Monochromatic Diameter Two Components in Edge Colorings of the Complete Graph*

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Abstract

Gyárfás conjectured that in every r-edge-coloring of the complete graph K_n there is a monochromatic component on at least $n/(r-1)$ vertices which has diameter at most three. We show that for $r = 3, 4, 5$ and 6 'diameter three' is best possible in this conjecture constructing colorings where every monochromatic diameter two subgraph has strictly less than $n/(r-1)$ vertices.

1 Monochromatic components and diameter

How many people do we need to ensure that there are three who each know each other or three who don't know each other? This question was answered long ago [7], and it kicked off the field of Ramsey theory. If we consider the people as vertices of a graph, and their relationship to each other to be an edge, we have a 2-edge-coloring of K_n, the complete graph on n vertices, where the colors represent the relationships 'know each other' and 'don't know each other'.

Ramsey theory aims to find large monochromatic structures in edge colorings of a graph. The fundamental Ramsey's Theorem [7] states that the *Ramsey number* $R(G_1, G_2, \ldots, G_r)$, *i.e., is the smallest* n *such that in every* r*-edge-coloring of* K_n *there is a monochromatic* G_i *in some color* $1 \leq i \leq r$ always exists.

The introduced problem is solved by $R(K_3, K_3) = 6$, i.e., 6 is the minimum number such that in every 2-edge-coloring of K_6 there is a monochromatic K_3. In other words, in every set of pairwise relationships among six people, there must either be a group of three that know each other or a group of three who are strangers to each other.

A simple remark by Erdős and Rado [4] states that any 2-coloring of the edges of K_n has a monochromatic spanning component, i.e., which contains all n vertices. For general r Gyárfás proved that the largest monochromatic component in an r-edge-coloring of K_n has size $\geq n/(r-1)$.

Theorem 1 (Gyárfás [3])**.** *The size of the largest monochromatic component in an* r*-edge-coloring of* K_n *is at least* $n/(r-1)$ *and equality holds if* $(r-1)^2|n$ *and there is an affine plane of order* $r-1$.

An affine plane of order $r-1$ is a collection of r even partitions of $(r-1)^2$ points into $r-1$ parts (lines) such that every pair of points is contained on a unique line. Each partition is called a parallel class of lines. The optimum is obtained by coloring the pairs in all $r-1$ disjoint lines of a parallel class the same, using for all r parallel classes different colors. Then each vertex in the plane is replaced by $\frac{n}{(r-1)^2}$ vertices, edges between different 'clusters' inherit the color of the associated edge and edges within clusters are colored arbitrarily. That is, each monochromatic component has the same $(r-1)\frac{n}{(r-1)^2} = n/(r-1)$ size.

If there is no affine plane of order $r-1$ then the following holds.

Theorem 2 (Füredi [2])**.** *If an affine plane of order* $r-1$ *does not exist, then the size of the largest monochromatic component in an* r*-edge-coloring of* K_n *is* $\geq \dfrac{n}{r-1-(r-1)^{-1}}$.

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Notice that this lower bound is significantly larger than the tight bound in case affine planes exist. It is well known that affine planes of prime power orders exist, but it still remains an important open problem – maybe the most important and difficult one in algebraic combinatorics – whether they only exist for prime powers. That is, to find the size of the largest monochromatic component in general is extremely difficult.

Can we say something more about these 'giant' (having at least $n/(r-1)$ vertices) monochromatic components?

A double star is a graph obtained by connecting the centers of two vertex disjoint stars. A triple star is similar, just that a third vertex disjoint star is connected to a double star.

The following possible strengthening of Theroem 1 has been posed by Gyárfás.

Problem 1 (Problem 4.2 in [4])**.** *For $r \geq 3$, is there a monochromatic double star of size asymptotic to $n/(r-1)$ in every r-coloring of K_n?*

This problem is still unsolved. The diameter of a graph G is the maximum distance between any two vertices, where the distance is the length of the shortest path between the two vertices. Clearly, the diameter of a double (triple) star is ≤ 3 (≤ 4, resp.) and in the same paper Gyárfás posed the following weaker version Problem 1.

Problem 2 (Problem 4.3 in [4])**.** *Given positive numbers n, r, is there a constant d (perhaps $d = 3$) such that in every r-coloring of K_n there is a monochromatic subgraph of diameter at most d with at least $n/(r-1)$ vertices?*

This was proved in affirmative for three colors by Mubayi [6].

