \frac{n}{n_a}) = \frac{1}{n_a}$, that is, $C_0 = \frac{1}{n_a - n}$. Consequently, we obtain $\frac{1}{n_a - n} = \frac{1}{\alpha}$, which is equivalent to $(\alpha - \tau)(n + \alpha - p) = 0$. Due to (1.2), it follows that $\alpha = \tau$. We can apply again Proposition 3.1(ii) to obtain the existence of $c > 0$ such that $\omega(x) = c\sigma(x)$ for every $x \in E$, and the $\sigma^{1/\alpha}$ is concave in E. In this way, (WSI) reduces to

$$\left(\int_E |u(x)|^{\frac{n_a}{n_a - 1}}\, \sigma(x)\, dx\right)^{1 - \frac{1}{n_a}} \leqslant \tilde{K}_0 \int_E |\nabla u(x)|\, \sigma(x)\, dx \ \text{ for all } u \in \dot{W}^{1,1}_\sigma(\mathbb{R}^n), \tag{3.21}$$

where

$$\tilde{K}_0 = \frac{1}{n_a} \left(\int_{B\cap E} \sigma(y)\, dy\right)^{-\frac{1}{n_a}}. \tag{3.22}$$

The constant $\tilde{K}_0$ in (3.21) is sharp. Indeed, according to [**6**, rel. (1.14), p. 2977], one has $P_\sigma(B, E) = (n + \alpha) \int_{B\cap E} \sigma(x) dx$. Since $n_a = n + \alpha$, by considering $u(x) := \mathbb{1}_{B\cap E}(x)$, it yields

$$\begin{aligned}\int_E |\nabla u(x)|\, \sigma(x)\, dx = P_\sigma(B, E) = n_a \int_{B\cap E} \sigma(x) dx &= \tilde{K}_0^{-1} \left(\int_{B\cap E} \sigma(x)\, dx\right)^{1 - \frac{1}{n_a}} \\ &= \tilde{K}_0^{-1} \left(\int_E |u(x)|^{\frac{n_a}{n_a - 1}} \sigma(x)\, dx\right)^{1 - \frac{1}{n_a}},\end{aligned}$$

which gives equality in (3.21).

Case 2: condition C-0 holds, $p = 1$ and $n_a = +\infty$.

We must have equality in the proof of Lemma 2.1(i), Case 4. Thus we have $\Delta_A \phi(x) = 0$ for a.e. $x \in E$, which contradicts again the Monge–Ampère equation (2.5). Thus, no equality can be obtained in (WSI).

Case 3: condition C-1 holds and $p = 1$.

The discussion is similar to Case 3 with $p > 1$, obtaining that equality in (WSI) implies that both ω and σ are constant, $\omega \equiv c_\omega > 0$, $\sigma \equiv c_\sigma > 0$, and $c_\omega^{\frac{1}{q}} = C_1 c_\sigma$, where $C_1 > 0$ is the constant in condition C-1. Therefore, (WSI) becomes the (usual) sharp isoperimetric inequality on the cone E. This concludes the proof of Theorem 1.3. □

4. *Examples and applications*

In this section, we illustrate the application of Theorems 1.1–1.3 to various examples.

4.1. *Weights satisfying conditions C-0 and C-1*

4.1.1. *Monomial weights.* We first discuss the validity of condition C-0 for *monomial* weights to recover from our statements the results of [**5, 9, 25**] . More precisely, let $\tau_i \in \mathbb{R}$ and $\alpha_i \geqslant 0$, $i = 1, \dots, n$; $\tau = \tau_1 + \cdots + \tau_n$ and $\alpha = \alpha_1 + \cdots + \alpha_n$ be such that

$$\gamma_i := \frac{\tau_i}{p'} + \frac{\alpha_i}{p} \geqslant 0 \quad \text{and} \quad \beta_i := \frac{\alpha_i}{p} - \frac{\tau_i}{q} \geqslant 0, \quad i = 1, \dots, n, \tag{4.1}$$

where $q = \frac{p(\tau+n)}{\alpha+n-p}$ with the property that if $\gamma_i = 0$ for some $i \in \{1, \dots, n\}$, then $\tau_i = \alpha_i = 0$.
We consider the convex cone

$$E = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0 \text{ whenever } \frac{\tau_i}{p'} + \frac{\alpha_i}{p} > 0 \right\}, \tag{4.2}$$

and the weights $\omega(x) = x_1^{\tau_1} \cdots x_n^{\tau_n}$ and $\sigma(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $x = (x_1, \dots, x_n) \in E$.

PROPOSITION 4.1. *Assume $n_a > n$. Let $E \subseteq \mathbb{R}^n$ be the convex cone given in (4.2) and $\omega(x) = x_1^{\tau_1} \cdots x_n^{\tau_n}$ and $\sigma(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for every $x = (x_1, \dots, x_n) \in E$. Then condition C-0 holds with the constant*

$$C_0 = \frac{n_a}{n_a - n} \left(\left(\frac{\beta_1}{\gamma_1}\right)^{\beta_1} \cdots \left(\frac{\beta_n}{\gamma_n}\right)^{\beta_n} \right)^{\frac{n_a}{n_a - n}}. \tag{4.3}$$

Here we use the convention $0^0 = 1$.

Proof. We first assume that $n_a < \infty$. Let $x = (x_1, \dots, x_n) \in E$ and $y = (y_1, \dots, y_n) \in E$ be fixed. By the scaling invariance relation (1.4) and the form of β_i, we have that

$$\beta_1 + \cdots + \beta_n + \frac{n}{n_a} = 1.$$

Then, by using the weighted AM-GM inequality, it follows that

$$\begin{aligned}
&\left(\left(\frac{\sigma(y)}{\sigma(x)}\right)^{1/p} \left(\frac{\omega(x)}{\omega(y)}\right)^{1/q} \right)^{n_a/(n_a - n)} \\
&= \left(\left(\frac{y_1}{x_1}\right)^{\beta_1} \cdots \left(\frac{y_n}{x_n}\right)^{\beta_n} \right)^{\frac{1}{\beta_1 + \cdots + \beta_n}} \\
&= \left(\left(\frac{\beta_1}{\gamma_1}\right)^{\beta_1} \cdots \left(\frac{\beta_n}{\gamma_n}\right)^{\beta_n} \right)^{\frac{1}{\beta_1 + \cdots + \beta_n}} \left(\frac{\gamma_1}{\beta_1}\frac{y_1}{x_1}\right)^{\frac{\beta_1}{\beta_1 + \cdots + \beta_n}} \cdots \left(\frac{\gamma_n}{\beta_n}\frac{y_n}{x_n}\right)^{\frac{\beta_n}{\beta_1 + \cdots + \beta_n}} \\
&\leqslant \frac{1}{\beta_1 + \cdots + \beta_n} \left(\left(\frac{\beta_1}{\gamma_1}\right)^{\beta_1} \cdots \left(\frac{\beta_n}{\gamma_n}\right)^{\beta_n} \right)^{\frac{1}{\beta_1 + \cdots + \beta_n}} \left(\gamma_1 \frac{y_1}{x_1} + \cdots + \gamma_n \frac{y_n}{x_n} \right) \\
&= C_0 \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot y,
\end{aligned}$$

where

$$C_0 = \frac{1}{\beta_1 + \cdots + \beta_n} \left(\left(\frac{\beta_1}{\gamma_1}\right)^{\beta_1} \cdots \left(\frac{\beta_n}{\gamma_n}\right)^{\beta_n} \right)^{\frac{1}{\beta_1 + \cdots + \beta_n}},$$

which ends the proof.

