Small exotic 4-manifolds from lines and quadrics in $\mathbb{CP}^2$

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Abstract

We construct potentially new manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ via rational blowdown surgery along certain 4-valent plumbing graphs. This way all the graph classes from [4] have a representative which admits a rational blowdown leading to an exotic manifold. We emphasize the simplicity of the constructions which boils down to finding a good configuration of complex lines and quadrics in $\mathbb{CP}^2$, and deciding which intersections to blow up.

1   Introduction

Smooth 4-manifold topology is a very intriguing field which has been transformed by several techniques and constructions in the past decades. Constructing different smooth structures on any given smoothable 4-manifold is still not a trivial problem, and for many of them it is not known whether there are different smooth structures, let alone if there is an infinite number of smoothings.

The problem we will be focusing on in this paper is the construction of *small* exotic 4-manifolds, meaning manifolds with small Euler characteristic and signature, homeomorphic but not diffeomorphic to some standard 4-manifolds. Donaldson first proved that a certain smooth 4-manifold admits two different smooth structures [6], by using his newly constructed invariants to distinguish Dolgachev surfaces which are homeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. Since then there were several papers providing increasingly more intricate constructions of even smaller exotic manifolds [12, 17, 23, 18, 1, 2]. In this note we prove the following:

Main Theorem. *There exists a configuration of complex lines and quadrics in $\mathbb{CP}^2$, and graphs from classes $\mathcal{B}^4$ and $\mathcal{C}^4$ shown in Figure 1, which can be used to produce exotic $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ via rational blowdowns.*

Examples of non-standard smooth structures on these manifolds were already known [6, 12], as well as the general technique we are using - the *rational blowdown surgery* introduced by Fintushel and Stern [7]. In its most general

form, this surgery technique replaces an adequate embedded plumbing with some rational homology ball, simplifying the topology in a controlled way. In our considerations all plumbings are neighbourhoods of spheres pairwise intersecting transversely in at most one point, and the plumbing graph is a tree.

The novelty is using particular plumbings from two graph classes $\mathcal{B}^4$ and $\mathcal{C}^4$ from [4] shown in Figure 1, previously unknown to produce exotic manifolds via rational blowdown. This way we show that each class of graphs from [4] has a representative which admits a rational blowdown leading to an exotic manifold, which might eventually advance the understanding of smoothings of singularities discussed there.
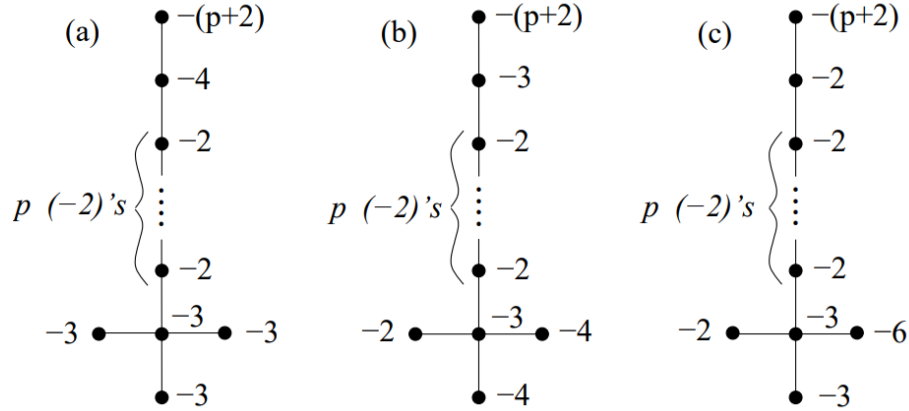

Figure 1: Classes $\mathcal{A}^4$, $\mathcal{B}^4$ and $\mathcal{C}^4$

Here it is worth emphasizing that we are actually not looking at a pencil of curves, blowing it up, deforming the monodromies, and rationally blowing down. Rather, we start with a good configuration of degree 1 and 2 curves (complex lines and quadrics) in $\mathbb{CP}^2$ which are all already spheres by the genus-degree formula. Then we blow up some intersection points, and some additional generic points until we get a required configuration of intersecting spheres embedded in $\mathbb{CP}^2$ blown up some number of times. After rationally blowing down this configuration in a symplectic way, we determine the homeomorphism type and concisely show that the diffeomorphism type is not standard.