Theorem 3 (Mubayi [6])**.** *Every 3-edge-coloring of K_n contains a monochromatic component of diameter ≤ 4 on at least $\lceil n/2 \rceil$ ($n/2+1$ if $n \equiv 2 \pmod 4$) vertices.*

Problem 2 in general with r colors has been solved by Ruszinkó with $d = 5$.

Theorem 4 (Ruszinkó [8])**.** *In every r-edge-coloring of K_n there is a monochromatic connected subgraph of diameter at most 5 on at least $n/(r-1)$ vertices.*

The proof relies on a theorem of Mubayi, which states that a complete bipartite graph on n vertices colored with r colors has a monochromatic double star of size n/r.

Theorem 4 has been improved by Letzter to diameter 4.

Theorem 5 (Letzter [5])**.** *In every r-edge-coloring of K_n there is a monochromatic triple star on at least $n/(r-1)$ vertices.*

Summarizing, it is not yet known if a diameter at most three monochromatic subgraph on at least $n/(r-1)$ vertices does exist.

The purpose of this paper is to explore, in Problem 2, the statement "*perhaps $d = 3$.*"

Is $d = 3$ the strongest possible conjecture? That is, can we find colorings with no 'giant' diameter 2 components? A theorem of Erdős and Fowler answers this question in affirmative for two colors.

Theorem 6 (Erdős, Fowler [1])**.** *Every 2-edge-coloring of K_n contains a monochromatic component of diameter ≤ 2 on at least $3n/4$ vertices.*

They give the following example to show that the bound given in Theorem 6 is sharp. Partition the set of vertices evenly into parts A_1, A_2, A_3, A_4 of size $\leq \lceil n/4 \rceil$. For $j > i$ color all edges red between A_i and A_j if $j - i = 1$, else color them blue. Color the edges inside each A_i arbitrarily, see Figure 1.

This example is sharp because the largest monochromatic diameter 2 component of this graph must be a subset of some $A_i \cup A_j \cup A_k$ with $i \neq j \neq k$, which means it has size $\leq 3n/4$. That is, a spanning monochromatic subgraph of diameter two does not necessarily exist in any 2-coloring of K_n.

We extend this result of Erdős and Fowler for $r = 3, 4, 5$ and 6 colors showing in Theorems 7, 8, 9 and 10, respectively, that a monochromatic subgraph of diameter two on at least $n/(r-1)$ vertices does not necessarily exist.

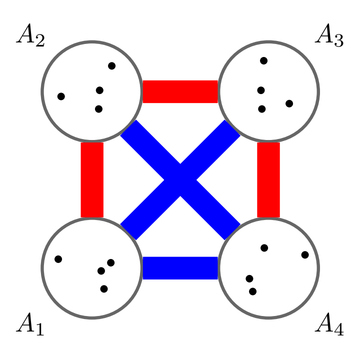

Figure 1: A coloring of K_4 with small diameter 2 components

2 Colorings With No Large Diameter Two Components

The 2-coloring in Figure 1 is essentially a partitioning of K_4 into two Hamiltonian paths, that is, paths that span the vertex set, and then "blowing up" the coloring for general n. We shall use a similar approach to prove Theorems 7, 8, 9 and 10. A *k-factorization* of a graph G is a partition on the edges of G into disjoint k-factors, i.e., spanning subgraphs with each vertex having degree k.

In order to r-color complete graphs with no large monochromatic diameter 2 components first choose suitably 'small' complete graphs K_{n_r} where the number of vertices n_r depends on the number of colors r. Then partition the edges of K_{n_r} into r k-factors that avoid large $(\geq n/(r-1))$ diameter two components. Coloring each factor a different color, there will be obviously no monochromatic connected diameter 2 subgraph in K_{n_r}. Then (blow up step) distribute n vertices into n_r clusters as evenly as possible. All the edges between two clusters inherit the color of the associated edge in K_{n_r}. The edges inside the clusters are colored arbitrarily. This way we obtain an r-coloring of K_n where every monochromatic connected diameter 2 subgraph is of size $< n/(r-1)$. First we present this method in the case of three colors.

