GROUP ALGEBRAS WHOSE GROUP OF UNITS IS POWERFUL
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Abstract

A \( p \)-group is called powerful if every commutator is a product of \( p \)th powers when \( p \) is odd and a product of fourth powers when \( p = 2 \). In the group algebra of a group \( G \) of \( p \)-power order over a finite field of characteristic \( p \), the group of normalized units is always a \( p \)-group. We prove that it is never powerful except, of course, when \( G \) is abelian.

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Throughout this note \( G \) is a finite \( p \)-group and \( F \) is a finite field of characteristic \( p \). Let

\[
V(FG) = \left\{ \sum_{g \in G} \alpha_g g \in FG \left| \sum_{g \in G} \alpha_g = 1 \right. \right\}
\]

be the group of normalized units of the group algebra \( FG \). Clearly \( V(FG) \) is a finite \( p \)-group of order

\[
|V(FG)| = |F|^{|G|-1}.
\]

A \( p \)-group is called powerful if every commutator is a product of \( p \)th powers when \( p \) is odd and a product of fourth powers when \( p = 2 \). The notion of powerful groups was introduced in [5] and it plays an important role in the study of finite \( p \)-groups (for example, see [2, 4] and [7]). Our main result is the following.

**Theorem.** The group of normalized units \( V(FG) \) of the group algebra \( FG \) of a group \( G \) of \( p \)-power order over a finite field \( F \) of characteristic \( p \), is never powerful except, of course, when \( G \) is abelian.

In view of the fact that a pro-\( p \)-group is powerful if and only if it is the limit of finite powerful groups, this has an immediate consequence.
COROLLARY. The group of normalized units $V(F[[G]])$ of the completed group algebra $F[[G]]$ of a pro-$p$-group $G$ over a finite field $F$ of characteristic $p$, is never powerful except, of course, when $G$ is abelian.

We denote by $\zeta(G)$ the center of $G$. We say that $G = A \Rightarrow B$ is a central product of its subgroups $A$ and $B$ if $A$ and $B$ commute elementwise and $G = \langle A, B \rangle$, provided also that $A \cap B$ is the center of (at least) one of $A$ and $B$. If $H$ is a subgroup of $G$, then we denote by $\mathfrak{J}(H)$ the ideal of $FG$ generated by the elements $h - 1$ where $h \in H$. Set $(a, b) = a^{-1}b^{-1}ab$, where $a, b \in G$. Denote by $|g|$ the order of $g \in G$. Put $\Omega_k(G) = \langle u \in G \mid u^{b^k} = 1 \rangle$ and $\hat{\mathcal{H}} = \sum_{g \in H} g \in FG$. If $H \leq G$ is a normal subgroup of $G$, then $FG/\mathfrak{J}(H) \cong F[G/H]$ and

$$V(FG)/(1 + \mathfrak{J}(H)) \cong V(F[G/H]).$$

(1)

We freely use the fact that every quotient of a powerful group is powerful [2, Lemma 2.2(ii)].

PROOF. We prove the theorem by assuming that counterexamples exist, considering one of minimal order, and deducing a contradiction. Suppose then that $G$ is a counterexample of minimal order. If $G$ had a nonabelian proper factor group $G/H$, that would be a smaller counterexample, for, by (1), $V(F[G/H])$ would be a homomorphic image of the powerful group $V(FG)$. Thus all proper factor groups of $G$ are abelian, that is, $G$ is just nilpotent of class 2 in the sense of Newman [6].

As Newman noted in the lead-up to his Theorem 1, this means that the derived group has order $p$ and the center is cyclic. Of course it follows that all $p$th powers are central, so the Frattini subgroup $\Phi(G)$ is central and also cyclic.

Suppose $p > 2$. Then a finite $p$-group with only one subgroup of order $p$ is cyclic [3, Theorem 12.5.2], so $G$ must have a noncentral subgroup $B = \langle b \rangle$ of order $p$. Now $(b, a) = c \neq 1$ for some $a \in G$ and some $c \in G'$. Of course $\langle c \rangle = G' \leq \zeta(G)$, $a^{-1}b^{i}a = b^{i}c^{i} = c^{i}b^{i}$ and $b^{i}\hat{B} = \hat{B}$ for all $i$, so

$$(a\hat{B})^{2} = a^{2}(1 + a^{-1}ba + \cdots + a^{-1}b^{p-1}a)\hat{B}$$

$$= a^{2}(1 + cb + \cdots + c^{p-1}b^{p-1})\hat{B}$$

$$= a^{2}\hat{G}^{2}\hat{B}.$$  

(2)

Noting that

$$(\hat{G}')^{2} = 0,$$  

(3)

we get

$$(a\hat{B})^{3} = a^{2}\hat{G}'\hat{B} \cdot a\hat{B} = a^{2}\hat{G}'a^{-1} \cdot (a\hat{B})^{2}$$

$$= a^{2}\hat{G}'a^{-1} \cdot a^{2}\hat{G}'\hat{B} = a^{3}(\hat{G}')^{2}\hat{B} = 0.$$  

(4)

