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EDGE-ORDERED RAMSEY NUMBERS

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ABSTRACT. We introduce and study a variant of Ramsey numbers for *edge-ordered graphs*, that is, graphs with linearly ordered sets of edges. The *edge-ordered Ramsey number* $\overline{R}_e(\mathfrak{G})$ of an edge-ordered graph $\mathfrak{G}$ is the minimum positive integer N such that there exists an edge-ordered complete graph $\mathfrak{K}_N$ on N vertices such that every 2-coloring of the edges of $\mathfrak{K}_N$ contains a monochromatic copy of $\mathfrak{G}$ as an edge-ordered subgraph of $\mathfrak{K}_N$.

We prove that the edge-ordered Ramsey number $\overline{R}_e(\mathfrak{G})$ is finite for every edge-ordered graph $\mathfrak{G}$ and we obtain better estimates for special classes of edge-ordered graphs. In particular, we prove $\overline{R}_e(\mathfrak{G}) \leq 2^{O(n^3 \log n)}$ for every bipartite edge-ordered graph $\mathfrak{G}$ on n vertices. We also introduce a natural class of edge-orderings, called *lexicographic edge-orderings*, for which we can prove much better upper bounds on the corresponding edge-ordered Ramsey numbers.

1. INTRODUCTION

An *edge-ordered graph* $\mathfrak{G} = (G, \prec)$ consists of a graph $G = (V, E)$ and a linear ordering $\prec$ of the set of edges E. We sometimes use the term *edge-ordering of* G for the ordering $\prec$ and also for $\mathfrak{G}$. An edge-ordered graph $(G, \prec_1)$ is an *edge-ordered subgraph* of an edge-ordered graph $(H, \prec_2)$ if G is a subgraph of H and $\prec_1$ is a suborder of $\prec_2$. We say that $(G, \prec_1)$ and $(H, \prec_2)$ are *isomorphic* if there is a graph isomorphism between G and H that also preserves the edge-orderings $\prec_1$ and $\prec_2$.

For a positive integer k, a *k-coloring* of the edges of a graph G is any function that assigns one of the *k colors* to each edge of G. The *edge-ordered Ramsey number* $\overline{R}_e(\mathfrak{G})$ of an edge-ordered graph $\mathfrak{G}$ is the minimum positive integer N such that there exists an edge-ordering $\mathfrak{K}_N$ of the complete graph K_N on N vertices such that every 2-coloring of the edges of $\mathfrak{K}_N$ contains a monochromatic copy of $\mathfrak{G}$ as an edge-ordered subgraph of $\mathfrak{K}_N$. More generally, for two edge-ordered graphs $\mathfrak{G}$ and $\mathfrak{H}$, we use $\overline{R}_e(\mathfrak{G}, \mathfrak{H})$ to denote the minimum positive integer N such that there exists an edge-ordering $\mathfrak{K}_N$ of K_N such that every 2-coloring of the edges of

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$\mathfrak{K}_N$ with colors red and blue contains a red copy of $\mathfrak{G}$ or a blue copy of $\mathfrak{H}$ as an edge-ordered subgraph of $\mathfrak{K}_N$.

To our knowledge, Ramsey numbers of edge-ordered graphs were not considered in the literature. On the other hand, Ramsey numbers of graphs with ordered vertex sets have been quite extensively studied recently; for example, see [**2, 3, 6**]. For questions concerning extremal problems about vertex-ordered graphs consult the recent surveys [**14, 15**]. A *vertex-ordered graph* $\mathcal{G} = (G, \prec)$ (or simply an *ordered graph*) is a graph G with a fixed linear ordering $\prec$ of its vertices. We use the term *vertex-ordering* of G to denote the ordering $\prec$ as well as the ordered graph $\mathcal{G}$. An ordered graph $(G, \prec_1)$ is a *vertex-ordered subgraph* of an ordered graph $(H, \prec_2)$ if G is a subgraph of H and $\prec_1$ is a suborder of $\prec_2$. We say that $(G, \prec_1)$ and $(H, \prec_2)$ are *isomorphic* if there is a graph isomorphism between G and H that also preserves the vertex-orderings $\prec_1$ and $\prec_2$. Unlike in the case of edge-ordered graphs, there is a unique vertex-ordering $\mathcal{K}_N$ of K_N up to isomorphism. The *ordered Ramsey number* $\overline{R}(\mathcal{G})$ of an ordered graph $\mathcal{G}$ is the minimum $N \in \mathbb{N}$ such that every 2-coloring of the edges of $\mathcal{K}_N$ contains a monochromatic copy of $\mathcal{G}$ as a vertex-ordered subgraph of $\mathcal{K}_N$.