Theorem 7. *There exists a 3-edge-coloring of K_n with the largest monochromatic diameter ≤ 2 subgraph of size $\leq 3\lceil n/7\rceil$.*

Proof. Partition the vertices of K_n into $A_1, A_2, \ldots, A_7$, $\lfloor n/7\rfloor \leq |A_i| \leq \lceil n/7\rceil$ and $\sum_{i=1}^{7} |A_i| = n$. Color the edges between A_i and A_j $c = min\{|i-j|, 7-|i-j|\}$ and within A_i arbitrarily for $i = 1 \ldots 7$, see Figure 2. The largest monochromatic diameter 2 subgraph in this 3-coloring may clearly contain vertices from at most three clusters, i.e., its size is $\leq 3\lceil n/7\rceil$. This is strictly less than $n/(r-1) = n/2$ for $r = 3$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5.

□

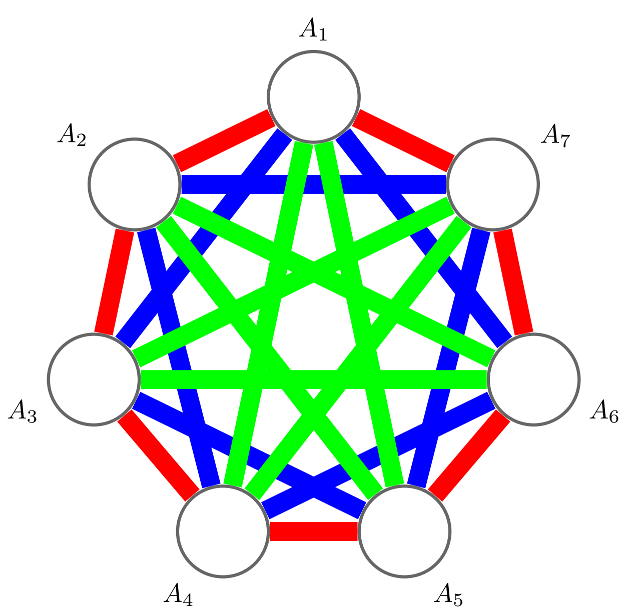

Figure 2: A 3-edge-coloring of K_7 used to color K_n

Here we performed a 2-factorization of K_7, more specifically a decomposition into three Hamiltonian cycles. It is well known that for any odd ℓ, K_ℓ can be decomposed into Hamiltonian cycles. In order to obtain suitable colorings for larger r, first we choose a suitable n_r, decompose the edges of K_{n_r} into Hamiltonian cycles and define the factors to be the unions of particular Hamiltonian cycles. All edges in a given factor will be colored the same. Also, we will be choosing n_r to be prime, which allows us to define our cycles as in the proof of Theorem 7, i.e., $C_i = \{(j,k) : i = min\{|k-j|, n_r - |k-j|\}\}$.

Theorem 8. *There exists a 4-edge-coloring of K_n with the largest monochromatic diameter ≤ 2 subgraph of size $\leq 5\lceil n/17\rceil$.*

Proof. Let $n_4 = 17$, and decompose K_{17} into 8 Hamiltonian cycles as above, i.e., $C_i = \{(j,k) : i = min\{|k-j|, 17-|k-j|\}\}$. The unions of pairs of Hamiltonian cycles $G_1 = (C_1 \cup C_2)$, $G_2(C_3 \cup C_6)$, $G_3(C_4 \cup C_8)$, $G_4(C_5 \cup C_7)$, form a 4 factorization of K_{17}. Notice that G_i-s are isomorphic where the isomorphism is simply renumbering the vertex v as $v \cdot i^{-1}$ mod 17. Color all the edges i in G_i and blow-up K_{17} to K_n as we did before (see Figure 3). All that's left to show is that G_1 has no diameter 2 subgraphs with at least 6 vertices.