When $n_a = +\infty$ (that is, $p = q$, which is equivalent to $\alpha = p + \tau$), we have that $\beta_1 + \cdots + \beta_n = 1$. The same proof as before using the AM-GM inequality shows that condition C-0 holds (see (2.2) in Remark 2.1(ii)) with the constant

$$C_0 = \left(\frac{\beta_1}{\gamma_1}\right)^{\beta_1} \cdots \left(\frac{\beta_n}{\gamma_n}\right)^{\beta_n},$$

which agrees with (4.3) whenever $n_a \to \infty$. □

From the last proposition, we have the following corollary of our main theorems.

Corollary 4.1. *Let $\tau_i \in \mathbb{R}$ and $\alpha_i \geqslant 0$, $i = 1, \dots, n$, and $\tau = \tau_1 + \cdots + \tau_n$ and $\alpha = \alpha_1 + \cdots + \alpha_n$. Consider the convex cone given in (4.2) and the weights $\omega(x) = x_1^{\tau_1} \cdots x_n^{\tau_n}$ and $\sigma(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $x = (x_1, \dots, x_n) \in E$. If conditions (1.2), (1.3) and (4.1) hold and $q = \frac{p(\tau+n)}{\alpha+n-p}$, then (WSI) holds. In addition, if $\omega = \sigma$, then the constant K_0 arising in (WSI) is optimal.*

Proof. The first conclusion follows directly from Theorems 1.1 and 1.2 taking into account Proposition 4.1.

To obtain the second conclusion, we use Theorem 1.3(i). Note that when $\tau_i = \alpha_i$, $i = 1, \dots, n$, one has that $n_a = n + \alpha_1 + \cdots + \alpha_n$, $\beta_i = \frac{\alpha_i}{n_a}$ and $\gamma_i = \alpha_i$, $i = 1, \dots, n$, while the convex cone introduced in (4.2) becomes

$$E = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0 \text{ whenever } \alpha_i > 0\}.$$

In this case, the constant in Proposition 4.1 reduces to $C_0 = \frac{1}{n_a - n}$. □

Remark 4.1. The first conclusion of Corollary 4.1 covers the main result in [**9**, Theorem 1] with a slightly different notation. The second conclusion shows that the main results [**5**, Theorem 1.3] and [**25**, Theorem 4.2, $\theta = 1$] are also particular cases of our results.

Remark 4.2. Let $E = (0, \infty)^n$ for any $n \geqslant 2$.

(a) If $\omega(x_1, \dots, x_n) = \sqrt[n]{x_1 \cdots x_n}$ and $\sigma(x_1, \dots, x_n) = x_1 + \cdots + x_n$, $(x_1, \dots, x_n) \in E$, the pair (ω, σ) does not satisfy condition C-0. However, since $\omega \leqslant \sigma/n$, Proposition 4.1 provides (a nonoptimal) (WSI) for the weights ω and σ; the corresponding constant $K_0 > 0$ in (WSI) can be obtained by using the monomial setting, see [**5, 25**].

(b) Conversely, if $\omega(x_1, \dots, x_n) = x_1 + \cdots + x_n$ and $\sigma(x_1, \dots, x_n) = \sqrt[n]{x_1 \cdots x_n}$, $(x_1, \dots, x_n) \in E$, it turns out that the pair (ω, σ) satisfies condition C-0 if and only if $n = 2$.

4.1.2. *Radial weights.* Using Theorems 1.1 and 1.2 as building blocks, we obtain further consequences that are suitable for other applications. The first consequence is the following domain additivity property of (WSI).

Corollary 4.2. *Let $M \in \mathbb{N}$ be a positive integer and assume that E is an open set in $\mathbb{R}^n$ of the form $E = (\cup_{i=1}^M E_i) \cup E_0$, where E_i are pairwise disjoint convex cones for $i = 1, \dots, M$*

and E_0 is a set of measure zero. Let $\omega, \sigma : \cup_i E_i \to (0,\infty)$ be two homogeneous weights such that their restrictions $(\omega_{|_{E_i}}, \sigma_{|_{E_i}})$ satisfy the conditions of Theorem 1.1 or Theorem 1.2 for all $i = 1, \dots, M$. Then (WSI) holds on the set E.

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$. Applying Theorem 1.1 or Theorem 1.2 on the domain E_i, we obtain

$$\left(\int_{E_i} |u(x)|^q\, \omega(x)\, dx\right)^{1/q} \leqslant K_i \left(\int_{E_i} |\nabla u(x)|^p\, \sigma(x)\, dx\right)^{1/p} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n),$$

for all $i = 1, \dots M$.

Since E_i are pairwise disjoint and E_0 has measure zero, it follows from Minkowski's inequality that

$$\left(\int_{E} |u(x)|^q\, \omega(x)\, dx\right)^{1/q} = \left(\sum_{i=1}^{M} \int_{E_i} |u(x)|^q\, \omega(x)\, dx\right)^{1/q} \leqslant \sum_{i=1}^{M} \left(\int_{E_i} |u(x)|^q\, \omega(x)\, dx\right)^{1/q}$$
$$\leqslant \sum_{i=1}^{M} K_i \left(\int_{E_i} |\nabla u(x)|^p\, \sigma(x)\, dx\right)^{1/p} \leqslant K_0 \left(\int_{E} |\nabla u(x)|^p\, \sigma(x)\, dx\right)^{1/p},$$

where $K_0 = M^{\frac{1}{p'}} \max_{i=1,\dots,M} K_i > 0$. □

With the domain additivity property of (WSI), we now consider radial weights and deduce a particular case of the inequality of Caffarelli, Kohn and Nirenberg [**7**, Inequality (1.4)], a case also called Hardy–Littlewood–Sobolev's inequality. To do this, we first prove the following.

COROLLARY 4.3. *Let us assume that the parameters p, q, α, τ satisfy conditions (1.2)–(1.4) and $\frac{\tau}{p'} + \frac{\alpha}{p} > 0$. Then there exists $K_0 > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$, one has*

$$\left(\int_{\mathbb{R}^n} |u(x)|^q\, |x|^\tau\, dx\right)^{1/q} \leqslant K_0 \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p\, |x|^\alpha\, dx\right)^{1/p}. \tag{4.4}$$

Proof. By standard arguments, we can find $M \in \mathbb{N}$ and pairwise disjoint convex cones $E_i i = 1, \dots, M$ such that $\mathbb{R}^n = (\cup_{i\in M} E_i) \cup E_0$ where E_0 is the union of the boundaries of E_i (and therefore a null measure set). Moreover, we can choose E_i so small that for all $x, y \in E_i$ we have that $x \cdot y \geqslant \frac{1}{2}|x| \cdot |y|$.