2

2 The curve configuration

The configuration of curves in $\mathbb{CP}^2$ that we start with is sketched in Figure 2 below. It will consist of two quadrics and four complex lines intersecting in a certain way, and it is derived by studying the configuration in the master thesis of Ta The Ahn [3] where an example from class $\mathcal{A}^4$ was used in an exotic construction.
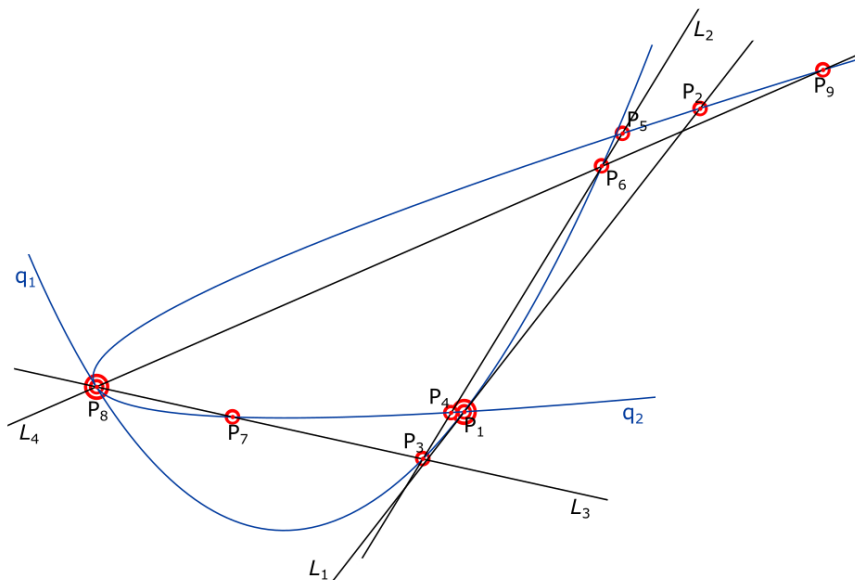

Figure 2: Sketch of the curve configuration

First, take two irreducible quadrics q_1 and q_2 which are tangent at one point and have two more transverse intersections. We give an example of such two quadrics, defined in standard projective coordinates in $\mathbb{CP}^2$ by homogeneous degree 2 equations:

$$z_1^2 + z_2^2 + z_3^2 = 0$$

$$z_1 z_2 + 2\sqrt{2}i \cdot z_2 z_3 + z_1 z_3 = 0$$

Their common tangency is the point $[1 : \frac{\sqrt{2}}{2}i : \frac{\sqrt{2}}{2}i]$ which we further denote by P_8, and the two other intersections are $[-(1+\sqrt{3})\sqrt{2}i : -(2+\sqrt{3}) : 1]$ and $[-(1-\sqrt{3})\sqrt{2}i : -(2-\sqrt{3}) : 1]$. One general way to find two quadrics that intersect this way is by deforming equations of an irreducible quadric and a quadric consisting of a tangent to the irreducible one and a generic line.

After constructing q_1 and q_2, we take the tangent line to q_1 at one of the transverse intersection points with q_2, denote this point by P_1 and line by L_1.

This tangent line intersects q_2 in another point, denote it P_2. Now take a generic line L_2 which intersects q_1 in points we name P_3 and P_6, and intersects q_2 in P_4 and P_5. Denote by L_3 the line passing through P_8 and P_3, and by L_4 the line going through P_8 and P_6. The other intersections of L_3 and L_4 with q_2 are denoted by P_7 and P_9 respectively.

3 Blowing up and the incidence graph

We blow up $\mathbb{CP}^2$ as shown in Figure 2, starting from the point P_1 to P_9. One red circle around a point means one blow up and two circles mean we did two consecutive blow ups completely removing the intersections at the points of tangency. Exceptional curves e_1 and e_2 correspond to the point P_1, e_3 corresponds to P_2, and so on, e_9 and e_{10} correspond to P_8, and e_{11} to P_9.

