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$T\bar{T}$-deformation and long range spin chains

Balázs Pozsgay[1], Yunfeng Jiang[2], Gábor Takács[3]

[1] *MTA-BME Quantum Dynamics and Correlations Research Group, Department of Theoretical Physics, Budapest University of Technology and Economics, 1521 Budapest, Hungary*
[2] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland.*
[3] *BME "Momentum" Statistical Field Theory Research Group, Department of Theoretical Physics, Budapest University of Technology and Economics, 1521 Budapest, Hungary*

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Abstract

We point out that two classes of deformations of integrable models, developed completely independently, have deep connections and share the same algebraic origin. One class includes the $T\bar{T}$-deformation of 1+1 dimensional integrable quantum field theory and related solvable irrelevant deformations proposed recently. The other class is a specific type of long range integrable deformation of quantum spin chains introduced a decade ago, in the context of $\mathcal{N} = 4$ super-Yang-Mills theory. We show that the detailed structures of the two deformations are formally identical and therefore share many features. Both deformations preserve integrability and lead to non-local deformed theories, resulting in a change of the corresponding factorized S-matrices. We also prove a factorisation formula for the expectation value of the operators which trigger the deformation on the lattice; similar results in quantum field theory play an essential role in the solvability of such deformations. We point out that the long range deformation is a natural counterpart of the $T\bar{T}$-deformation for integrable spin chains, and argue that this observation leads to interesting new avenues to explore.

1 Introduction

Recently, a class of solvable irrelevant deformations of quantum field theories (QFT) have attracted considerable attention. One of the most well-studied example of such solvable deformations is the so-called $T\bar{T}$ deformation [1, 2] (see also [3, 4]), which can be defined for any 2d QFT. The $T\bar{T}$ deformation has many distinguished features, among which the following two are most relevant to us:

1. **Solvability and integrability**. Usually, irrelevant deformations of QFT are highly ambiguous and complicated due to appearance of an infinite number of counter-terms. In contrast,

the $T\bar{T}$ deformation is under much better control and solvable in an appropriate sense. In particular, when the original theory is an integrable quantum field theory (IQFT), the deformation preserves integrability. This allows for exact analytical results, especially when the original theory is a conformal field theory (CFT) or IQFT (cf. the reviews [5, 6] and references therein).

2. **Non-locality**. The deformed QFT becomes non-local, which can be seen classically by reformulating the $T\bar{T}$ deformation as coupling to various 2d gravity theories including the Jackiw-Teitelboim type gravity [7, 8], massive ghost-free gravity [9] and random geometry [10, 11]. At the quantum level it can be seen from the fact that the asymptotic density of states exhibit a Hagedorn behaviour instead of the usual Cardy growth [12, 13, 14]. Therefore the non-locality of $T\bar{T}$ deformed theories makes them different from usual local QFT and leads to a novel type of UV behaviour.

Other solvable irrelevant deformations including the $J\bar{T}$ deformation [15], deformation by higher spin irrelevant operators constructed from KdV currents [16, 17] and their various combinations [18, 19], which also share these two features. A further characteristics shared by these deformations is that the irrelevant operators which trigger such deformations are constructed from certain bi-local combination of conserved currents. Exploiting the conservation of the currents and translational invariance leads to a factorisation formula for the expectation value of such irrelevant operators. This simple yet crucial observation was first pointed out by Zamolodchikov in 2004 [20] for the $T\bar{T}$ operator and lies at the heart of solvability of these deformations.

Given the recent development of solvable irrelevant deformations for QFT, a natural and intriguing question is whether such deformations exist for *integrable lattice models* such as quantum spin chains, as it was stated in the paper [1]: *"Connections of such "effective theories" with lattice integrability seems an interesting question to explore."*. In this work we point out that not only do such deformations exist for integrable quantum spin chains, but they had been constructed already a decade ago, although in disguise! The $T\bar{T}$-like deformation corresponds to a specific type of integrable long-range deformation studied in [21], where the generator of the deformation is a bi-local combination of the charges.

Let us discuss this point in more detail. The spin chain analogue of the $T\bar{T}$-deformation in integrable QFT should capture the two important features mentioned above, namely it must *preserve integrability* and the deformed theory should *become non-local* in an appropriate sense. Usual integrable quantum spin chains such as the Heisenberg spin chain involve only nearest neighbour interactions. Spin chains that involve interactions between more sites are called long-range spin chains and are much less studied. Interestingly, a wide class of long range deformed spin chains with the desired properties were introduced by Bargheer, Beisert and Loebbert in [21] (see also [22, 23]). The motivation of these studies was to find a systematic way to construct the dilatation operator in planar $\mathcal{N}=4$ super-Yang-Mills theory at higher loop orders. For that purpose, the requirements are essentially the same: the dilatation operators at higher orders should preserve integrability and the interacting range should increase order by order in the 't Hooft coupling. The emerging non-locality is not very severe: the deformed Hamiltonian and higher conserved charges remain extensive and quasi-local (see the main text for details).

According to [21], there are two main types of non-trivial long-range deformations which preserve integrability: the boost-type and the bi-local-type. The boost-type deformation modifies the dispersion relation, while the bi-local-type deformation modifies the factorised S-matrix of the quasi-particle excitations.

The boost-type deformations were treated in detail by one of the authors in [24], where it was realised that they involve certain generalised current operators. The physical meaning of these generalised currents is rather simple: they describe the flow of a conserved quantity under a time evolution generated by some other conserved charge. As special cases they also include the physical current operators, which describe the flow under the fundamental spin chain Hamiltonian.

Another motivation for the study of such current operators originates from the recently introduced Generalised Hydrodynamics (GHD), which describes the large scale dynamics of integrable models out of equilibrium. To the leading order in a hydrodynamic expansion it captures the ballistic part of the transport [25, 26]. The theory is based on a local density approximation and the continuity relations for the conserved charges, and it leads to a generalised Boltzmann-type equation involving all charges. For this purpose it is essential to know the mean values of the currents in any finite density state. A physically motivated conjecture for the currents was given in [25, 26]. For massive integrable QFT it was already proven in [25], and a proof for the spin chains was given in [27]. Later it was realised in [24] that the connection to the boost-deformed long range spin chains provides a rather simple derivation of the results of [27].