Let us assume first that $n_a > n$. Using that $\nabla(|x|^\alpha) = \alpha\, x\, |x|^{\alpha-2}$ and $\nabla(|x|^\tau) = \tau\, x\, |x|^{\tau-2}$, condition C-0 on E_i, $i = 1, \dots, M$, can be written as

$$\frac{|y|}{|x|} \leqslant C_0 \left(\frac{\tau}{p'} + \frac{\alpha}{p}\right) \frac{x \cdot y}{|x|^2},\ \ x, y \in E_i. \tag{4.5}$$

Using the estimate $x \cdot y \geqslant \frac{1}{2}|x| \cdot |y|$, $x, y \in E_i$, we see that the above relation is satisfied for $x, y \in E_i$ with a properly chosen constant $C_0 > 0$. The conclusion now follows by Corollary 4.2. In the case $n_a = n$, we can argue in a similar way. □

We note that the condition $\frac{\tau}{p'} + \frac{\alpha}{p} > 0$ in Corollary 4.3 is not assumed in [**7**]. However, it turns out that by applying Corollary 4.3 with appropriate values of τ, α and q, we will obtain [**7**, Inequality (1.4) with $a = 1$] for the full range of exponents. In fact, with the notation from [**7**], let $p \geqslant 1$, $r > 0$, $\beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{r}+\frac{\gamma}{n}>0, \tag{4.6}$$

$$0\leqslant \beta-\gamma\leqslant 1 \tag{4.7}$$

and

$$\frac{1}{r}+\frac{\gamma}{n}=\frac{1}{p}+\frac{\beta-1}{n}. \tag{4.8}$$

We shall then prove the following desired inequality.

Corollary 4.4. *Under assumptions (4.6)–(4.8), there exists $K_0=K_0(p,\beta,\gamma)>0$ such that for all $u\in C_0^\infty(\mathbb{R}^n)$, one has*

$$\left(\int_{\mathbb{R}^n}|u(x)|^r\,|x|^{\gamma r}\,dx\right)^{1/r}\leqslant K_0\left(\int_{\mathbb{R}^n}|\nabla u(x)|^p\,|x|^{\beta p}\,dx\right)^{1/p}. \tag{4.9}$$

Proof. Let $d>1$ be fixed that will be specified later, and let

$$\tau:=n(d-1)+\gamma rd,\quad \alpha:=(n-p)(d-1)+\beta pd\quad\text{and}\quad q:=r.$$

We claim the parameters p,q,α,τ satisfy conditions (1.2), (1.3) and (1.4). First, a straightforward computation shows that the balance condition (1.4) is equivalent to condition (4.8) which determines the value of r in terms of p, β and γ.

Inequality $p<\alpha+n$ in (1.2) is equivalent to $n+p(\beta-1)>0$ which holds true due to (4.6) and (4.8). The second inequality in (1.2), that is, $\alpha\leqslant\tau+p$, is equivalent to $(\beta-1)p\leqslant\gamma r$. Adding n to both sides to the last inequality, it follows from (4.8) that the resulting inequality is equivalent to $p\leqslant r$. Again by (4.8), $r=\frac{pn}{(\beta-1)p+n-\gamma p}$. Hence $p\leqslant r$ is equivalent to $\beta-1\leqslant\gamma$ which holds from (4.7). Thus (1.2) holds.

To show (1.3), we observe that from (4.8), $\alpha\geqslant(1-\frac{p}{n})\tau$ is equivalent to $\frac{\beta}{r}\geqslant\frac{\gamma}{p}(1-\frac{p}{n})$, which again by (4.8) is equivalent to $(\beta-\gamma)(\frac{1}{r}+\frac{\gamma}{n})\geqslant 0$, which in turn holds true from (4.6) and (4.7).

To apply Corollary 4.3, it remains to check the inequality $\frac{\tau}{p'}+\frac{\alpha}{p}>0$, which for the chosen exponents can be written equivalently as $d(n-1+\beta+\frac{\gamma r}{p'})-n+1>0$. From (4.6) and (4.8), it follows that $n-1+\beta+\frac{\gamma r}{p'}=n(\frac{1}{r}+\frac{\gamma}{n})(1+\frac{r}{p'})>0$. So choosing $d>1$ large enough, we obtain that $d(n-1+\beta+\frac{\gamma r}{p'})-n+1>0$ as desired.

Therefore, from Corollary 4.3 there exists $K_0>0$ such that for every $v\in C_0^\infty(\mathbb{R}^n)$, one has

$$\left(\int_{\mathbb{R}^n}|v(x)|^r\,|x|^{n(d-1)+\gamma rd}\,dx\right)^{1/r}\leqslant K_0\left(\int_{\mathbb{R}^n}|\nabla v(x)|^p\,|x|^{(n-p)(d-1)+\beta pd}\,dx\right)^{1/p}. \tag{4.10}$$

In addition, by an approximation argument, the last inequality is also valid for every $v\in C_0^1(\mathbb{R}^n)$.

On the other hand, for any fixed $d>1$, if $T:\mathbb{R}^n\to\mathbb{R}^n$ is the map defined by $T(x)=|x|^{d-1}x$, then the determinant of its Jacobian is

$$\det J_T(x)=d|x|^{n(d-1)},\ x\neq 0,$$

see Lam and Lu [19]. For any $u\in C_0^\infty(\mathbb{R}^n)$, we introduce $Ru(x)=d^{-\frac{1}{p'}}u(T(x))$ (with the usual convention that $\frac{1}{p'}=0$ when $p=1$). Due to [19, Lemma 2.2], a change of variable gives that for every $t,\mu\in\mathbb{R}$ and every continuous function $f:\mathbb{R}\to\mathbb{R}$, one has

$$\int_{\mathbb{R}^n}\frac{f\left(d^{-\frac{1}{p'}}u(x)\right)}{|x|^t}dx=d\int_{\mathbb{R}^n}\frac{f(Ru(x))}{|x|^{n+td-nd}}dx \tag{4.11}$$

and

$$\int_{\mathbb{R}^n} \frac{|\nabla Ru(x)|^p}{|x|^{d(p+\mu-n)+n-p}} dx \leqslant \int_{\mathbb{R}^n} \frac{|\nabla u(x)|^p}{|x|^{\mu}} dx. \tag{4.12}$$

If we apply (4.11) and (4.12) with $t := -\gamma r$, $\mu := -\beta p$ and $f(s) = |s|^r$, then using (4.10) with $v := Ru \in C_0^1(\mathbb{R}^n)$, we obtain precisely (4.9). □

4.1.3. *Further examples of weights.* In this section, we further illustrate conditions C-0 and C-1. A sufficient condition for C-0 to hold is the following.

Proposition 4.2. *Let $E \subseteq \mathbb{R}^n$ be an open convex cone and let $\omega, \sigma : E \to (0, \infty)$ be differentiable weights satisfying (1.1)–(1.4), and $n_a > n$. If*

$$F(x) = \omega(x)^{\delta}\, \sigma(x)^{\gamma}$$

is concave in E with $\delta = -\frac{1}{q}\frac{n_a}{n_a-n}$, $\gamma = \frac{1}{p}\frac{n_a}{n_a-n}$ and $\nabla\omega(x)\cdot y \geqslant 0$ for every $x, y \in E$, then the pair (ω, σ) satisfies condition C-0 with constant $C_0 = \frac{n_a}{n_a-n}$.