Therefore $|1 + a\hat{B}| = p$. We know from 4.12 of [7] that $\Omega_1(V(FG))$ has exponent $p$, so it must be that $(1 + a\hat{B}b)^{p} = 1$ as well. However,

$$b^{i}ab^{-i} = a(a, b^{-i}) = ac^{i} = c^{i}a,$$  

(5)
which allows one to calculate that
\[(1 + a\widehat{B})b^p = (1 + a\widehat{B})(1 + bab^{-1}\widehat{B}) \cdots (1 + b^{p-1}ab^{-p+1}\widehat{B}) \cdot b^p\]
\[= (1 + a\widehat{B})(1 + ca\widehat{B}) \cdots (1 + c^{p-1}a\widehat{B}) \text{ by (5)}\]
\[= 1 + \widehat{G}'(a\widehat{B}) + \frac{1}{2}(p - 1)\widehat{G}'(a\widehat{B})^2 \text{ by (4)}\]
\[= 1 + \widehat{G}'(a\widehat{B}) + \frac{1}{2}(p - 1)(\widehat{G}')^2 a^2\widehat{B} \text{ by (2)}\]
\[= 1 + \widehat{G}'(a\widehat{B}) \text{ by (3)}\]
\[\neq 1.\]

(To see that the third line is equal to the second, it helps to think in terms of polynomials with $a\widehat{B}$ as the indeterminate and $FG'$ as the coefficient ring, the critical point being that in the third line the coefficients of all positive powers of $a\widehat{B}$ are integer multiples of $\widehat{G}'$.) This contradiction completes the proof when $p > 2$.

Next, we turn to the case $p = 2$. Then $G' = \langle c \mid c^2 = 1 \rangle$ and the ideal $\mathfrak{J}(G')$ is spanned by the elements of the form $\widehat{G}'g$, while $FG$ is spanned by the elements $h$ of $G$. It is clear that $\widehat{G}'g$ and $h$ commute, because
\[
\widehat{G}'gh = \widehat{G}'(ghg^{-1}h^{-1})hg \quad \text{and} \quad \widehat{G}'(ghg^{-1}h^{-1}) = \widehat{G}',
\]
so $\mathfrak{J}(G')$ is central in $FG$ and $1 + \mathfrak{J}(G')$ is central in $V(FG)$. As $(\widehat{G}')^2 = 0$, it also follows that $(\mathfrak{J}(G'))^2 = 0$ and so the square of every element of $1 + \mathfrak{J}(G')$ is 1. As $V(FG)/(1 + \mathfrak{J}(G')) \cong V(F[G/G'])$, the derived group $V'$ of $V(FG)$ lies in $1 + \mathfrak{J}(G')$, a central subgroup of exponent 2. It follows that in $V(FG)$ all squares are central.

Let $w \in V'$. By [5, Proposition 4.1.7], this is the fourth power of some element $u$ of $V(FG)$. Write $u$ as $\sum_{g \in G} \alpha_g g$ with each $\alpha_g$ in $F$. In the commutative quotient modulo $\mathfrak{J}(G')$, $u^2 = \sum_{g \in G} \alpha_g^2 g^2$, hence
\[u^2 = v + \sum_{g \in G} \alpha_g^2 g^2 \]
for some $v$ in $\mathfrak{J}(G')$. Of course then $v$ and all the $g^2$ are central in $FG$ and $v^2 = 0$, so we may conclude that $w = u^4 = \sum_{g \in G} \alpha_g^4 g^4$.

In particular, as $V(FG)$ is not abelian, the exponent of $G$ must be larger than 4. Recall that $\Phi(G)$ is central, the center is cyclic, and $|G'| = 2$, so [1, Theorem 2] applies and for this case gives the structure of $G$ as
\[G = G_0 \mathcal{Y} G_1 \mathcal{Y} \cdots \mathcal{Y} G_r\]
where $G_1, \ldots, G_r$ are dihedral groups of order 8 and $G_0$ is either cyclic of order at least 8 (and in this case $r > 0$) or an $M(2^m+2)$ with $m > 1$, where
\[M(2^m+2) = \langle a, b \mid a^{2^m+1} = b^2 = 1, a^b = a^{1+2^m} \rangle.\]
One of the conclusions we need from this is that every fourth power in $G$ is already a fourth power in $G_0$, thus every element of $V'$ is an element of $FG_0^4$. In particular, when $w$ is the unique nontrivial element of $G'$, the linear independence of $G$ as subset of $FG$ implies that $w$ itself is the fourth power of some element of $G_0$.

It is easy to verify that, in $M(2^{m+2})$ with $m \geq 1$, the inverse of the element $1 + a + b$ is

$$(a^{2m-3} + a^{-3} + a^{-2} + a^{-1}) + (a^{2m-2} + a^{2m-2} + a^{-3})b$$

and so

$$(1 + a + b, a) = (1 + a^{2m-2} + a^{-2}) + (a^{2m-2} + a^{2m-1} + a^{-2} + a^{-1})b.$$  

Of course the left-hand side is an element of $V'$, but the right-hand side is not an element of $\langle a \rangle$. When $G_0 \cong M(2^{m+2})$, this shows that there is an element in $V'$ which does not lie in $FG_0^4$. When $G_0$ is cyclic, then $G_1 \cong M(2^{m+2})$ with $m = 1$, and we have an element in $V'$ which does not even lie in $FG_0$. In either case, we have reached the promised contradiction and the proof of the theorem is complete. \hfill \Box

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References


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