For an n-vertex graph G, let $R(G)$ be the Ramsey number of G. It is easy to see that $R(G) \leq \overline{R}(\mathcal{G})$ and $R(G) \leq \overline{R}_e(\mathfrak{G})$ for each vertex-ordering $\mathcal{G}$ of G and edge-ordering $\mathfrak{G}$ of G. We also have $\overline{R}(G) \leq \overline{R}(\mathcal{K}_n) = R(K_n)$ and thus ordered Ramsey numbers are always finite. Proving that $\overline{R}_e(\mathfrak{G})$ is always finite seems to be more challenging; see Theorem 2.1.

The Turán numbers of edge-ordered graphs were recently introduced in [**7**]. The authors of [**7**] proved, for example, a variant of the Erdős–Stone–Simonovits Theorem for edge-ordered graphs, and also investigated the Turán numbers of small edge-ordered paths, star forests, and 4-cycles; see the last section of [**15**].

2. Our results

We study the growth rate of edge-ordered Ramsey numbers with respect to the number of vertices for various classes of edge-ordered graphs. As our first result, we show that edge-ordered Ramsey numbers are always finite and thus well-defined.

Theorem 2.1. *For every edge-ordered graph $\mathfrak{G}$, the number $\overline{R}_e(\mathfrak{G})$ is finite.*

Theorem 2.1 also follows from a recent deep result of Hubička and Nešetřil [**9**, Theorem 4.33] about Ramsey numbers of general relational structures. In comparison, our proof of Theorem 2.1 is less general, but it is much simpler and produces better and more explicit bound on $\overline{R}_e(\mathfrak{G})$. It is a modification of the proof of Theorem 12.13 [**12**, Page 138], which is based on the Graham–Rothschild Theorem [**8**]. In fact, the proof of Theorem 2.1 yields a stronger induced-type statement where additionally the ordering of the vertex set is fixed. Theorem 2.1 can also be extended to k-colorings with $k > 2$.

Due to the use of the Graham–Rothschild Theorem, the bound on the edge-ordered Ramsey numbers obtained in the proof of Theorem 2.1 is still enormous. It follows from a result of Shelah [**13**, Theorem 2.2] that this bound on $\overline{R}_e(\mathfrak{G})$ is primitive recursive, but it grows faster than, for example, a tower function of any fixed height. Thus we aim to prove more reasonable estimates on edge-ordered Ramsey numbers, at least for some classes of edge-ordered graphs.

As our second main result, we show that one can obtain a much better upper bound on edge-ordered Ramsey numbers of two edge-ordered graphs, provided that one of them is bipartite. For $d \in \mathbb{N}$, we say that a graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d.

Theorem 2.2. *Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n' vertices and let $\mathfrak{G}$ be a bipartite edge-ordered graph with m edges and with both parts containing n vertices. If $d \le n$ and $n' \le t^{d+1}$ for $t = 3n^{10}m!$, then*

$$\overline{R}_e(\mathfrak{H}, \mathfrak{G}) \le (n')^2 t^{d+1}.$$

In particular, if $\mathfrak{G}$ is a bipartite edge-ordered graph on n vertices, then $\overline{R}_e(\mathfrak{G}) \le 2^{O(n^3 \log n)}$. We believe that the bound can be improved. In fact, it is possible that $\overline{R}_e(\mathfrak{G})$ is at most exponential in the number of vertices of $\mathfrak{G}$ for every edge-ordered graph $\mathfrak{G}$. We note that, for every graph G and its vertex-ordering $\mathcal{G}$, both the standard Ramsey number $R(G)$ and the ordered Ramsey number $\overline{R}(\mathcal{G})$ grow at most exponentially in the number of vertices of G.

In general, the difference between edge-ordered Ramsey numbers and ordered Ramsey numbers with the same underlying graph can be very large. Let M_n be a *matching* on n vertices, that is, a graph formed by a collection of $n/2$ disjoint edges. There are ordered matchings $\mathcal{M}_n = (M_n, <)$ with super-polynomial ordered Ramsey numbers $\overline{R}(\mathcal{M}_n)$ in n [**2, 6**]. In fact this is true for almost all ordered matchings on n vertices [**6**]. On the other hand, all edge-orderings of M_n are isomorphic as edge-ordered graphs and thus $\overline{R}_e(\mathfrak{M}_n) = R(M_n) \le O(n)$ for every edge-ordering $\mathfrak{M}_n$ of M_n.