In the present work we treat the long range deformations of the bi-local type for which the generator of the deformation is a certain bi-local combination of the conserved charges. As explained below, after proper rewriting the perturbing operators take essentially the same form as the perturbing irrelevant operators in QFT: they are composed of an anti-symmetric combination of charge densities and generalised currents, a connection which went unnoticed in [21]. Applying Zamolodchikov's argument to the lattice case we prove a factorisation formula for the perturbing operators on the lattice. This demonstrates that the algebraic construction is completely the same as for the $T\bar{T}$ deformation in the QFT. As a direct consequence, the S-matrix is deformed by a phase factor involving the charge eigenvalues, which echoes the fact that the $T\bar{T}$-deformation modifies the S-matrix of QFT by multiplying it with a CDD factor.

The paper is composed as follows. In section 2, we give a brief review of the salient features of the solvable irrelevant deformations. In section 3, we introduce local integrable spin chains and a class of operators constructed from specific combinations of the current and charge density operators. We then prove a lattice version of the factorisation formula for the mean values of these operators. In section 4, we discuss integrable long-range deformations of local spin chains. We mainly focus on the bi-local-type deformation and point out the intimate relationship between this deformation and the solvable irrelevant deformations of QFT. In section 5, we consider the deformed expectation value of conserved charges in the finite volume using asymptotic Bethe Ansatz. We confirm that the factorisation formula still holds in the finite volume. Finally we conclude in section 6 and discuss future interesting directions to explore.

2 $T\bar{T}$-deformation of QFT

In this section we provide a short review of the solvable irrelevant deformations of QFT that are related to the discussion in this work; the reader is invited to consult the original papers for further details.

Definitions We give the definition of general solvable deformation in Lagrangian formalism following [16, 28]. For a given 2d QFT described by an action S_0, consider two conserved currents $J_\mu^{(1)}$ and $J_\mu^{(2)}$ satisfying $\partial^\mu J_\mu^{(a)} = 0$. Using these currents we construct the following composite operator

$$\mathcal{O} \equiv \epsilon^{\mu\nu} J_\mu^{(1)} J_\nu^{(2)}. \tag{2.1}$$

Taking $J_\mu^{(1)} = T_{1\mu}$ and $J^{(2)} = T_{2\mu}$, the operator $\mathcal{O}$ is called the $T\bar{T}$ operator. More precisely, we have

$$\mathcal{O} = \epsilon^{\mu\nu} T_{1\mu} T_{2\nu} = T_{11}T_{22} - T_{12}T_{21} = \det T_{\mu\nu} \tag{2.2}$$

which can be written in the complex coordinates as [1]

$$\det T_{\mu\nu} = -(T\bar{T} - \Theta^2) \tag{2.3}$$

where T, $\bar{T}$ and Θ are defined as

$$\begin{aligned} T &= -\frac{1}{2}(T_{11} - T_{22} - 2iT_{12}), \\ \bar{T} &= -\frac{1}{2}(T_{11} - T_{22} + 2iT_{12}), \\ \Theta &= \frac{1}{2}(T_{11} + T_{22}). \end{aligned} \tag{2.4}$$

For CFT, $\Theta = 0$ and the operator takes the form of $T\bar{T}$.

Using the composite operator $\mathcal{O}$ we can define a family of theories parametrized by a parameter λ

$$\frac{dS_\lambda}{d\lambda} = \int d^2x\, \mathcal{O}_\lambda(x). \tag{2.5}$$

We stress that under this deformation the currents $J_\mu^{(a)}$ remain conserved, but their explicit form changes depending on λ. Therefore the corresponding operator $\mathcal{O}_\lambda$ is also deformed and depends on λ.

Factorization formula The solvability of the quantum theory for the class of theory (2.5) is based on the factorization formula of the expectation value of the composite operator. To derive it, consider the theory on a cylinder where the spacial direction is a circle of length L, while the temporal direction is non-compact. For a generic eigenstate of the Hamiltonian denoted by $|\psi\rangle$, translational invariance and conservation of the current implies

$$\langle\psi|\mathcal{O}_\lambda|\psi\rangle = \epsilon^{\mu\nu}\langle\psi|J_\mu^{(1)}J_\nu^{(2)}|\psi\rangle = \epsilon^{\mu\nu}\langle\psi|J_\mu^{(1)}|\psi\rangle\langle\psi|J_\nu^{(2)}|\psi\rangle. \tag{2.6}$$

As a result, the expectation value of the composite operator $\mathcal{O}_\lambda$ can be written in terms of the expectation values of the currents along the whole flow. The factorization formula was first derived by Zamolodchikov for $\mathcal{O}_\lambda$ being the $T\bar{T}$ operator [20].

Deformed spectrum The factorization formula leads to a flow equation for the deformed energy. Denoting the deformed energy by $\langle\psi|H_\lambda|\psi\rangle = E(\lambda, L)$, and using the definition (2.5) and the Hellmann-Feynman theorem yields

$$\frac{\partial}{\partial\lambda}E = L\,\langle\psi|\mathcal{O}_\lambda|\psi\rangle = L\,\epsilon^{\mu\nu}\langle\psi|J_\mu^{(1)}|\psi\rangle\langle\psi|J_\nu^{(2)}|\psi\rangle. \tag{2.7}$$

Given some convenient expressions for the expectation values of the conserved currents, the above equation can be used to determine the deformed spectrum. For example, taking $\mathcal{O}_\lambda$ to be the $T\bar{T}$ operator, the right hand side of (2.7) can be written in terms of the energy and momentum, so the flow equation reads

$$\frac{\partial E}{\partial\lambda} = \frac{1}{2}\left(E\partial_L E + \frac{P^2}{L}\right), \tag{2.8}$$

where P is the momentum of the state $|\psi\rangle$. Eqn. (2.8) is nothing else but the inviscid Burger's equation in one dimension, which can be solved by applying the method of characteristics. When the original theory is a CFT, the deformed energy $E(\lambda, L)$ can be found explicitly.