In the process of blowing up a point, any curve passing through this point can be transformed in a certain way (see e.g. [9, 21]), and the result is called the *proper transform* of the curve. One effect is that proper transforms of the curves which intersect transversely in the point that is blown up, no longer intersect in that point. Another is that the homology class of the proper transform is the homology class of the initial curve minus the class of the exceptional curve. In our example, after the initial 11 blow ups, the homology classes of proper transforms of the curves and their self-intersections are as follows:

$\widetilde{q_1} = q_1 - e_1 - e_2 - e_4 - e_7 - e_9 - e_{10}$	$\widetilde{q_1} \cdot \widetilde{q_1} = -2$
$\widetilde{q_2} = q_2 - e_1 - e_3 - e_5 - e_6 - e_8 - e_9 - e_{10} - e_{11}$	$\widetilde{q_2} \cdot \widetilde{q_2} = -4$
$\widetilde{L_1} = L_1 - e_1 - e_2 - e_3$	$\widetilde{L_1} \cdot \widetilde{L_1} = -2$
$\widetilde{L_2} = L_2 - e_4 - e_5 - e_6 - e_7$	$\widetilde{L_2} \cdot \widetilde{L_2} = -3$
$\widetilde{L_3} = L_3 - e_4 - e_8 - e_9$	$\widetilde{L_3} \cdot \widetilde{L_3} = -2$
$\widetilde{L_4} = L_4 - e_7 - e_9 - e_{11}$	$\widetilde{L_4} \cdot \widetilde{L_4} = -2$

Table 1: Homology classes and self-intersections of curves after 11 blow ups

We can now form the incidence graph of the new configuration by representing curves as vertices, with an edge connecting vertices if there is an intersection between those two curves, as shown in Figure 3.

Two different ways of further blowing up intersection points in this configuration eventually give embedded plumbings from classes $\mathcal{B}^4$ and $\mathcal{C}^4$ of 4-valent graphs from [4], and this is shown in the beginnings of the next two sections. Then we use the fact that these plumbings admit rational blowdown surgeries, and that they can be done in a symplectic way. Finally, we find the homeomorphism types of the resulting manifolds, and prove that they are exotic. The Main Theorem stated in the introduction is comprised of Theorem 1 in section 4 and Theorem 2 in section 5.
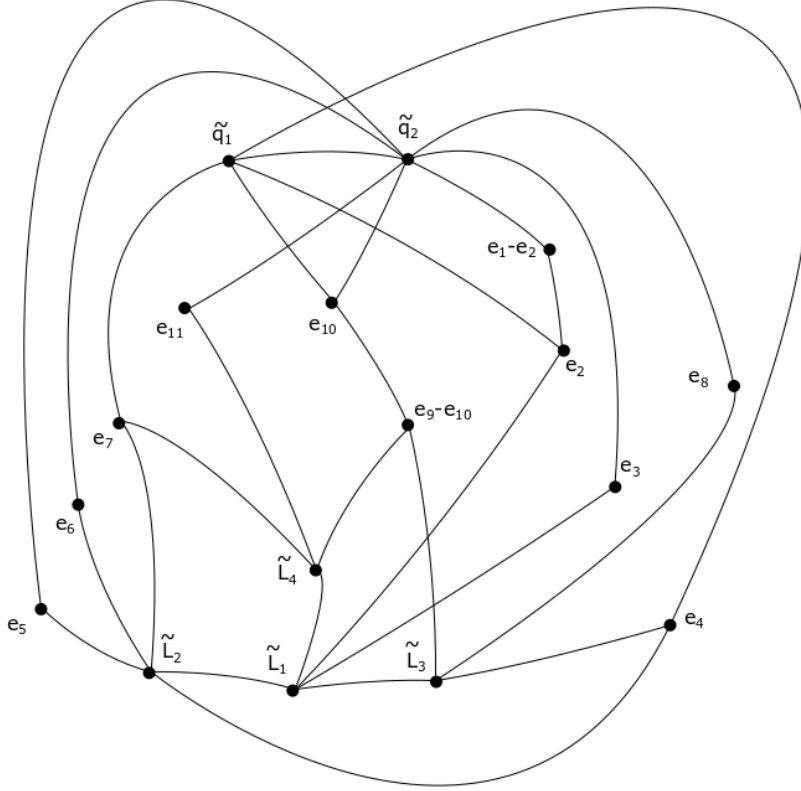
Figure 3: The incidence graph of the curve configuration after 11 blow ups