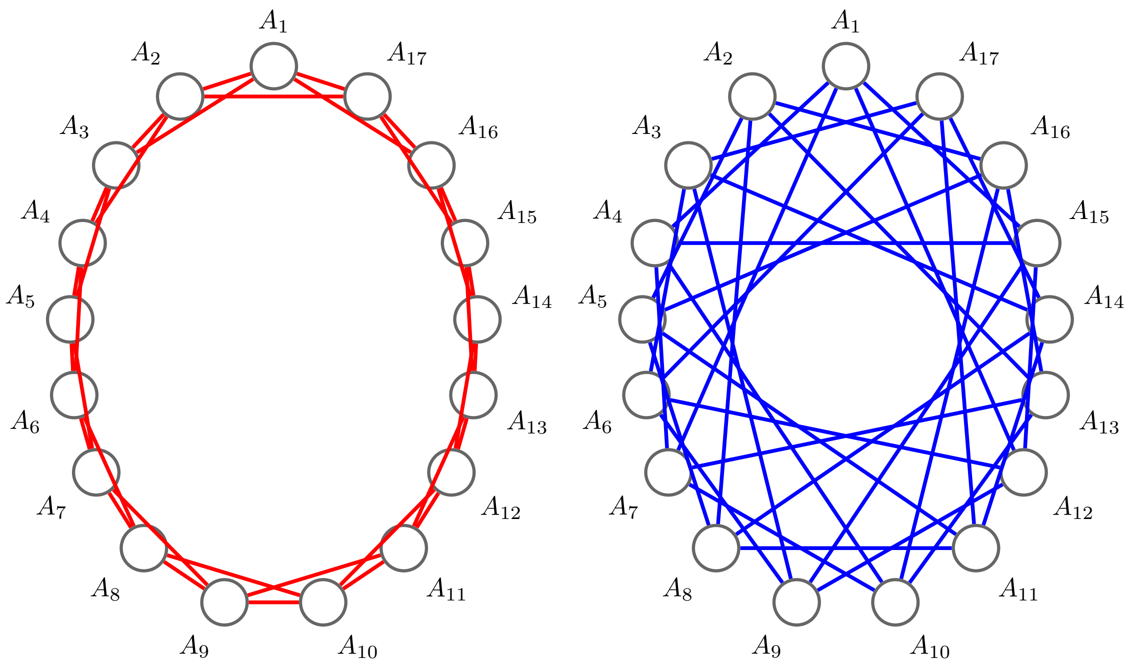

Figure 3: $C_1 \cup C_2$ is Isomorphic to $C_3 \cup C_6$

In contrary, assume there is a collection of 6 vertices in G_1 that don't satisfy this. Call one of the vertices v. Clearly, there are four vertices in each direction reachable in two steps in G_1 from v, for a total of 8 possibilities; the remaining 5 vertices must be in these spots. Name the vertex farthest to the left v_L and the one farthest to the right v_R. As there are only 4 spots on each side, both sides must have at least one vertex, meaning v is between v_L and v_R. So there are 4 vertices between v_L and v_R, i.e., their distance is at least 3.

This implies that no six vertices in a color class can each be at distance two from each other, so the largest monochromatic diameter 2 subgraph has 5 vertices of K_{17} to be "blown up", having size $\leq 5\lceil n/17 \rceil$. This is strictly less than $n/(r-1) = n/3$ for $r = 4$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5.

□

Theorem 9. *There exists a* 5*-edge-coloring of* K_n *with the largest monochromatic diameter* ≤ 2 *subgraph of size* $\leq 7\lceil n/31 \rceil$*.*

Proof. Decompose K_{31} into 15 Hamiltonian cycles $C_i = \{(j,k) : i = min\{|k-j|, 31-|k-j|\}\}$, $i = 1 \ldots, 15$ and then color as follows (see Figure 4):

- Color $C_1 \cup C_2 \cup C_3$ red.
- Color $C_4 \cup C_5 \cup C_6$ blue.
- Color $C_7 \cup C_8 \cup C_9$ green.
- Color $C_{10} \cup C_{11} \cup C_{13}$ purple.
- Color $C_{12} \cup C_{14} \cup C_{15}$ orange.

Notice that these color classes are not isomorphic, but there are only two different isomorphism classes. Therefore, one can relatively easily check, that no 8 vertices in any of the color classes induce a diameter 2 subgraph. Theorem 9 follows by our standard 'blow up' technique. $7\lceil n/31 \rceil$ is strictly less than $n/(r-1) = n/4$ for $r = 5$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5.

□

Theorem 10. *There exists a* 6*-edge-coloring of* K_n *with the largest monochromatic diameter* ≤ 2 *subgraph of size* $\leq 9\lceil n/47 \rceil$*.*

Proof. Decompose K_{47} into 23 Hamiltonian cycles $C_i = \{(j,k) : i = min\{|k-j|, 31-|k-j|\}\}$, $i = 1, 2, \ldots, 23$ and then color as follows (see Figure 5):

- Color $C_1 \cup C_2 \cup C_3 \cup C_4$ red.
- Color $C_5 \cup C_{10} \cup C_{15} \cup C_{20}$ blue.
- Color $C_6 \cup C_{12} \cup C_{18} \cup C_{23}$ green.
- Color $C_7 \cup C_{14} \cup C_{21} \cup C_{19}$ purple.
- Color $C_9 \cup C_{11} \cup C_{13} \cup C_{17}$ orange.
- Color $C_8 \cup C_{16} \cup C_{12}$ yellow.