Deformed S-matrix and CDD factor The S-matrix of the QFT is deformed in a simple way under the solvable irrelevant deformation [1] (see also [4, 29]). This fact is particularly powerful for IQFT since the factorized S-matrix plays an essential role in computing many physical quantities. Let us denote the factorized S-matrix by $S_{ij}^{kl}(\theta)$ where $\theta \equiv \theta_i - \theta_j$ and θ_i are the rapidities of the particles. Under $T\bar{T}$ deformation the S-matrix is deformed as

$$S_{ij}^{kl}(\theta) \mapsto S_{ij}^{kl}(\theta) e^{i\delta_{ij}^{(\lambda)}(\theta_i,\theta_j)}, \tag{2.9}$$

where

$$\delta_{ij}^{(\lambda)}(\theta_i,\theta_j) = \lambda\, \epsilon_{\mu\nu} p_i^\mu p_j^\nu \tag{2.10}$$

and $p_i^\mu = (E_i, P_i)$ is the 2-momentum of the particle. For a massive relativistic QFT

$$p_i^\mu = (m_i \cosh\theta_i, m_i \sinh\theta_i) \tag{2.11}$$

and the corresponding phase factor takes the form

$$\delta_{ij}^{(\lambda)}(\theta) = \lambda m_i m_j \sinh(\theta). \tag{2.12}$$

For massless particles

$$\delta_{ij}^{(\lambda)}(\theta) = \frac{\lambda}{2} M_i M_j e^{\theta} = -2\lambda\, p_i^{(+)} p_j^{(-)}, \tag{2.13}$$

where $p_i^{(+)}$ and $p_j^{(-)}$ are the momenta of left-moving and right-moving massless particles. Analogous phase factors for the irrelevant deformation triggered by bi-locals of higher KdV charges are given in [1], while the phase factor corresponding to JT_a deformation was found recently in [30].

In the following we demonstrate that all these features can be generalised in a natural way to long range deformations of spin chains of bi-local type.

3 Local spin chains

We consider integrable local spin chains given by Hamiltonian

$$H = \sum_x h(x), \tag{3.1}$$

where $h(x)$ is a local operator acting on the nearest neighbour sites x and $x+1$. We consider both finite and infinite spin chains, with periodic boundary conditions in the finite case.

As a concrete example we consider integrable spin chains with $SU(N)$ symmetry, where at each site the local space is $\mathbb{C}^N$ and

$$h(x) = P_{x,x+1} - 1, \tag{3.2}$$

where P is the permutation operator exchanging the local spaces on sites x and $x+1$; for $N=2$ this corresponds to the XXX Heisenberg spin chain. However, we stress that the discussions below are quite general and work for other important classes of integrable spin chains such as the XXZ and XYZ chains.

Integrable spin chains have a family of infinitely many conserved charges Q_α in involution. They are extensive operators, i.e. they can be written as

$$Q_\alpha = \sum_x q_\alpha(x), \quad \alpha = 1, \ldots, \infty, \tag{3.3}$$

where $q_\alpha(x)$ are the charge densities. In the following we use the notation $|\mathcal{O}(x)|$ for the range of the local operator $\mathcal{O}(x)$. The charge densities can be chosen such that $|q_\alpha(x)| = \alpha$, and specifically we have $H \sim Q_2$, where the proportionality factor depends on conventions.

The local charges are usually obtained from a commuting set of transfer matrices, which are built from local Lax operators [31]. Alternatively, they can also be obtained with help of the boost operator [32, 33, 34, 35], which is the approach we use below.

The boost operators are formal operator that exists only on the infinite chain. For an extensive local operator

$$M = \sum_{x=-\infty}^{\infty} m(x) \tag{3.4}$$

we define the boosted operator as the formal sum

$$\mathcal{B}[M] = \sum_{x=-\infty}^{\infty} x m(x). \tag{3.5}$$

Then we have [32, 33, 34, 35]

$$Q_{\alpha+1} = i[\mathcal{B}[Q_2], Q_\alpha] + \text{constant}. \tag{3.6}$$

The constant part can be chosen in various ways; one possibility is to require that the charges have zero eigenvalues on a specifically chosen ferromagnetic reference state.

The current operators associated to the charges are defined through the continuity equation

$$\frac{d}{dt} q_\alpha(x) = i\,[H, q_\alpha(x)] = J_\alpha(x) - J_\alpha(x+1), \tag{3.7}$$

where the choice for the shift in x is just a matter of convention.

The essential difference between the lattice and the QFT is that space derivatives are discrete and there is no Lorentz invariance here; in particular, the pair $(q_\alpha(x), J_\alpha(x))$ does not form a vector. Nevertheless, as explained below they play a very similar role in the deformations as their QFT counterparts J_μ, which are Lorentzian 2-vectors.

We also introduce the generalised currents, that describe the flow of charge Q_α under the time evolution dictated by Q_β:

$$i\,[Q_\beta, q_\alpha(x)] = J_{\alpha,\beta}(x) - J_{\alpha,\beta}(x+1). \tag{3.8}$$

These operators also play an important role in the long range deformations. The analogous construction in QFT would correspond to using the higher conserved charges to generate Hamiltonian time evolution of the system. Such quantum integrable hierarchies exist e.g. for the quantum KdV system [36] and the quantum Benjamin-Ono equation [37].

3.1 Factorization on the lattice

Now we construct a certain combination of the current and charge density operators, such that the mean value of the resulting local operator factorises. We adopt the arguments of [10, 20] to the lattice case. Let us fix the indices $\alpha \neq \beta$ and consider the operator

$$J_\alpha(x) q_\beta(0) - q_\alpha(x) J_\beta(1). \tag{3.9}$$

Setting $x = 0$ or $x = 1$ we can recognise the lattice version of the anti-symmetric combination (2.1) from QFT.