4   Exotic $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ via a graph from class $\mathcal{B}^4$

Start by Figure 4 where we highlighted nodes and edges which will form the required subgraph. The homology classes of curves at this point are in Table 1. Blowing up the intersection of curves $\widetilde{q}_1$ and e_2, their self-intersections drop to -3 and -2, and we get a new exceptional sphere e_{12}. Doing the same with the intersection between $\widetilde{L}_2$ and e_4, their self-intersections drop to -4 and -2 and we get e_{13}. After three additional blow ups needed to achieve the self-intersections required for the rational blowdown surgery, we arrive to the subgraph shown in Figure 5 which is of type $\mathcal{B}^4$ with $p = 2$ using notation of Figure 1: we can first blow up a generic point of $\widetilde{L}_1$, creating an exceptional curve e_{14}, and then two different generic points of $\widetilde{L}_4$, making two new exceptional curves e_{15} and e_{16}.
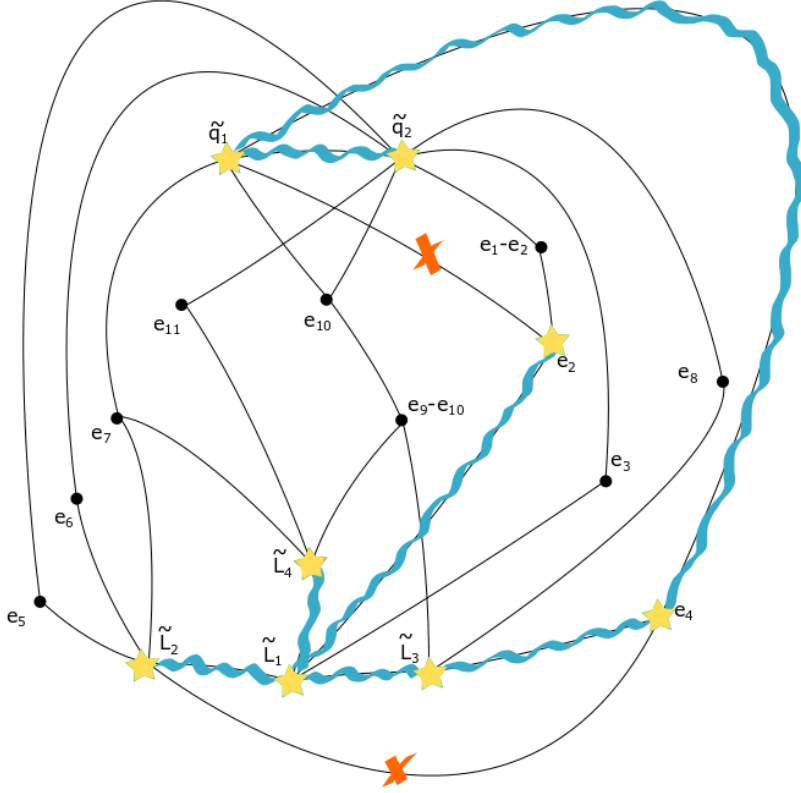
5

Figure 4: Yellow stars are vertices and blue curly lines are edges which form a subgraph from class $\mathcal{B}^4$ presented in Figure 5. Orange X's show which 2 intersections to blow up, whereas some additional blow ups used for adjusting the self-intersections to match the vertex markings in Figure 5 are not visible here but described in the main text.

Denote the final classes by $u_1 = \widetilde{L_2} - e_{13}$, $u_2 = \widetilde{L_1} - e_{14}$, $u_3 = \widetilde{L_4} - e_{15} - e_{16}$, $u_4 = e_2 - e_{12}$, $u_5 = \widetilde{L_3}$, $u_6 = e_4 - e_{13}$, $u_7 = \widetilde{q_1} - e_{12}$ and $u_8 = \widetilde{q_2}$. Therefore, after 16 blow ups, we have the plumbing P from Figure 5 embedded in $\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}$, and the homology classes of plumbing spheres are in Table 2:

$$u_1 = h - e_4 - e_5 - e_6 - e_7 - e_{13}$$
$$u_2 = h - e_1 - e_2 - e_3 - e_{14}$$
$$u_3 = h - e_7 - e_9 - e_{11} - e_{15} - e_{16}$$
$$u_4 = e_2 - e_{12}$$
$$u_5 = h - e_4 - e_8 - e_9$$
$$u_6 = e_4 - e_{13}$$
$$u_7 = 2h - e_1 - e_2 - e_4 - e_7 - e_9 - e_{10} - e_{12}$$
$$u_8 = 2h - e_1 - e_3 - e_5 - e_6 - e_8 - e_9 - e_{10} - e_{11}$$