We consider a special class of edge-orderings, which we call *lexicographic edge-orderings*, for which we can prove much better upper bounds on their edge-ordered Ramsey numbers and which seem to be quite natural.

An ordering $\prec$ of edges of a graph $G = (V, E)$ is *lexicographic* if there is a one-to-one correspondence $f\colon V \to \{1, \dots, |V|\}$ such that any two edges $\{u, v\}$ and $\{w, t\}$ of G with $f(u) < f(v)$ and $f(w) < f(t)$ satisfy $\{u, v\} \prec \{w, t\}$ if either $f(u) < f(w)$ or if $(f(u) = f(w) \;\&\; f(v) < f(t))$. We say that such mapping f is *consistent* with $\prec$. Note that, for every vertex u, the edges $\{u, v\}$ with $f(u) < f(v)$ form an interval in $\prec$. Also observe that there is a unique (up to isomorphism) lexicographic edge-ordering $\mathfrak{K}_n^{\mathrm{lex}}$ of K_n. Setting $\{u, v\} \prec' \{w, t\}$ if either $f(u) < f(w)$ or if $(f(u) = f(w) \;\&\; f(v) > f(t))$ we obtain the *max-lexicographic* edge-ordering $\prec'$ of G.

For a linear ordering $<$ on some set X, we use $<^{-1}$ to denote the *inverse ordering* of $<$, that is, for all $x, y \in X$, we have $x <^{-1} y$ if and only if $y < x$.

The lexicographic and max-lexicographic edge-orderings are natural, as Nešetřil and Rödl [**11**] showed that these orderings are canonical in the following sense.

Theorem 2.3 ([**11**])**.** *For every $n \in \mathbb{N}$, there is a positive integer $T(n)$ such that every edge-ordered complete graph on $T(n)$ vertices contains a copy of K_n such that* *the edges of this copy induce one of the following four edge-orderings: lexicographic edge-ordering $\prec$, max-lexicographic edge-ordering $\prec'$, $\prec^{-1}$, or $(\prec')^{-1}$.*

Theorem 2.3 is also an unpublished result of Leeb; see [**10**].

It is thus natural to consider the following variant of edge-ordered Ramsey numbers, which turns out to be more tractable than general edge-ordered Ramsey numbers. The *lexicographic edge-ordered Ramsey number* $\overline{R}_{\mathrm{lex}}(\mathfrak{G})$ of a lexicographically edge-ordered graph $\mathfrak{G}$ is the minimum N such that every 2-coloring of the edges of $\mathfrak{K}_N^{\mathrm{lex}}$ contains a monochromatic copy of $\mathfrak{G}$ as an edge-ordered subgraph of $\mathfrak{K}_N^{\mathrm{lex}}$. Observe that $\overline{R}_e(\mathfrak{G}) \leq \overline{R}_{\mathrm{lex}}(\mathfrak{G})$ for every lexicographically edge-ordered graph $\mathfrak{G}$.

For every lexicographically edge-ordered graph $\mathfrak{G} = (G, \prec)$, the lexicographic edge-ordered Ramsey number $\overline{R}_{\mathrm{lex}}(\mathfrak{G})$ can be estimated from above with the ordered Ramsey number of some vertex-ordering of G. More specifically,

$$\overline{R}_{\mathrm{lex}}(\mathfrak{G}) \leq \min_f \overline{R}(\mathcal{G}_f), \tag{1}$$

where the minimum is taken over all one-to-one correspondences $f\colon V \to \{1, \dots, |V|\}$ that are consistent with the lexicographic edge-ordering $\mathfrak{G}$ and $\mathcal{G}_f$ is the vertex-ordering of G determined by f. Since $\overline{R}(\mathcal{K}_n) = R(K_n)$, it follows from (1) and from the well-known bound $R(K_n) \leq 2^{2n}$ that the numbers $\overline{R}_{\mathrm{lex}}(\mathfrak{G})$ are always at most exponential in the number of vertices of G. In fact, we have $\overline{R}_{\mathrm{lex}}(\mathfrak{K}_n^{\mathrm{lex}}) = \overline{R}(\mathcal{K}_n) = R(K_n)$ for every n. The equality is achieved in (1), for example, for graphs with a unique vertex-ordering determined by the lexicographic edge-ordering. Such graphs include graphs where each edge is contained in a triangle. Additionally, combining (1) with a result of Conlon et al. [**6**, Theorem 3.6] gives the estimate

$$\overline{R}_{\mathrm{lex}}(\mathfrak{G}) \leq 2^{O(d \log_2^2 (2n/d))}$$

for every d-degenerate lexicographically edge-ordered graph $\mathfrak{G}$ on n vertices. In particular, $\overline{R}_{\mathrm{lex}}(\mathfrak{G})$ is at most quasi-polynomial in n if d is fixed.