Let us also fix an arbitrary eigenstate $|\Psi\rangle$ of the model; the computations can be performed both in finite or in infinite volume.

Theorem 1. *The function*

$$C(x) = \langle\Psi|J_\alpha(x)q_\beta(0) - q_\alpha(x)J_\beta(1)|\Psi\rangle \tag{3.10}$$

does not depend on x.

Proof. Applying a lattice derivative and using the translational invariance of the correlator yields

$$C(x+1) - C(x) = \langle\Psi|\Big(J_\alpha(x+1) - J_\alpha(x)\Big)q_\beta(0)|\Psi\rangle + \langle\Psi|q_\alpha(x)\Big(J_\beta(1) - J_\beta(0)\Big)|\Psi\rangle. \tag{3.11}$$

From the definition (3.7) we get

$$\begin{aligned} C(x+1) - C(x) &= -i\left[\langle\Psi|[H, q_\alpha(x)]q_\beta(0)|\Psi\rangle + \langle\Psi|q_\alpha(x)[H, q_\beta(0)]|\Psi\rangle\right] = \\ &= -i\langle\Psi|[H, q_\alpha(x)q_\beta(0)]|\Psi\rangle = 0. \end{aligned} \tag{3.12}$$

where in the last step we used the fact that $|\Psi\rangle$ is an eigenvector of H. □

Denoting the constant value of $C(x)$ by C, using translational invariance and the fact $|\Psi\rangle$ is an eigenvector of the conserved charges we get

$$LC = \sum_{x=1}^{L} C(x) = \langle\Psi|J_\alpha(x)|\Psi\rangle\langle\Psi|Q_\beta|\Psi\rangle - \langle\Psi|Q_\alpha|\Psi\rangle\langle\Psi|J_\beta(x)|\Psi\rangle. \tag{3.13}$$

Here the current mean values don't depend on x; we just kept the dependence on x to signal that J_α and J_β are intensive quantities, while the Q's are extensive.

Dividing by L and using the charge densities again we get

$$\langle\Psi|J_\alpha(x)q_\beta(0) - q_\alpha(x)J_\beta(1)|\Psi\rangle = \langle\Psi|J_\alpha|\Psi\rangle\langle\Psi|q_\beta|\Psi\rangle - \langle\Psi|q_\alpha|\Psi\rangle\langle\Psi|J_\beta|\Psi\rangle, \tag{3.14}$$

where now we deleted the x-dependence on the r.h.s. As in the field theory case [20, 10], the derivation only depends on the conservation equation (3.7) and translational invariance.

This argument can be extended to the generalised currents defined in (3.8). Introducing the three index local operators

$$K_{\alpha,\beta,\gamma}(x) = J_{\alpha,\gamma}(x)q_\beta(x) - q_\alpha(x)J_{\beta,\gamma}(x+1) \tag{3.15}$$

we obtain:

Theorem 2. *The mean values of $K_{\alpha,\beta,\gamma}(x)$ factorize as*

$$\langle\Psi|K_{\alpha,\beta,\gamma}(x)|\Psi\rangle = \langle\Psi|J_{\alpha,\gamma}|\Psi\rangle\langle\Psi|q_\beta|\Psi\rangle - \langle\Psi|q_\alpha|\Psi\rangle\langle\Psi|J_{\beta,\gamma}|\Psi\rangle. \tag{3.16}$$

Note that (3.14) and (3.16) are precisely the lattice counterparts of the factorization formula in QFT (2.6). To our knowledge this result is new. We give a few more comments about the factorisation in Section 6.

Now we show that this particular combination of operators arises from a simple commutation relation, which is analogous to the boost relation (3.6). Once again, the commutation is only formally defined in infinite volume.

Let us pick two indices $\alpha \neq \beta$ and define the formal sum

$$X = \sum_{x<y} q_\alpha(x)q_\beta(y). \tag{3.17}$$

Theorem 3. *The following commutator generates the factorising local operators:*

$$i[X, Q_\gamma] = \sum_x K_{\alpha,\beta,\gamma}(x). \tag{3.18}$$

Proof. The commutator can be computed as

$$\sum_{x<y} \{ i q_\alpha(x)[q_\beta(y), Q_\gamma] + i[q_\alpha(x), Q_\gamma] q_\beta(y) \} \tag{3.19}$$

which can be rearranged as

$$\sum_x i q_\alpha(x) \left[\sum_{y>x} q_\beta(y), Q_\gamma \right] + \sum_y i \left[\sum_{x<y} q_\alpha(x), Q_\gamma \right] q_\beta(y). \tag{3.20}$$

Integrating (3.8) yields

$$\begin{aligned} i\left[Q_\beta, \sum_{y>x} q_\alpha(y) \right] &= J_{\alpha,\beta}(x+1) \\ i\left[Q_\beta, \sum_{y<x} q_\alpha(y) \right] &= -J_{\alpha,\beta}(x). \end{aligned} \tag{3.21}$$

and substituting these into (3.20) after the appropriate changes yields

$$-\sum_x q_\alpha(x) J_{\beta,\gamma}(x+1) + \sum_y J_{\alpha,\gamma}(y) q_\beta(y), \tag{3.22}$$

which proves the theorem. □

Just as in the QFT case, the structural form of $K_{\alpha,\beta,\gamma}$ is given by a determinant

$$K_{\alpha,\beta,\gamma}(x) = \begin{vmatrix} J_{\alpha,\gamma}(x) & q_\alpha(x) \\ J_{\beta,\gamma}(x+1) & q_\beta(x) \end{vmatrix} \tag{3.23}$$

with a special choice for the operator ordering. The case of the stress-energy tensor (and thus $\det T$) encountered in QFT is obtained by formally setting $Q_\alpha = Q_\gamma = H$ and $Q_\beta = P$, where P is the total momentum. However, it is not possible to generate the precise analogue of T on the lattice, because there is no local operator corresponding to the momentum density. This is also consistent with the fact that Lorentz invariance is lost on the lattice.