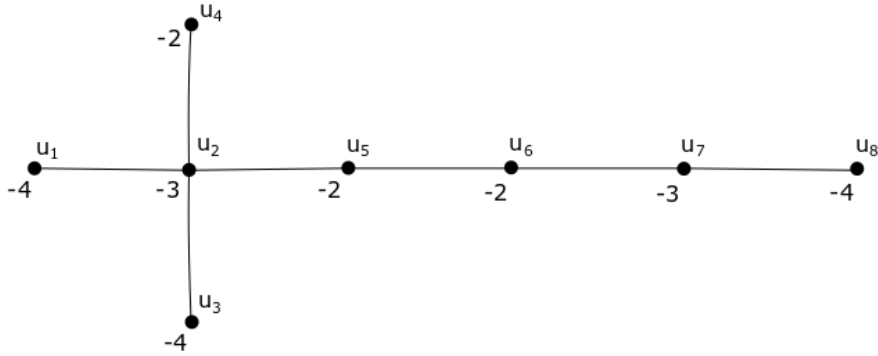Table 2: Homology classes of spheres of the plumbing P

Figure 5: Plumbing graph P from class $\mathcal{B}^4$

As our plumbing is from the class $\mathcal{B}^4$, by [4, Theorem 1.6], we can perform the rational blowdown along P granting:

$$X = (\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} - intP) \cup B$$

where B is the rational homology ball smoothing of the normal surface singularity defined on pp. 1296-1297 of [4] using results of [22].

An important point is that we can assume that the rational blowdown can be performed *symplectically*, which follows from the main result of [16]. First, all the plumbing spheres of P can be assumed to be symplectic submanifolds as proper transforms of complex submanifolds, and second, our plumbing graph is a negative definite tree [4]. Then, from [16, Theorem 1.1], the appropriate neighbourhood of the plumbing can be replaced by B so that (X, ω_X) is symplectic, and denoting $V = \mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} - intP$, there is a symplectomorphism $\phi_V : (V, \omega_X|_V) \longrightarrow (V, \omega|_V)$, where ω is any symplectic structure on $\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}$ that we started with.

Of course, this way we get a well-defined underlying smooth structure on the new manifold X. The main goal of this section is to prove the following:

7

Theorem 1. X *is homeomorphic but not diffeomorphic to* $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$.

Proof. Propositions 1 and 2 in upcoming subsections prove the theorem. $\qquad\square$

4.1 The topology of X

To find the homeomorphism type of X, we use the foundational result of Freedman [8], which along with Donaldson's theorem [5] implies that :

Two smooth simply connected 4-manifolds are homeomorphic if and only if their Euler characteristics, signatures, and parity of the intersection forms are equal.

First we need to prove that X is simply connected, and to do so we will have three standard applications of Van Kampen's theorem. The main part is to prove that for the inclusion $i : \partial P \hookrightarrow \mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} - intP$, the homomorphism i_* induced on fundamental groups is a trivial map.

From [15, Theorem 5.1], the boundary ∂P is a Seifert fibered 3-manifold with a Seifert ivariant $\{0; (1,3),(2,1),(4,1),(4,1),(25,18)\}$. Its fundamental group is described by [10, Theorem 6.1] which implies:

Lemma 1. $\pi_1(\partial P)$ *has a presentation given by generators* $q_0, q_1, q_2, q_3, q_4, h$ *and relations:*

- $q_0 q_1 q_2 q_3 q_4 = 1$

- $[h, q_i] = 1$ *for all* $i = 0, 1, 2, 3, 4$

- $q_0 h^3 = 1$, $q_1^2 h = 1$, $q_2^4 h = 1$, $q_3^4 h = 1$, $q_4^{25} h^{18} = 1$

Furthermore, the classes of q_1, q_2 *and* q_3 *can be chosen to be normal circles to spheres* u_4, u_1 *and* u_3 *respectively.*

Lemma 2. $i_*(\pi_1(\partial P))$ *is trivial.*

Proof. We denoted $V = \mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} - intP$, meaning V is the complement of the plumbing. The normal circle to the sphere u_3 can be contracted along the sphere which intersects it in a single point, and we can choose e_{15} (or e_{16}) and contract that normal circle in V. Therefore, the corresponding generator trivializes through the inclusion, $i_*(q_3) = 1$. Relation $q_3^4 h = 1$ from Lemma 1 gives $i_*(h) = 1$ and then $q_0 h^3 = 1$ implies $i_*(q_0) = 1$.

Looking at Figure 4, we can see that $\widetilde{L_2}$ and $\widetilde{L_4}$ do not intersect each other but intersect the sphere e_7 in one point each, and their proper transforms u_1 and u_3 do the same in the final picture. As e_7 is disjoint from the rest of the plumbing, normal circles to u_1 and u_3, namely q_2 and q_3, can be isotoped in e_7 to bound an annulus in V. Therefore, $i_*(q_2) = i_*(q_3)$, so $i_*(q_2) = 1$ as well.