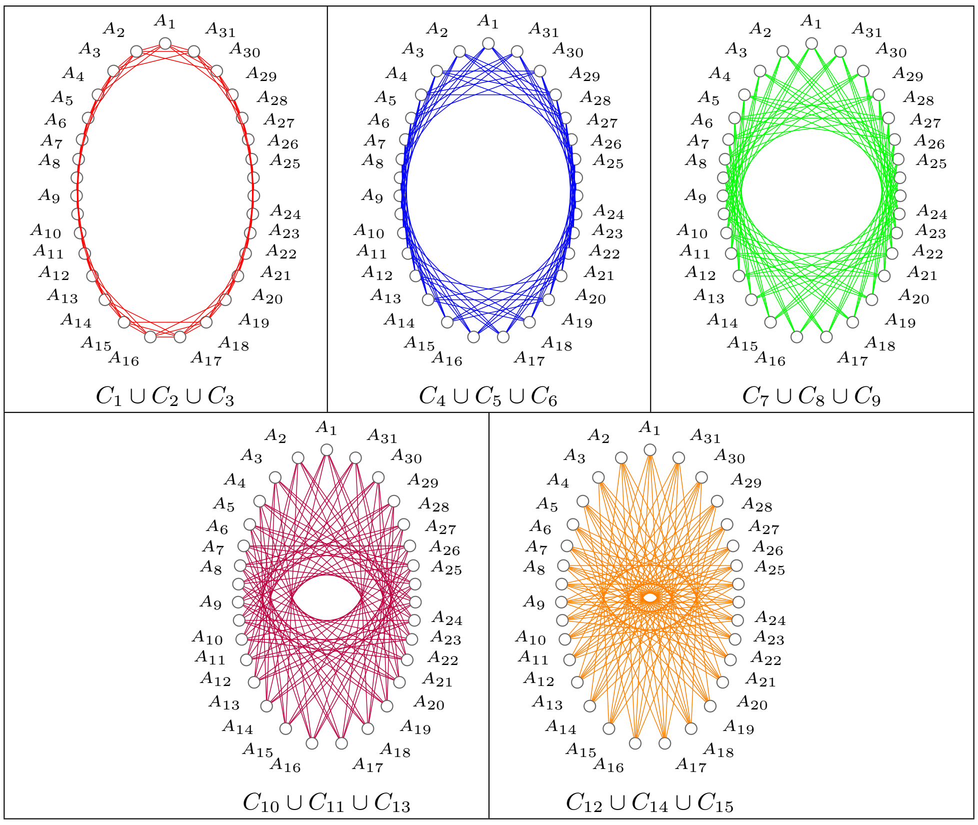

Figure 4: The 5-edge coloring for K_n made by blowing up K_{31}

That no 10 vertices in any of the color classes induce a diameter 2 subgraph is verified computationally. Theorem 10 follows by our standard 'blow up' technique. $9\lceil n/47 \rceil$ is less than $n/(r-1) = n/5$ for $r = 6$, the size of the largest monochromatic component of diameter at most 4 existing by Theorem 5. □

3 Final Remarks

By Erdős and Fowler (Theorem 6) and Letzter (Theorem 5) in case of two colors, the size of the largest monochromatic diameter two subgraph existing in every 2 coloring is significantly (i.e., by a linear in number of vertices term) less then the size of the largest monochromatic diameter four subgraph. We showed that the same phenomena holds if the number of colors is $3, 4, 5$ or 6.

Based on this we conjecture the following.

Conjecture 1. *For arbitrary number of colors r, the size of the largest monochromatic diameter two subgraph existing in every r coloring of K_n is significatly less than the size of the largest monochromatic diameter four subgraph existing in every r coloring of K_n.*

So Gyárfás's suggestion of *probably* $d = 3$ in Problem 2 seems to be accurate.