On the other hand, one of the advantages of working on the lattice is that there is no need to regularise the product of operators e.g. by point-splitting. Nevertheless, the prescription is not entirely trivial as it involves a shift in the x coordinate whose precise form depends on the definition of the currents given in (3.8).

We also note that simple re-definitions of the generating operator X lead essentially to the same result. For example we could have taken

$$X = \sum_{x<y+l} q_\alpha(x) q_\beta(y) \tag{3.24}$$

with some $l \in \mathbb{Z}$. This would give

$$i[X, Q_\gamma] = \sum_x \left[J_{\alpha,\gamma}(x+l) q_\beta(x) - q_\alpha(x+l) J_{\beta,\gamma}(x+1) \right]. \tag{3.25}$$

Theorem 1 states that the mean values of this operator do not depend on l, therefore l could be chosen arbitrary. In particular l can be chosen large enough so that the supports of the two operators in (3.24) do not overlap in space, for which case there is no issue with the operator ordering.

4 Long range spin chains

In this section we explain how to deform the local spin chains with the factorizing operators described above. The framework for this was developed in [38, 21], although the relation of the perturbing operators to the generalised currents was not realised back then. We restrict ourselves to the infinite volume situation in what follows, except in Section 5 where we return to finite volumes.

We introduce a deformation parameter κ and set out to find the deformations the commuting set of charges which satisfy the following conditions:

1. The deformed charges allow a power series expansion in κ

$$Q_\alpha^\kappa = \sum_{j=0}^{\infty} \frac{\kappa^j}{j!} Q_\alpha^{(j)} \tag{4.1}$$

   with the initial conditions

$$Q_\alpha^{\kappa=0} = Q_\alpha, \tag{4.2}$$

   where Q_α are the local charges of the original (local) spin chain.

2. The charges continue to form a commuting family

$$[Q_\alpha^\kappa, Q_\beta^\kappa] = 0, \tag{4.3}$$

   which ensures that integrability is preserved by the deformation.

3. We also require that the resulting operators remain extensive and quasi-local[1], written as

$$Q_\alpha^\kappa = \sum_{x=-\infty}^{\infty} q_\alpha^\kappa(x). \tag{4.4}$$

   in terms of appropriate charge densities $q_\alpha^\kappa(x)$.

4. Furthermore the deformation of the infinite volume eigenstates can be expressed as a power series

$$|\Psi^\kappa\rangle = \sum_{j=0}^{\infty} \frac{\kappa^j}{j!} \left|\Psi^{(j)}\right\rangle, \tag{4.5}$$

   where $\left|\Psi^{(0)}\right\rangle$ are the original eigenstates.

These requirements can easily be satisfied by postulating the following formal generating equations [38, 21]:

$$\begin{aligned} \frac{d}{d\kappa}|\Psi^\kappa\rangle &= -iX(\kappa)|\Psi^\kappa\rangle \\ \frac{d}{d\kappa}Q_\alpha^\kappa &= i[X(\kappa), Q_\alpha^\kappa], \end{aligned} \tag{4.6}$$

where $X(\kappa)$ is a formal operator to be specified later, typically dependent on κ.

The generating equation naturally preserves the commutation of the charges due to

$$\frac{d}{d\kappa}[Q_\alpha^\kappa, Q_\beta^\kappa] = i[X(\kappa), [Q_\alpha^\kappa, Q_\beta^\kappa]], \tag{4.7}$$

[1] Following [39, 40] we call an extensive operator $A = \sum_x a(x)$ quasi-local, if the Hilbert-Schmidt (HS) norm of its traceless part grows at most linearly with the volume, and if its HS overlap with any local operator is finite in the $L \to \infty$ limit. way. Quasi-local operators can include pieces with arbitrary long range, but the amplitudes of these terms typically decay exponentially with the range.

where the Jacobi identity was exploited, and the eigenvalues of the charges are also unchanged:

$$Q^\kappa_\alpha|\Psi^\kappa\rangle = \Lambda^\Psi_\alpha|\Psi^\kappa\rangle, \tag{4.8}$$

where Λ^Ψ_α are the eigenvalues corresponding to the state $|\Psi\rangle$ in the original model.

These simple consequences of the generating equation do not depend on the form of $X(\kappa)$, which is instead constrained by the physical requirement that the deformed charges should remain quasi-local.

In [21] three families of deformations satisfying this requirement were identified[2]:

1. **Local operators.** Take

$$X = \sum_{x=-\infty}^{\infty} \mathcal{O}(x) \tag{4.9}$$

with $\mathcal{O}$ being any short range operator. This deformation describes a "physical" similarity transformation corresponding to a change of basis:

$$Q^\kappa_\alpha = e^{i\kappa X} Q^0_\alpha e^{-i\kappa X}, \tag{4.10}$$

and can be extended immediately to the case when $\mathcal{O}(x)$ is quasi-local.

2. **Boost operators.** The choice

$$X(\kappa) = -\mathcal{B}[Q^\kappa_\alpha] \tag{4.11}$$

for some α also generates quasi-local charges. This deformation was treated in detail in [24], where it was shown that the κ-derivative of the charges is

$$\frac{d}{d\kappa} Q^\kappa_\beta = \sum_x J^\kappa_{\alpha,\beta}(x), \tag{4.12}$$

where we defined the κ-deformed generalized current operators as

$$i\left[Q^\kappa_\beta, q^\kappa_\alpha(x)\right] = J^\kappa_{\alpha,\beta}(x) - J^\kappa_{\alpha,\beta}(x+1). \tag{4.13}$$

3. **Bi-local operators.** We choose the generator in the form

$$X(\kappa) = \sum_{x<y} q^\kappa_\alpha(x) q^\kappa_\beta(y). \tag{4.14}$$

Repeating the computations of the previous Section the κ-derivatives can be written as

$$\frac{d}{d\kappa} Q^\kappa_\gamma = \sum_x K^\kappa_{\alpha,\beta,\gamma}(x), \tag{4.15}$$

where the deformed K-operators are

$$K^\kappa_{\alpha,\beta,\gamma}(x) = J^\kappa_{\alpha,\gamma}(x) q^\kappa_\beta(x) - q^\kappa_\alpha(x) J^\kappa_{\beta,\gamma}(x+1). \tag{4.16}$$

This gives a recursion relation, which together with (4.14) can be used to generate the deformed charges.