Proof. From the form of F, the pair (ω, σ) satisfies C-0 if and only if

$$\frac{F(y)}{F(x)} \leqslant C_0 \left(\frac{1}{p'} \frac{\nabla\omega(x)}{\omega(x)} + \frac{1}{p} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot y, \quad \forall x, y \in E.$$

To prove the last inequality, we see that by (1.1), F is homogenous of degree $\delta\tau + \gamma\alpha$. Hence $\nabla F(x) \cdot x = (\delta\tau + \gamma\alpha)\, F(x)$ for every $x \in E$. By the concavity of F in E, we have that

$$F(y) - F(x) \leqslant \nabla F(x) \cdot (y - x) \quad \text{for all } x, y \in E. \tag{4.13}$$

Since from the balance condition (1.4) $\delta\tau + \gamma\alpha - 1 = 0$, it follows from (4.13) that

$$\frac{F(y)}{F(x)} \leqslant \left(\delta \frac{\nabla\omega(x)}{\omega(x)} + \gamma \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot y \quad \text{for all } x, y \in E. \tag{4.14}$$

On the other hand, by assumption $\nabla\omega(x) \cdot y \geqslant 0$ and $\delta < 0$, so we get

$$\frac{F(y)}{F(x)} \leqslant \left(\frac{1}{p} \frac{n_a}{n_a - n} \frac{\nabla\sigma(x)}{\sigma(x)} \right) \cdot y \quad \text{for all } x, y \in E.$$

Thus, $\nabla\sigma(x) \cdot y \geqslant 0$ for every $x, y \in E$. Using again that $\nabla\omega(x) \cdot y \geqslant 0$ for every $x, y \in E$, we obtain from (4.14) that C-0 holds with $C_0 = \frac{n_a}{n_a-n}$. □

To illustrate Proposition 4.2 we show the following example. Let $E = (0, \infty)^n$ with $n \geqslant 2$, $0 < \alpha < p$ and $1 \leqslant p < \alpha + n$. If $\omega \equiv 1$, $\sigma(x) = (\frac{x_1 \cdots x_n}{x_1 + \cdots + x_n})^{\alpha/(n-1)}$, then $n_a = \frac{pn}{p-\alpha}$,

$$F(x) = \omega(x)^{\delta}\, \sigma(x)^{\gamma} = \left(\frac{x_1 \cdots x_n}{x_1 + \cdots + x_n} \right)^{\frac{1}{n-1}}$$

in Proposition 4.2, which is concave in E, see Marcus and Lopes [**22**]. Therefore, the pair (ω, σ) satisfies C-0 with $C_0 = p/\alpha$ and from Theorem 1.1, (WSI) holds for these weights with $q = \frac{p\,n}{\alpha+n-p}$.

We conclude this part by giving an example of weights for which condition C-1 holds. Let $E = (0, \infty)^n$, $n \geqslant 2$, $\tau \geqslant 0$, and $1 \leqslant p < n$. If $\omega(x) = (x_1 + \cdots + x_n)^{\tau}$ and $\sigma(x) = |x|^{\tau(1-p/n)}$, then $n_a = n$ and $q = \frac{np}{n-p}$. Since

$$\sup_{x \in E} \frac{\omega(x)^{1/q}}{\sigma(x)^{1/p}} = n^{\frac{\tau}{2q}} \in (0, \infty),$$

condition C-1 holds, and from Theorem 1.1(ii), we get that (WSI) holds for these weights. In particular, if $\tau = 0$, then (WSI) reduces to the sharp Sobolev inequality of Talenti [**30**] on the cone E.

4.2. *Weighted PDEs*

4.2.1. *Spectral gap.* In this subsection, we provide an estimate of the spectral gap for a weighted eigenvalue problem. More precisely, we have the following.

PROPOSITION 4.3. *Let $E \subseteq \mathbb{R}^n$ be an open convex cone and let $\Omega \subset \mathbb{R}^n$ be an open bounded set such that $\Omega \cap E \neq \emptyset$. Let $\omega, \sigma : \overline{E} \to [0, \infty)$ be two continuous nonzero weights which are differentiable in E, satisfying (1.1) with $\alpha = \tau + 2$, condition C-0 for some $C_0 > 0$, and $\sigma|_{\partial E} = 0$. Then any eigenvalue $\lambda > 0$ of the problem*

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = \lambda \omega u & \text{in} \quad \Omega \cap E, \\ u = 0 & \text{on} \quad \partial\Omega \cap \overline{E}. \end{cases} \tag{P}$$

verifies

$$\lambda \geqslant \frac{1}{4C_0^2} \sup_{v \in C_0^\infty(\Omega)\setminus\{0\}, v \geqslant 0} \frac{\left(\int_E v(y)\omega(y)^{-\frac{1}{2}}\sigma(y)^{\frac{1}{2}}\,dy\right)^2}{\int_E v(y)\,|y|^2 dy \int_E v(y) dy} > 0.$$

Proof. Let us multiply the first equation in (P) by $u \neq 0$; an integration and the divergence theorem gives that

$$-\int_{\partial(\Omega\cap E)} \sigma(x)\frac{\partial u}{\partial \mathbf{n}}(x)u(x)ds(x) + \int_\Omega |\nabla u(x)|^2\sigma(x)dx = \lambda \int_\Omega u(x)^2\omega(x)dx. \tag{4.15}$$

Since $\partial(\Omega \cap E) \subseteq \partial\Omega \cup \partial E$, the first integral in the left-hand side vanishes either for $\sigma|_{\partial E} = 0$ or for the Dirichlet boundary condition $u = 0$ on $\partial\Omega \cap \overline{E}$. Therefore, equation (4.15) reduces to

$$\int_\Omega |\nabla u(x)|^2\sigma(x)d = \lambda \int_\Omega u(x)^2\omega(x)dx. \tag{4.16}$$

Since $\tau + n > 0$ (by the locally integrability of ω) and $\alpha = \tau + 2$, assumptions (1.2)–(1.4) are immediately verified with the choices $p = q = 2$. In particular, $n_a = +\infty$ and we can apply Theorem 1.1(i), obtaining

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u(x)|^2\sigma(x)dx}{\displaystyle\int_\Omega u(x)^2\omega(x)dx} \geqslant K_0^{-2},$$

where the constant $K_0 > 0$ appears in the statement of Theorem 1.1/(i). The rest is a simple computation. $\square$

REMARK 4.3. Due to (4.15), a similar spectral gap estimate can be obtained in the same way also for the Neumann boundary value condition. Moreover, the case $p \neq 2$ can be also handled using the operator $\operatorname{div}(\sigma|\nabla u|^{p-2}\nabla u)$ in problem (P).

4.2.2. *A variational problem.* Applying a variational method, we prove the following result.

PROPOSITION 4.4. *Let $E \subseteq \mathbb{R}^n$ be an open convex cone and let $\Omega \subset \mathbb{R}^N$ be an open bounded set such that $\Omega \cap E \neq \emptyset$. Let $\omega, \sigma : \overline{E} \to [0, \infty)$ be two nonzero weights continuous* *in $\overline{E}$, differentiable a.e. in E, and satisfying (1.1)–(1.4) with $\alpha < \tau + 2$, condition C-0, and $\sigma|_{\partial E} = 0$. Then for every $r \in (2, q)$, the problem*

$$\begin{cases} -\mathrm{div}(\sigma\nabla u) + \sigma u = \omega u^{r-1} & \text{in} \quad \Omega \cap E, \\ u \geqslant 0 & \text{in} \quad \Omega \cap E, \\ u = 0 & \text{on} \quad \partial\Omega \cap \overline{E}, \end{cases} \tag{$\mathcal{P}$}$$

has a nonzero weak solution in the weighted Sobolev space $W^{1,2}_\sigma(\Omega)$.