We think that our method could be used to prove Conjecture 1 for an arbitrary number of colors. We can show that a prime n_r for every $r \geq 3$ that meets our needs, i.e. it can be factored into appropriately sized classes with small stars does exist. One possible approach is to ensure that each color class is isomorphic to one that is easily proven not to have large diameter 2 components, as we did in Theorem 8. However we have yet to find a way to partition the cycles into color classes in a way that ensures the graph will have no large monochromatic diameter two components.

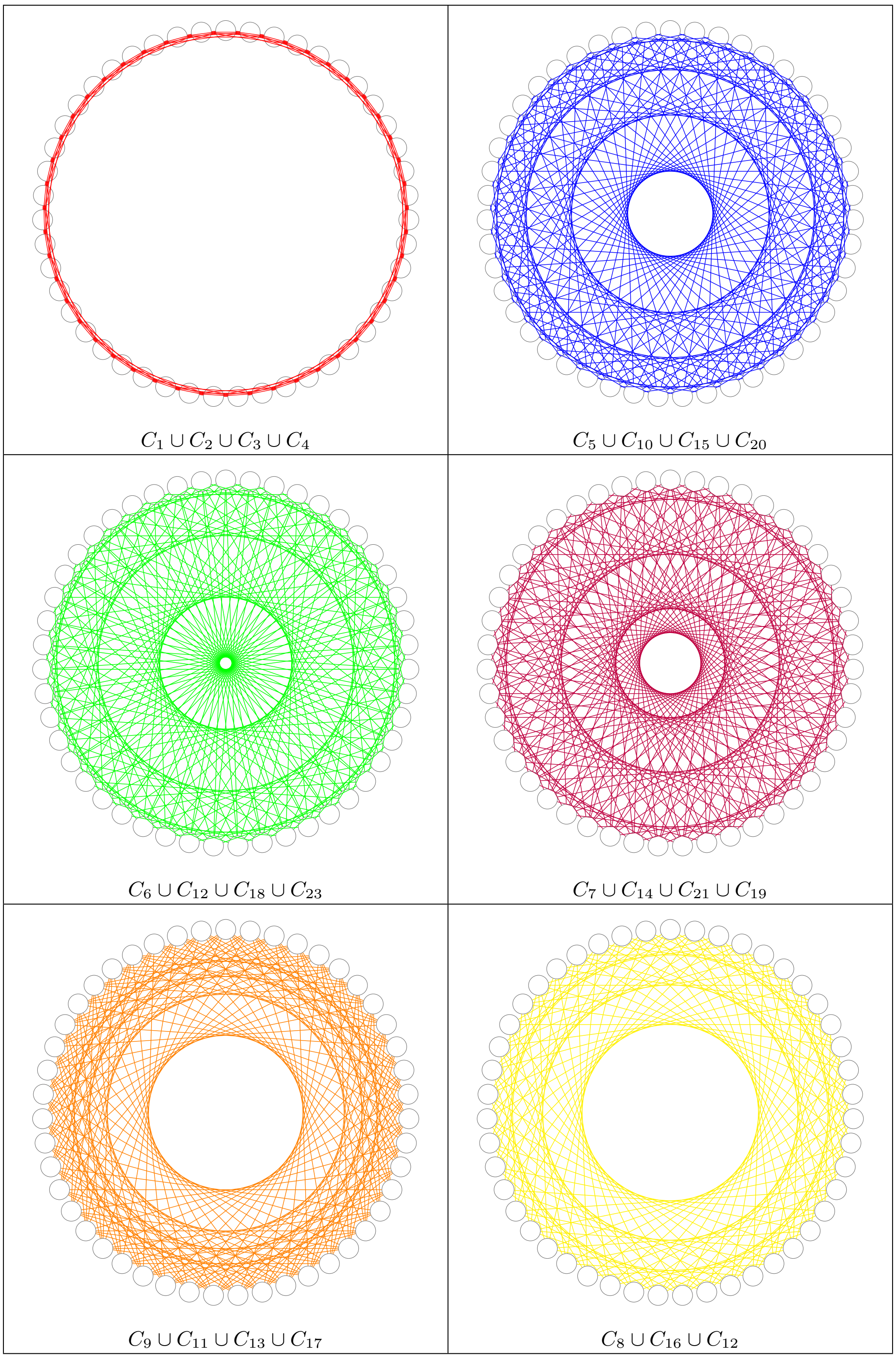

Figure 5: The 6-edge coloring for K_n made by blowing up K_{47}

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