[2]The quasi-locality of the deformed charges was not proven there, but it is clearly satisfied at least in some neighbourhood of $\kappa = 0$.

In all of these cases there is a large gauge freedom inherent in specifying these generators. For example, the charge density operators are not unique, as any re-definition of the form

$$q_\alpha(x) \quad \to \quad q_\alpha(x) + D(x+1) - D(x). \tag{4.17}$$

leaves the charges invariant. Furthermore, instead of (4.14) we could have a shifted version as in (3.24). However, it can be shown that these choices do not affect the main conclusions: they always result in adding extensive local operators to X, and do not affect the finite volume mean values of the perturbing operators.

It is an important property that the factorization (3.16) holds even for the deformed operators given by (4.16). This can be seen by repeating the steps of the derivation of (3.16), and noticing that it does not use the locality properties of the operators, only the global commutation of the charges and the local continuity equations.

4.1 Action on Bethe states

In this subsection we consider the deformation of the eigenstates dictated by the generating equation (4.6). This sets the lattice case apart from QFT, where it is not clear how the eigenstates are deformed. Most local integrable spin chains can be solved by Bethe Ansatz and the eigenstates can be written down explicitly. For simplicity we restrict ourselves to systems with a simple (non-nested) Bethe Ansatz; the extension to nested cases is rather straightforward (c.f. [21, 24]). Furthermore, we do not specify the details of the model; concrete formulas pertaining to the XXX and XXZ models can be found for example in [27, 24].

The infinite volume (un-normalized) Bethe states with N particles can be written as

$$|\boldsymbol{\lambda}_N\rangle = \sum_{x_1<x_2<\dots<x_N} \sum_{\sigma\in S_N} \prod_{j>k} f(\lambda_{\sigma_j} - \lambda_{\sigma_k}) \prod_{j=1}^{N} e^{ip_{\sigma_j} x_j} |x_1, \dots, x_N\rangle, \tag{4.18}$$

where $|x_1, \dots, x_N\rangle$ are the basis states with the particles occupying positions x_j. We have $p_j = p(\lambda_j)$, where $p(\lambda)$ is the one-particle quasi-momentum and λ is the rapidity parameter. The summation $\sigma \in S_N$ runs over all permutations of the rapidities. The function $f(\lambda)$ describes the interaction between the particles with the scattering phase shift given by

$$S(\lambda) = e^{i\delta(\lambda)} = \frac{f(\lambda)}{f(-\lambda)}. \tag{4.19}$$

In the above form the Bethe states are symmetric with respect to the exchange of rapidities.

The Bethe states are eigenvectors of the set of commuting charges with eigenvalues given by

$$Q_\alpha|\boldsymbol{\lambda}_N\rangle = \Lambda_\alpha(\boldsymbol{\lambda}_N)|\boldsymbol{\lambda}_N\rangle, \qquad \Lambda_\alpha(\boldsymbol{\lambda}_N) = \sum_{j=1}^{N} h_\alpha(\lambda_j). \tag{4.20}$$

The specific form of the one-particle eigenvalues is not relevant for us; explicit formulas can be found in [27, 24].

The total quasi-momentum of the state can be expressed as

$$P = \sum_{j=1}^{N} p(\lambda_j). \tag{4.21}$$

The deformation of the eigenstates can be understood simply from the generating equation (4.6) [21, 24]. The Bethe states retain their functional form for large separations of the particles, but the propagation factors and the relative phases change. Furthermore, the wave function acquires additional correction terms for small separations. More precisely, for any finite l contact terms appear for separations of l sites in higher order terms $\sim \kappa^c$ where $c \sim l$.

The boost-type deformations with charge Q_α generate a change of the one-particle momentum. It was shown in [24] that if this is the only ingredient in the deformation, then

$$p^\kappa(\lambda) = p(\lambda) + \kappa h_\alpha(\lambda) \tag{4.22}$$

holds to all orders in κ. The boost operators are one-particle irreducible, therefore they do not change the scattering matrix of the particles. This property and the deformation of the momentum was used in [24] to derive the finite volume mean values of the current operators; the results are summarized in Section 5 below.

The bi-local type deformations change the S-matrix of the spin chain (4.19), with the κ-derivative of the scattering phase given by

$$\frac{d}{d\kappa}\delta^\kappa(u,v) = h_\alpha(v)h_\beta(u) - h_\alpha(u)h_\beta(v). \tag{4.23}$$

This equation can be understood intuitively by looking at the deformation of the states. The various terms of the Bethe wave function get multiplied with different phases corresponding to the relative positions of the particles, eventually resulting in the above anti-symmetric combination of the one-particle charge eigenvalues. Note that the correction terms generally result in the S-matrix depending on the two rapidities separately instead of only on their difference, which in QFT corresponds to breaking Lorentz invariance.

The charge eigenvalues do not change under deformation, therefore the all-orders result is simply

$$\delta^\kappa(u,v) = \delta(u,v) + \kappa\left[h_\alpha(v)h_\beta(u) - h_\alpha(u)h_\beta(v)\right], \tag{4.24}$$

and the deformation of S-matrix can be written is

$$S(u,v) \mapsto S(u,v)e^{i\kappa(h_\alpha(v)h_\beta(u)-h_\alpha(u)h_\beta(v))}, \tag{4.25}$$

which is precisely the lattice version of the QFT relation (2.9)!