Proof. We first recall that the weighted Sobolev space $W^{1,2}_\sigma(\Omega)$ is the set of all measurable functions such that $u \in L^2_\sigma(\Omega \cap E)$ and $|\nabla u| \in L^2_\sigma(\Omega \cap E)$ with the norm

$$\|u\|_{W^{1,2}_\sigma(\Omega)} = \left(\int_{\Omega\cap E} |\nabla u(x)|^2 \sigma(x)dx + \int_{\Omega\cap E} u(x)^2\sigma(x)dx\right)^{1/2}.$$

By our assumptions, Theorem 1.1 implies that the space $W^{1,2}_\sigma(\Omega)$ is continuously embedded into $L^q_\omega(\Omega \cap E)$, where $q = \frac{2(\tau+n)}{\alpha+n-2}$ is the critical exponent. We also note that $2 < q$ since $\alpha < \tau + 2$. Thus, it follows from the boundedness of Ω that $W^{1,2}_\sigma(\Omega)$ is compactly embedded into $L^r_\omega(\Omega \cap E)$ for any $r \in (2, q)$.

Fix $r \in (2, q)$. Instead of $(\mathcal{P})$, we consider first the problem

$$\begin{cases} -\mathrm{div}(\sigma\nabla u) + \sigma u = \omega u_+^{r-1} & \text{in} \quad \Omega \cap E, \\ u = 0 & \text{on} \quad \partial\Omega \cap \overline{E}, \end{cases} \tag{$\mathcal{P}_+$}$$

where we used the notation $u_+ = \max\{u, 0\}$.

The energy functional $\mathcal{E} : W^{1,2}_\sigma(\Omega) \to \mathbb{R}$ associated with problem $(\mathcal{P}_+)$ is defined by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2_{W^{1,2}_\sigma(\Omega)} - \frac{1}{r}\int_{\Omega\cap E} (u(x))_+^r \omega(x)dx.$$

Standard arguments imply that $\mathcal{E}$ is well defined (since $W^{1,2}_\sigma(\Omega)$ is continuously embedded into $L^r_\omega(\Omega \cap E)$) and $\mathcal{E} \in C^1(W^{1,2}_\sigma(\Omega); \mathbb{R})$; moreover, its differential is given by

$$\mathcal{E}'(u)(v) = \int_{\Omega\cap E} (\nabla u(x) \cdot \nabla v(x) + u(x)v(x))\sigma(x)dx - \int_{\Omega\cap E} (u(x))_+^{r-1}\omega(x)v(x)dx,$$

for all $u, v \in W^{1,2}_\sigma(\Omega)$. In fact, using the divergence theorem together with the Dirichlet boundary condition $u = 0$ on $\partial\Omega \cap \overline{E}$ and $\sigma|_{\partial E} = 0$, it follows that

$$\mathcal{E}'(u)(v) = \int_{\Omega\cap E} (-\mathrm{div}(\sigma(x)\nabla u(x)) + \sigma(x)u(x))v(x)dx - \int_{\Omega\cap E} \omega(x)(u(x))_+^{r-1}v(x)dx.$$

In particular, $u \in W^{1,2}_\sigma(\Omega)$ is a critical point of $\mathcal{E}$ if and only if u is a weak solution of problem $(\mathcal{P}_+)$.

We are going to prove that $\mathcal{E}$ satisfies the Palais–Smale condition on $W^{1,2}_\sigma(\Omega)$. In order to complete this, we consider a sequence $\{u_k\}_k \subset W^{1,2}_\sigma(\Omega)$ such that $\mathcal{E}'(u_k) \to 0$ as $k \to \infty$ and $|\mathcal{E}(u_k)| \leqslant C$ $(k \in \mathbb{N})$ for some $C > 0$, and our aim is to prove that there exists a subsequence of $\{u_k\}_k$ which converges strongly in $W^{1,2}_\sigma(\Omega)$ to some element $u \in W^{1,2}_\sigma(\Omega)$. We notice that

$$r\mathcal{E}(u_k) - \mathcal{E}'(u_k)(u_k) = \left(\frac{r}{2} - 1\right)\|u_k\|^2_{W^{1,2}_\sigma(\Omega)}, \quad k \in \mathbb{N}.$$

Since $\mathcal{E}'(u_k) \to 0$, we have $|\mathcal{E}'(u_k)(u_k)| \leqslant 1$ for large enough values of k. Therefore, for large $k \geqslant 1$, one has that $|r\mathcal{E}(u_k) - \mathcal{E}'(u_k)(u_k)| \leqslant rC + \|u_k\|_{W^{1,2}_\sigma(\Omega)}$. Because $r > 2$, the latter relation implies that $\{u_k\}_k$ is bounded in $W^{1,2}_\sigma(\Omega)$. In particular, we may extract a subsequence of $\{u_k\}_k$ (denoted in the same way) which converges weakly to an element $u \in W^{1,2}_\sigma(\Omega)$, and strongly to u in $L^r_\omega(\Omega\cap E)$. The latter follows from the fact that $W^{1,2}_\sigma(\Omega)$ is compactly embedded into $L^r_\omega(\Omega\cap E)$. A simple computation shows that

$$\|u_k-u\|^2_{W^{1,2}_\sigma(\Omega)}=\mathcal{E}'(u_k)(u_k-u)-\mathcal{E}'(u)(u_k-u)+\int_{\Omega\cap E}((u_k)_+^{r-1}-u_+^{r-1})(u_k-u)\omega dx,\ \ k\in\mathbb{N}.$$

Since $\mathcal{E}'(u_k)\to 0$ as $k\to\infty$ and $\{u_k\}_k$ is bounded in $W^{1,2}_\sigma(\Omega)$, one has that $\mathcal{E}'(u_k)(u_k-u)\to 0$ as $k\to\infty$. Since $\{u_k\}_k$ converges weakly to u, one has that $\mathcal{E}'(u)(u_k-u)\to 0$ as $k\to\infty$. Moreover, since $\{u_k\}_k$ converges strongly to u in $L^r_\omega(\Omega\cap E)$, Hölder's inequality implies

$$\left|\int_{\Omega\cap E}((u_k)_+^{r-1}-u_+^{r-1})(u_k-u)\omega dx\right|\leqslant\left(\|u_k\|^{r-1}_{L^r_\omega(\Omega\cap E)}+\|u\|^{r-1}_{L^r_\omega(\Omega\cap E)}\right)\|u_k-u\|_{L^r_\omega(\Omega\cap E)}\to 0$$

as $k\to\infty$. Summing up, it follows that $\|u_k-u\|^2_{W^{1,2}_\sigma(\Omega)}\to 0$ as $k\to\infty$, which means that $\{u_k\}_k$ strongly converges to u in $W^{1,2}_\sigma(\Omega)$.

We shall prove that $\mathcal{E}$ satisfies the mountain pass geometry, that is, there exist $w_0\in W^{1,2}_\sigma(\Omega)$ and $\rho>0$ such that $\|w_0\|_{W^{1,2}_\sigma(\Omega)}>\rho$ and

$$\inf_{\|u\|_{W^{1,2}_\sigma(\Omega)}=\rho}\mathcal{E}(u)>\mathcal{E}(0)\geqslant\mathcal{E}(w_0). \tag{4.17}$$