We note that the bound (1) is not always tight. For example, $R(K_{1,n}) = \overline{R}_{\mathrm{lex}}(\mathfrak{K}_{1,n})$ for every edge-ordering $\mathfrak{K}_{1,n}$ of $K_{1,n}$, as any two edge-ordered stars $K_{1,n}$ are isomorphic as edge-ordered graphs. However, the Ramsey number $R(K_{1,n})$ is known to be strictly smaller than $\overline{R}(\mathcal{K}_{1,n})$ for n even and for any vertex-ordering $\mathcal{K}_{1,n}$ of $K_{1,n}$; see [**4**] and [**1**, Observation 11 and Theorem 12].

Using the inequality (1) we obtain asymptotically tight estimate on the following lexicographic edge-ordered Ramsey numbers of paths. The *edge-monotone path* $\mathfrak{P}_n = (P_n, \prec)$ is the edge-ordered path on vertices $v_1, \dots, v_n$, where $\{v_1, v_2\} \prec \cdots \prec \{v_{n-1}, v_n\}$.

Proposition 2.4. *For every integer $n > 2$, we have $\overline{R}_{\mathrm{lex}}(\mathfrak{P}_n) \leq 2n - 3 + \sqrt{2n^2 - 8n + 11}$.*

The proof of Proposition 2.4 uses the fact that the one-to-one correspondence f consistent with the lexicographic edge-ordering of P_n is not determined uniquely. Indeed, we can choose the mapping f so that it determines the vertex-ordering $\mathcal{P}_n$

of P_n where edges are between consecutive pairs of vertices. Such vertex-ordering $\mathcal{P}_n$ is called *monotone path.* However, it is known that $\overline{R}(\mathcal{P}_n) = (n-1)^2+1$ [**5**] and thus we cannot apply (1) to this ordering to obtain a linear bound on $\overline{R}_{\text{lex}}(\mathfrak{P}_n)$. Instead we choose a different mapping f that determines a vertex-ordering of P_n with linear ordered Ramsey number.

Finally, we show an upper bound on edge-ordered Ramsey numbers of two graphs, where one of them is bipartite and suitably lexicographically edge-ordered. This result uses a stronger assumption about $\mathfrak{G}$ than Theorem 2.2, but gives much better estimate. For $m, n \in \mathbb{N}$, let $\mathfrak{K}^{\text{lex}}_{m,n}$ be the lexicographic edge-ordering of $K_{m,n}$ that induces a vertex-ordering, in which both parts of $K_{m,n}$ form an interval.

Theorem 2.5. *Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n' vertices and let $\mathfrak{G}$ be an edge-ordered subgraph of $\mathfrak{K}^{\text{lex}}_{n,n}$. Then*

$$\overline{R}_e(\mathfrak{H}, \mathfrak{G}) \leq (n')^2 n^{d+1}.$$

3. Open problems

Many questions about edge-ordered Ramsey numbers remain open, for example proving a better upper bound on edge-ordered Ramsey numbers than the one obtained in the proof of Theorem 2.1. For general upper bounds, it suffices to focus on edge-ordered complete graphs. It is possible that edge-ordered Ramsey numbers of edge-ordered complete graphs do not grow significantly faster than the standard Ramsey numbers.

Problem 3.1. Is there a constant C such that, for every $n \in \mathbb{N}$ and every edge-ordered complete graph $\mathfrak{K}_n$ on n vertices, we have $\overline{R}_e(\mathfrak{K}_n) \leq 2^{Cn}$?

It might also be interesting to consider sparser graphs and try to prove better upper bounds on their edge-ordered Ramsey numbers.

Another interesting open problem is to determine the growth rate of the function $T(n)$ from Theorem 2.3. The current upper bound on $T(n)$ is quite large as the proof of Nešetřil and Rödl [**11**] uses Ramsey's theorem for quadruples and $6! = 720$ colors.

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