5 Finite volume: asymptotic Bethe Ansatz

The long range deformations described in the previous Section are only defined in infinite volume. The reason for this is that consistency requires the presence of correction terms with arbitrary long range which appear at successively higher orders in κ. The problem in finite volume is that there is no general prescription to define these long range terms once they wrap around the chain, which is the famous *wrapping problem*. The long range deformation does not provide integrable finite volume Hamiltonians beyond the wrapping order, thus a truncation of the deformation series is necessary. Nevertheless, the spectrum can still be computed using the so-called *asymptotic Bethe Ansatz* up to exponentially small corrections in the volume. The idea is to use the infinite volume quantities to construct the Bethe wave functions, and set up the Bethe equations using this information. This procedure is justified up to some finite order in κ such that the perturbing operators still fit into the volume.

Often we are only interested in the first order correction terms, and then the only requirement is that the leading perturbing operator should fit into the volume [24]. As shown below, the mean values of such perturbing operators are obtained from the first order corrections and so the results for the mean values below are eventually exact.

We now collect the main asymptotic equations and derive the mean values of the perturbing operators. Focusing on the bi-local case we also show that the deformation of the scattering phase (4.23) is consistent with the factorisation (3.16).

In finite volume the original Bethe equations are

$$p_j L + \sum_{k\neq j} \delta(\lambda_j, \lambda_k) = 2\pi I_j, \tag{5.1}$$

where $I_j \in \mathbb{Z}$ are the momentum quantum numbers. The deformation is assumed to be continuous in κ, therefore the I_j not be changed. According to this the deformed asymptotic Bethe equations are

$$p_j^\kappa L + \sum_{k\neq j} \delta^\kappa(\lambda_j, \lambda_k) = 2\pi I_j, \tag{5.2}$$

where p^κ and δ^κ are given by (4.22) and (4.24). Here we used a notation reflecting that both quantities are changed at the same time. Whereas this is certainly possible by combining the boost and bi-local types of generators, in the following we treat the two types of deformations separately. We recall that the commutativity of the different types of deformations is discussed in detail in [21].

Within the asymptotic Bethe Ansatz the charge eigenvalues are computed as

$$Q_\gamma^\kappa |\boldsymbol{\lambda}_N\rangle = \Lambda_\gamma^\kappa(L)|\boldsymbol{\lambda}_N\rangle, \qquad \Lambda_\gamma^\kappa(L) = \sum_{j=1}^N h_\gamma(\lambda_j). \tag{5.3}$$

Therefore, the dependence of the charge eigenvalues on κ comes entirely from the change of the rapidities:

$$\frac{d}{d\kappa}\Lambda_\gamma^\kappa(L) = \sum_{j=1}^N h'_\gamma(\lambda_j)\frac{d\lambda_j}{d\kappa}. \tag{5.4}$$

This idea was used in [24] to derive the current mean values of the original model, with the result

$$\langle \boldsymbol{\lambda}_N | J_{\alpha,\beta} | \boldsymbol{\lambda}_N\rangle = \mathbf{h}'_\beta G^{-1} \mathbf{h}_\alpha, \tag{5.5}$$

where $\mathbf{h}_\alpha$ is a vector of length N with components $h_\alpha(\lambda_j)$, $\mathbf{h}'_\beta$ is defined analogously, and G is the Gaudin matrix of size $N\times N$ given by

$$G_{jk} = \frac{\partial}{\partial \lambda_k}\left(p_j L + \sum_{k\neq j}\delta(\lambda_j,\lambda_k)\right), \qquad j,k = 1\dots N. \tag{5.6}$$

Let us now also investigate the first order correction of $\langle \boldsymbol{\lambda}_N | Q_\gamma^\kappa | \boldsymbol{\lambda}_N\rangle$ under the bi-local transformation with Q_α and Q_β. Using perturbation theory this is given by

$$\frac{d}{d\kappa}\langle \boldsymbol{\lambda}_N | Q_\gamma^\kappa | \boldsymbol{\lambda}_N\rangle = L\langle \boldsymbol{\lambda}_N | K_{\alpha,\beta,\gamma}(x) | \boldsymbol{\lambda}_N\rangle. \tag{5.7}$$

Taking the derivative of the asymptotic Bethe equations (5.2) with respect to κ yields

$$G_{jk}\frac{d\lambda_k}{d\kappa} + \sum_{k\neq j}\frac{d\delta(\lambda_j,\lambda_k)}{d\kappa} = 0. \tag{5.8}$$

From (4.23)

$$G_{jk}\frac{d\lambda_k}{d\kappa} = \sum_{k\neq j}\left[h_\alpha(\lambda_j)h_\beta(\lambda_k) - h_\alpha(\lambda_k)h_\beta(\lambda_j)\right]. \tag{5.9}$$

The summation can be changed to include $j = k$, resulting in

$$G_{jk}\frac{d\lambda_k}{d\kappa} = \left[h_\alpha(\lambda_j)\Lambda^\kappa_\beta - \Lambda^\kappa_\alpha h_\beta(\lambda_j)\right]. \tag{5.10}$$

which can be rewritten in vector form as

$$\frac{d}{d\kappa}\boldsymbol{\lambda} = \Lambda^\kappa_\alpha G^{-1}\mathbf{h}_\beta - \Lambda^\kappa_\beta G^{-1}\mathbf{h}_\alpha. \tag{5.11}$$

Substituting into (5.7) results in

$$L\langle\boldsymbol{\lambda}_N|K_{\alpha,\beta,\gamma}(x)|\boldsymbol{\lambda}_N\rangle = \Lambda^\kappa_\beta\left(\mathbf{h}'_\gamma G^{-1}\mathbf{h}_\alpha\right) - \Lambda^\kappa_\alpha\left(\mathbf{h}'_\gamma G^{-1}\mathbf{h}_\beta\right). \tag{5.12}$$