To see this, let $c_{\omega,\sigma}>0$ be the constant in the Sobolev embedding $W^{1,2}_\sigma(\Omega)$ into $L^r_\omega(\Omega\cap E)$, that is, $\|u\|_{L^r_\omega(\Omega\cap E)}\leqslant c_{\omega,\sigma}\|u\|_{W^{1,2}_\sigma(\Omega)}$ for every $u\in W^{1,2}_\sigma(\Omega)$. Therefore, since $\|u_+\|_{L^r_\omega(\Omega\cap E)}\leqslant \|u\|_{L^r_\omega(\Omega\cap E)}$, it follows that

$$\begin{aligned}\mathcal{E}(u)&=\frac12\|u\|^2_{W^{1,2}_\sigma(\Omega)}-\frac1r\int_{\Omega\cap E}(u(x))_+^r\omega(x)dx=\frac12\|u\|^2_{W^{1,2}_\sigma(\Omega)}-\frac1r\|u_+\|^r_{L^r_\omega(\Omega\cap E)}\\&\geqslant\frac12\|u\|^2_{W^{1,2}_\sigma(\Omega)}-\frac1r\|u\|^r_{L^r_\omega(\Omega\cap E)}\\&\geqslant\frac12\|u\|^2_{W^{1,2}_\sigma(\Omega)}-\frac1r c^r_{\omega,\sigma}\|u\|^r_{W^{1,2}_\sigma(\Omega)}=\left(\frac12-\frac1r c^r_{\omega,\sigma}\|u\|^{r-2}_{W^{1,2}_\sigma(\Omega)}\right)\|u\|^2_{W^{1,2}_\sigma(\Omega)}.\end{aligned}\tag{4.18}$$

Since $r>2$, the number $\rho:=(\frac{r}{4c^r_{\omega,\sigma}})^{\frac{1}{r-2}}$ is well defined and $\rho>0$. Thus, for any $u\in W^{1,2}_\sigma(\Omega)$ with $\|u\|_{W^{1,2}_\sigma(\Omega)}=\rho$, the estimate (4.18) gives that

$$\mathcal{E}(u)\geqslant\left(\frac12-\frac1r c^r_{\omega,\sigma}\rho^{r-2}\right)\rho^2=\frac{\rho^2}{4}.$$

Therefore, since $\mathcal{E}(0)=0$, the left-hand side of (4.17) immediately holds.

On the other hand, let $w\in W^{1,2}_\sigma(\Omega)$ be any nonnegative, nonzero function. Since $r>2$, we may fix $t_0>0$ large enough such that

$$t_0>\max\left\{\frac{\rho}{\|w\|_{W^{1,2}_\sigma(\Omega)}},\left(\frac{r\|w\|^2_{W^{1,2}_\sigma(\Omega)}}{2\|w\|^r_{L^r_\omega(\Omega\cap E)}}\right)^{\frac{1}{r-2}}\right\}.$$

Accordingly, the function $w_0:=t_0w\in W^{1,2}_\sigma(\Omega)$ verifies $\|w_0\|_{W^{1,2}_\sigma(\Omega)}>\rho$ and

$$\mathcal{E}(w_0)=\mathcal{E}(t_0w)=\frac{t_0^2}{2}\|w\|^2_{W^{1,2}_\sigma(\Omega)}-\frac{t_0^r}{r}\|w\|^r_{L^r_\omega(\Omega\cap E)}<0,$$

which is the right-hand side of (4.17).

We are now in a position to apply the Mountain Pass Theorem, see, for example, Rabinowitz [**26**], which implies the existence of a critical point $u\in W^{1,2}_\sigma(\Omega)$ of $\mathcal{E}$ with the property that $\mathcal{E}(u)>0$ (thus $u\neq 0$), which is a weak solution to the problem $(\mathcal{P}_+)$.

It remains to prove that u is nonnegative and weakly solves the original problem $(\mathcal{P})$. By multiplying the first equation of $(\mathcal{P}_+)$ by $u_- = \min(u, 0)$, an integration on $\Omega \cap E$ implies $\|u_-\|_{W^{1,2}_\sigma(\Omega)} = 0$, that is, $u_- = 0$. Accordingly, $u \geqslant 0$ is a nonzero weak solution to the original problem $(\mathcal{P})$ as well, which completes the proof. $\square$

5. *Final comments and open questions*

5.1. *Necessity of conditions* (1.2), (1.3) *and* (1.4)

We start this section showing that by choosing appropriate test functions in (WSI), conditions (1.2)–(1.4) on the parameters are *necessary* for the validity of (WSI).

Condition (1.4) follows by scaling: if u verifies (WSI), then $u_\lambda(x) = u(\lambda x)$ also satisfies (WSI) for each $\lambda > 0$. Also, since $q > 0$, the left-hand side inequality in (1.2) follows immediately from (1.4) because $\tau + n > 0$ from the local integrability of ω.

Let us next prove the right-hand inequality in (1.2). Let φ be a smooth function defined for $t \geqslant 0$ satisfying $\varphi(t) = 0$ for $0 \leqslant t < 1$, $\varphi(t) = 1$ for $t \geqslant 2$, and $0 \leqslant \varphi(t) \leqslant 1$ for all $t > 0$. Also choose $h(t)$ smooth for $t \in \mathbb{R}$ with $h(t) = 1$ for $|t| \leqslant 1$, $h(t) = 0$ for $|t| \geqslant 2$ and $0 \leqslant h \leqslant 1$. Given $\epsilon > 0$, the function $u_\epsilon(x) = |x|^{-\beta} \log|x|\, \varphi(|x|/\epsilon)\, h(|x|)$ belongs to $C_0^\infty(\mathbb{R}^n)$ with support in the ring $\{\epsilon \leqslant |x| \leqslant 2\}$ for each $\beta \in \mathbb{R}$ and so u_ϵ satisfies (WSI). If $\beta = (\tau + n)/q$, we have for $\epsilon < 1/2$ that

$$
\begin{aligned}
\int_E |u_\epsilon(x)|^q\, \omega(x)\, dx &\geqslant \int_{E\cap\{2\epsilon\leqslant|x|\leqslant 1\}} |x|^{-\beta q} \left(\log\frac{1}{|x|}\right)^q \omega(x)\, dx \\
&= \int_{2\epsilon}^1 t^{-\beta q+\tau+n-1} \left(\log\frac{1}{t}\right)^q dt \int_{E\cap S^{n-1}} \omega(x)\, dx \\
&= \frac{1}{q+1}\left(\log\frac{1}{2\epsilon}\right)^{q+1} \int_{E\cap\mathbb{S}^{n-1}} \omega(x)\, dx.
\end{aligned}
$$

Let us now estimate $\int_E |\nabla u_\epsilon(x)|^p\, \sigma(x)\, dx$ from above. We have

$$
\begin{aligned}
\nabla u_\epsilon(x) &= (-\beta)|x|^{-\beta-1}\frac{x}{|x|} \log|x|\, \varphi(|x|/\epsilon)\, h(|x|) + |x|^{-\beta}\frac{1}{|x|}\frac{x}{|x|}\varphi(|x|/\epsilon)\, h(|x|) \\
&\quad + |x|^{-\beta} \log|x|\, \varphi'(|x|/\epsilon)\frac{1}{\epsilon}\frac{x}{|x|} h(|x|) + |x|^{-\beta}\log|x|\, \varphi(|x|/\epsilon)\, h'(|x|)\frac{x}{|x|}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|\nabla u_\epsilon(x)| &\leqslant |\beta|\, |x|^{-\beta-1}\, |\log|x||\, \chi_{\epsilon\leqslant|x|\leqslant 2}(x) + |x|^{-\beta-1}\, \chi_{\epsilon\leqslant|x|\leqslant 2}(x) \\
&\quad + \|\varphi'\|_\infty\, |x|^{-\beta} |\log|x||\, \frac{1}{\epsilon}\, \chi_{\epsilon\leqslant|x|\leqslant 2\epsilon}(x) + |x|^{-\beta}\, |\log|x||\, \|h'\|_\infty\, \chi_{1\leqslant|x|\leqslant 2}(x) \\
&\leqslant C_1\, |x|^{-\beta-1}(1 + |\log|x||)\, \chi_{\epsilon\leqslant|x|\leqslant 2}(x),
\end{aligned}
$$

with $C_1 > 0$ a constant depending only on β, $\|h'\|_\infty$, and $\|\varphi'\|_\infty$. Therefore

$$
\begin{aligned}
\int_E |\nabla u_\epsilon(x)|^p\, \sigma(x)\, dx &= \int_{E\cap\{\epsilon\leqslant|x|\leqslant 2\}} |\nabla u_\epsilon(x)|^p\, \sigma(x)\, dx \\
&\leqslant C_1^p \int_{E\cap\{\epsilon\leqslant|x|\leqslant 2\}} |x|^{-(\beta+1)p}(1 + |\log|x||)^p\, \sigma(x)\, dx := C_1^p\, I.
\end{aligned}
$$

Integrating in polar coordinates

$$I = \int_\epsilon^2 t^{-(\beta+1)p+n-1+\alpha}(1+|\log t|)^p\,dt \int_{E\cap S^{n-1}} \sigma(x)\,dx,$$

and from (1.4) and the choice of β, the exponent $-(\beta+1)p+n-1+\alpha = -1$. So

$$\begin{aligned} I &= C\int_\epsilon^2 t^{-1}(1+|\log t|)^p\,dt \leqslant 2^p\,C\int_\epsilon^2 t^{-1}(1+|\log t|^p)\,dt \\ &= 2^p\,C\left(\int_\epsilon^2 t^{-1}\,dt + \int_\epsilon^2 t^{-1}|\log t|^p\,dt\right) = 2^p\,C\,(I_1+I_2). \end{aligned}$$

Now $I_1 = \log(2/\epsilon)$ and

$$\begin{aligned} I_2 &= \int_\epsilon^1 t^{-1}|\log t|^p\,dt + \int_1^2 t^{-1}|\log t|^p\,dt \\ &= \int_\epsilon^1 t^{-1}\left(\log\frac{1}{t}\right)^p dt + c_p = \frac{1}{p+1}\left(\log\frac{1}{\epsilon}\right)^{p+1} + c_p. \end{aligned}$$

We then obtain the estimate

$$\int_E |\nabla u_\epsilon(x)|^p\,\sigma(x)\,dx \leqslant C_p\left(\left(\log\frac{1}{\epsilon}\right)^{p+1} + \log\frac{2}{\epsilon} + 1\right)$$

and since u_ϵ satisfies (WSI), it then follows from the estimate of the L^q-norm of u_ϵ that

$$\left(\log\frac{1}{2\epsilon}\right)^{1+\frac{1}{q}} \leqslant C\left(\left(\log\frac{1}{\epsilon}\right)^{p+1} + \log\frac{2}{\epsilon} + 1\right)^{1/p},$$

for all ϵ small with C independent of ϵ. Since the dominant term, as $\epsilon \to 0$, on the right-hand side of the last inequality is $(\log\frac{1}{\epsilon})^{1+\frac{1}{p}}$, we then get that $p \leqslant q$ which together with (1.4) yields the inequality on right-hand side of (1.2).

It remains to prove that (1.3) is necessary for (WSI). Fix $y_0 \in E\cap\mathbb{S}^{n-1}$. The idea is to construct a test function supported on a small ball whose center is along the direction y_0 that tends to infinity. Since E is open, we may pick $r_0 > 0$ small enough with $\overline{B}_{r_0}(y_0) \subset E$. Let

$$m_0 := \min_{\overline{B}_{r_0}(y_0)} \omega > 0, \quad M_0 := \max_{\overline{B}_{r_0}(y_0)} \sigma > 0,$$

fix a function $v \in C_0^\infty(B_1)\setminus\{0\}$, and define $u_\delta(x) = v(x-\delta y_0)$ for $\delta > 0$. Note that $u_\delta \in C_0^\infty(B_1(\delta y_0))$. Observe also, that if $\delta\,r_0 > 1$, then $B_1(\delta y_0) \subset B_{\delta r_0}(\delta y_0) \subset \delta(\overline{B}_{r_0}(y_0)) \subset E$, since E is a cone. Therefore, by (1.1) and the definitions of m_0, M_0, it follows that

$$\begin{aligned} \int_E |u_\delta(x)|^q\,\omega(x)\,dx &= \int_E |v(x-\delta y_0)|^q\,\omega(x)\,dx = \int_{B_1(\delta y_0)} |v(x-\delta y_0)|^q\,\omega(x)\,dx \\ &= \int_{B_1} |v(y)|^q\,\omega(y+\delta y_0)\,dy = \delta^\tau\int_{B_1} |v(y)|^q\,\omega\Big(\frac{y}{\delta}+y_0\Big)\,dy \\ &\geqslant \delta^\tau m_0\int_{B_1} |v(y)|^q\,dy. \end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}\int_E |\nabla u_\delta(x)|^p\,\sigma(x)\,dx &= \int_E |\nabla v(x-\delta y_0)|^p\,\sigma(x)\,dx = \int_{B_1(\delta y_0)} |\nabla v(x-\delta y_0)|^p\,\sigma(x)\,dx\\ &= \int_{B_1} |\nabla v(y)|^p\,\sigma(y+\delta y_0)\,dy = \delta^\alpha \int_{B_1} |\nabla v(y)|^p\,\sigma\Big(\frac{y}{\delta}+y_0\Big)\,dy\\ &\leqslant \delta^\alpha M_0 \int_{B_1} |\nabla v(y)|^p\,dy.\end{aligned}$$

Accordingly, if we plug in the function u_δ into (WSI) with $\delta > 1/r_0$, and use the last two estimates it follows that

$$\Big(\delta^\tau m_0 \int_{B_1} |v(y)|^q\,dy\Big)^{\frac{1}{q}} \leqslant K_0\Big(\delta^\alpha M_0 \int_{B_1} |\nabla v(y)|^p\,dy\Big)^{\frac{1}{p}}.$$

Letting $\delta \to \infty$, we obtain that $\frac{\tau}{q} \leqslant \frac{\alpha}{p}$. Now, using once again the dimensional balance condition (1.4), we see that the last inequality is equivalent to (1.3).

5.2. *Sobolev inequalities in the Heisenberg group*

In this part, we consider the connection between weighted Sobolev inequalities in Euclidean cones and Sobolev inequalities in Heisenberg groups. Our original purpose was in fact to prove Sobolev inequalities in the Heisenberg group with sharp constants.