The eigenvalues divided by L are nothing else but the mean values of the charge densities, while the terms in parentheses are exactly the mean values of the currents given in (5.5), so finally we obtain

$$\begin{aligned}&\langle\boldsymbol{\lambda}_N|K_{\alpha,\beta,\gamma}(x)|\boldsymbol{\lambda}_N\rangle = \\ &\qquad = \langle\boldsymbol{\lambda}_N|q_\beta(x)|\boldsymbol{\lambda}_N\rangle\langle\boldsymbol{\lambda}_N|J_{\alpha,\gamma}(x)|\boldsymbol{\lambda}_N\rangle - \langle\boldsymbol{\lambda}_N|q_\alpha(x)|\boldsymbol{\lambda}_N\rangle\langle\boldsymbol{\lambda}_N|J_{\beta,\gamma}(x)|\boldsymbol{\lambda}_N\rangle,\end{aligned} \tag{5.13}$$

which is exactly the factorisation condition (3.16) applied to the finite volume Bethe states.

Another approach is to start with the above factorisation equation (which was derived independently from Bethe Ansatz), and then to compute the deformation rule (4.23) by focusing on the two-particle case. Alternatively, this result can also be considered as an independent derivation of the current mean values, starting from the factorization (3.16) and the deformation (4.23) of the scattering phase. All these aspects can be summarised in the statement that the transformation rules are consistent with the algebra of the charges and the currents.

To conclude this Section let us return once more to the deformation rule (4.6), which guarantees that the charge eigenvalues are invariant under the deformation *in infinite volume*. This is in contrast with the finite volume case, where the charge eigenvalues do change according to (5.7). The difference is clearly due to the fact that the similarity transformation generated by (4.6) is not compatible with periodic boundary conditions. This phenomenon is completely analogous to what was observed in [10], where it was explained that the $T\bar{T}$-deformation is eventually trivial *up to boundary terms*.

In this Section we only treated the spin chains, but the asymptotic Bethe Ansatz can also be formulated for the integrable QFT's. However, in QFT the Bethe Ansatz is only asymptotic even in the undeformed case, since the finite volume results of the Bethe Ansatz receive exponential corrections due to virtual processes. Our formulas for the mean values of the current operators remain valid also in the QFT situation, up to additional exponential corrections which appear in addition to the contributions accounted for in (5.5) and (5.13).

6 Discussion

In this work we pointed out that the known bi-local type deformations of the integrable spin chains are formally equivalent to the $T\bar{T}$-type deformations of integrable QFT. The key step in this identification was the commutation relation (3.18) of Theorem 3. We also found that the generalised current operators play a central role, similar to the case of the boost-type deformations treated in [24]. Remarkably, the present framework yields the deformation of all conserved charges in a very straightforward way, which sets it apart from the known QFT computations which mostly focus on the Hamiltonian only.

A central result is the factorization of the mean values of the three index operators $K_{\alpha,\beta,\gamma}(x)$, see eq. (3.16). The idea behind this is essentially the same as the original observations of Zamolodchikov regarding the $T\bar{T}$-operator [20]. Nevertheless, as far as we know, this form has not yet been written down for lattice systems.

Let us also mention, that for the XXZ spin chain the study of factorisation of correlation functions is of independent interest and is by now well understood [41, 42, 43, 44]. In this specific model the mean values of all local operators can be factorised, and our result (3.16) can be seen as a special case. However, we also stress that our $K_{\alpha,\beta,\gamma}(x)$ are very specific operators that exhibit factorisation for *every local integrable spin chain* and the factorisation also holds in the presence of the integrable long-range deformation.

Our observations open up many interesting future directions to explore.

Continuum limit It is well-known that the continuum limit and the low energy physics of the spin chains can in general be described by a QFT. Therefore it is interesting to investigate the continuum limit of the deformed chains. While the bi-local deformation for the spin chain is rather general and does not depend on specific details of the model, the continuum limit is more subtle and it does depend on details regarding the states under consideration and the scaling to the continuum limit. All our perturbations are generally expected to correspond to irrelevant operators, therefore their coupling constants scale to zero under the Renormalization Group flow. Nevertheless it might be possible to construct special scaling limits, such that the resulting QFT's would correspond to the $T\bar{T}$ (or related) deformations. This connection can potentially be exploited to learn about the short-distance physics and the ultraviolet completion of the $T\bar{T}$ deformations.

Alternative formulations and relation to lattice gravity Even though the generating equation (4.6) provides all necessary details of the deformation, we believe that the understanding of the long range spin chains is far from complete. The present formulation is only tangentially related to the standard methods of lattice integrability, such as local Lax operators, integrable vertex models, etc. Even though there are certain special cases when a long range spin chain could indeed be related to the more conventional models and methods [45, 46, 47, 48], the generating equation (4.6) opens up a big parameter space, which has not yet been understood. It would be desirable to find an underlying integrable structure, which is rich enough so that it could naturally accommodate all long range deformations. Certain hints are provided by QFT itself: in the Lagrangian formalism the $T\bar{T}$ deformation turns out to be a pure boundary effect [10]; furthermore, it can be understood by coupling the theory two a certain 2D gravity. It would be interesting to find the natural lattice counterparts of these phenomena. One possible lattice interpretation of the $T\bar{T}$ deformation is found in the upcoming work [49], where it is shown that the coupling to 2D gravity can be simulated by allowing the inhomogeneity parameters of the spin chain to become dynamical variables.

Other observables and non-equilibrium dynamics The long range deformations lead to a whole family of integrable spin chains whose properties have so far barely been investigated. It is interesting to study their properties in more detail by computing physically interesting quantities such as form factors, correlation functions and entanglement entropy. The non-local nature of the long-range interacting spin chains is expected to result in a behaviour for certain quantities which is qualitatively different from short range integrable spin chains. Given the close connection between generalised hydrodynamics and long range deformation, it is also natural to consider quantum quenches for these spin chains and see how the deformation changes the non-equilibrium dynamics.

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