For simplicity, we consider the first Heisenberg group $\mathbb{H}^1$. Let us recall that $\mathbb{H}^1 = \mathbb{R}^3$ is endowed with its group operation given by

$$(x,y,z) * (x',y',z') := \Big(x+x', y+y', z+z' + \frac{1}{2}(xy'-yx')\Big).$$

In this setting, one considers the *left invariant horizontal vector fields* given by $X = \partial_x - \frac{1}{2}y\partial_z$ and $Y = \partial_y + \frac{1}{2}x\partial_z$ and the associated horizontal gradient $\nabla_H u = X(u)X + Y(u)Y$. For $p \in [1,4)$ we consider the Sobolev inequality

$$\Big(\int_{\mathbb{H}^1} |u|^q\Big)^{1/q} \leqslant C_p\Big(\int_{\mathbb{H}^1} |\nabla_H u|^p\Big)^{1/p},\ u \in C_0^\infty(\mathbb{H}^1), \tag{5.1}$$

where $C_p > 0$ and $q = \frac{4p}{4-p}$ is the Sobolev exponent given by scaling with Heisenberg dilations, where we have used the norm of the horizontal gradient for a function $u \in C_0^\infty(\mathbb{H}^1)$ given by $|\nabla_H u| = \sqrt{(Xu)^2 + (Yu)^2}$. In the following, let us consider the class of functions u that are *axially symmetric*:

$$u(x,y,z) = w(z, x^2+y^2).$$

Then, by changing variables, the Heisenberg Sobolev inequality (5.1) becomes equivalent to the Euclidean weighted Sobolev inequality

$$\Big(\int_{\mathbb{R}}\int_0^\infty w^q(x_1,x_2)dx_1dx_2\Big)^{1/q} \leqslant C_p\Big(\int_{\mathbb{R}}\int_0^\infty |\nabla w|^p(x_1,x_2)x_2^{p/2}dx_1dx_2\Big)^{1/p}. \tag{5.2}$$

This problem fits well into the framework of this paper. In fact, with our setup, the open convex cone we are working with is $E = \mathbb{R}\times(0,\infty)$, the weights being $\omega = 1$ and $\sigma(x_1,x_2) = x_2^{p/2}$

for $(x_1, x_2) \in E$; accordingly, $\tau = 0$ while $\alpha = p/2$, and the fractional dimension is $n_a = 4$. Applying Theorem 1.1(i) and Theorem 1.2(i), we obtain that (5.2) holds with constant

$$C_p = \begin{cases} \dfrac{3p}{4-p} \displaystyle\inf_{\int_E v(y)\,dy=1, v\in C_0^\infty(\mathbb{R}^2), v\geqslant 0} \dfrac{\left(\int_E v(y)\,|y|^{\frac{p}{p-1}}\,dy\right)^{\frac{p-1}{p}}}{\int_E v(y)^{\frac{3}{4}}(y_2)^{\frac{1}{2}}\,dy} & \text{if} \quad p \in (1,4), \\ \frac{5\pi^{\frac{5}{4}}}{2^{\frac{13}{4}}\Gamma^2\left(\frac{3}{4}\right)} & \text{if} \quad p = 1. \end{cases}$$

We do not know how to compute the explicit value of the constant C_p for $p > 1$. On the other hand, it is clear that this constant is not the sharp one for the inequality (5.2), see Theorem 1.3. It is in fact still an open question to determine the sharp constant in both inequalities (5.1) and (5.2) for general values of p. When $p = 2$, a sharp Sobolev inequality in the Heisenberg setting is due to Jerison and Lee [**15**] and it was proved also by a different method by Frank and Lieb [**14**]. Inequality (5.1) for $p = 1$ is equivalent with Pansu's isoperimetric inequality; the Pansu's optimal constant is claimed to be $C_{\text{opt}} = \frac{3^{\frac{3}{4}}}{4\sqrt{\pi}} < C_1$. There are several partial results related to Pansu's conjecture; we refer to the monograph of Capogna, Danielli, Pauls and Tyson [**8**] for a detailed account on this subject.

5.3. *Open questions*

We list here a few open problems related to results of this paper.

5.3.1. *Sharp Sobolev inequalities with different weights.* While the explicit computation of the constant K_0 in the statement of Theorem 1.2 can be done by a direct calculation of the integrals in the expression of K_0, the computation of the value of K_0 in the statement of Theorem 1.1, even in case of simple weights, is a nontrivial matter.

Motivated mainly by the Heisenberg setting from Section 5.2, it would be interesting to further investigate whether the method of optimal transport can be used to obtain sharp constants in weighted Sobolev inequalities with different weights.

Another challenging question is to obtain Gagliardo–Nirenberg type inequalities with different weights. We note that sharp Gagliardo–Nirenberg inequalities have been established by Del Pino and Dolbeault [**12, 13**] in the unweighted form, and by Lam [**17, 18**] for identical homogeneous weights.

5.3.2. *Condition C-0 and Bakry-Émery curvature-dimension condition.* When $\omega = \sigma$, condition C-0 is equivalent to the concavity of $\omega^{\frac{1}{\alpha}}$ that in turn characterizes the fact that the metric-measure space $(\mathbb{R}^n, d_E, \omega dx)$ satisfies the Bakry–Émery curvature-dimension condition $CD(0, n+\alpha)$ (see [**6**, Remark 1.4] for details). Here, d_E and ωdx are the usual Euclidean metric and the measure whose density with respect to the Lebesgue measure is ω, respectively. It would be an interesting problem to find a geometric interpretation of condition C-0 in terms of generalized curvature conditions of metric-measure spaces in the spirit of [**2, 21 23, 28, 29**].

5.3.3. *On Muckenhoupt–Wheeden's weighted inequality.* To give a broader view, we close the paper mentioning earlier Sobolev inequalities for two weights in all space proved by Muckenhoupt and Wheeden [**24**] via fractional integration. They proved the following result: if $0 < \gamma < n$, $1 < p < n/\gamma$, and $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, then

$$\|T_\gamma f\, V\|_{L^q(\mathbb{R}^n)} \leqslant C\, \|f\, V\|_{L^p(\mathbb{R}^n)} \tag{5.3}$$

for all functions f if and only if there exists $K > 0$ such that

$$\left(\fint_Q V(x)^q\,dx\right)^{1/q}\left(\fint_Q V(x)^{-p'}\,dx\right)^{1/p'} \leqslant K \tag{5.4}$$

for all cubes $Q \subset \mathbb{R}^n$. This condition is equivalent to V^q belongs to the Muckenhoupt class A_r, with $r = 1 + q/p'$. Here T_γ stands for the fractional integral of order γ given by

$$T_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\,dy.$$

Using a representation formula of functions in terms of the fractional integral of order one of its derivatives, Muckenhoupt and Wheeden [**24**, Theorem 9] deduced from (5.3) when $\gamma = 1$ a weighted Sobolev inequality of the form

$$\|f\,V\|_{L^q(\mathbb{R}^n)} \leqslant C\left(\|f\,V\|_{L^p(\mathbb{R}^n)} + \||\nabla f|\,V\|_{L^p(\mathbb{R}^n)}\right).$$

We note that Muckenhoupt–Wheeden's condition and our condition C-0 are rather independent from each other. Indeed, if $V : \mathbb{R}^n \to (0,\infty)$ is any differentiable, homogeneous function of degree $\alpha \in \mathbb{R}$ and $\omega(x) = V(x)^q$, $\sigma(x) = V(x)^p$ for every $x \in E = \mathbb{R}^n$, then $n_a = n$ and $\sup_{x\in\mathbb{R}^n} \frac{\omega(x)^{1/q}}{\sigma(x)^{1/p}} = 1$. We observe that the pair (ω,σ) satisfies inequality (1.7) in condition C-0 if and only if $V \equiv c$ for some $c > 0$. Hence with this choice of the weights, conditions (5.4) and C-0 are simultaneously satisfied in the whole $\mathbb{R}^n$ if and only if both weights are constant, that is, $\omega(x) = c^q$, $\sigma(x) = c^p$, $x \in \mathbb{R}^n$.

Since our results are on cones, they are not in general comparable to these but nevertheless they raise the following methodological question: is it possible to prove inequality (5.3), for example, when $V = 1$, by using optimal transport? This would be the analogue of the problem solved in [**11**] for fractional integrals. In particular, it may give optimal constants and extremal functions for the fractional integral inequality as in Lieb [**20**, Theorem 2.3].

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