From $q_0 q_1 q_2 q_3 q_4 = 1$ we are left with $i_*(q_1 q_4) = 1$, which we multiply by $i_*(q_1)$ on the left. Using $i_*(q_1)^2 = 1$ which holds since $q_1^2 h = 1$ and $i_*(h) = 1$, we

get $i_*(q_4) = i_*(q_1)$. So we have $i_*(q_4)^2 = 1$ as well, but by deducing $i_*(q_4)^{25} = 1$ from the last relation in Lemma 1, it follows that $i_*(q_4) = 1$. Finally, $i_*(q_1) = i_*(q_4) = 1$ concludes the result. $\qquad\square$

Lemma 3. *X is simply connected.*

Proof. X is constructed as the union of $V = \mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} - intP$ and some rational homology ball B glued along ∂P. Therefore Van Kampen's theorem gives us a presentation of its fundamental group through fundamental groups of the two pieces.

To determine $\pi_1(V)$ we also apply Van Kampen's theorem, this time to the decomposition $\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} = V \cup P$. The fundamental group of the plumbing P is trivial because it is homotopic to a wedge sum of several spheres. Also, $\pi_1(\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2})$ is trivial for any k because it can be built without 1-handles, so from $\pi_1(\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}) = \pi_1(V) *_{\pi_1(\partial P)} \pi_1(P)$ we get $1 = \pi_1(V)/i_*(\pi_1(\partial P))$. Now Lemma 2 concludes that $\pi_1(V)$ is a trivial group.

We denote the inclusion of the boundary ∂B into the rational homology ball B by $j : \partial B \hookrightarrow B$, and $N := \langle i_*(x) \cdot j_*(x)^{-1} | x \in \pi_1(\partial B) \rangle$. From Van Kampen's theorem and the triviality of $\pi_1(V)$, we have that $\pi_1(X) = \pi_1(V) *_N \pi_1(B) = \pi_1(B)/\langle j_*(x) | x \in \pi_1(\partial B) \rangle$. However, surjectivity of j_* comes from the fact that our rational homology ball was constructed as a complement of a certain (dual) plumbing P' from $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ for some $k > 0$ ([22, section 8.1] and [4, pp. 1296-1297]). More precisely, from another application of Van Kampen's theorem on $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2} = B \cup P'$, we get $1 = \pi_1(B)/\langle j_*(x) | x \in \pi_1(\partial B) \rangle$. Therefore, X is simply connected. $\qquad\square$

Proposition 1. *X is homeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$.*

Proof. To calculate $\chi(X)$ and $\sigma(X)$ we use the formulas:

$$\chi(X) = \chi(\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}) - \chi(P) + \chi(B) = 19 - 9 + 1 = 11$$

$$\sigma(X) = \sigma(\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}) - \sigma(P) + \sigma(B) = -15 - (-8) = -7$$

Rokhlin's theorem [19] implies that if the signature of a smooth simply connected 4-manifold is not divisible by 16, its intersection form must be odd, so this is the case for X. Therefore, the three invariants of X match the corresponding invariants of $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$. As X is simply connected by Lemma 3, it is homeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ as a consequence of Freedman's theorem. $\qquad\square$

4.2 Exoticness of X

To prove that X is not diffeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$, we will use its symplectic structure ω_X explained earlier (coming from [16]), and the following result:

Lemma 4 ([**13**], **Theorem D**). *There is a unique symplectic structure on* $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ *for all* $2 \leq m \leq 9$ *up to diffeomorphism and deformation.*

Remark. *We will slightly abuse notation denoting symplectic forms as their cohomology classes. Poincaré dual of* α *will be denoted by* $PD(\alpha)$.

A symplectic structure Ω on a 4-manifold M determines a contractible family $\mathcal{J}$ of Ω-compatible almost complex structures J on the cotangent bundle T^*M. The first Chern class is the same for all $J \in \mathcal{J}$ and it is called the symplectic canonical class $K_\Omega = c_1(T^*M, J)$.

The strategy of proving that X is exotic is as in [17], to calculate the cup product of the symplectic class and a compatible canonical class on both $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ and X, see that the signs of these products differ, and prove that this is impossible because of the uniqueness result stated in Lemma 4.

Lemma 5 essentially stated as [11, Lemma 5.4] presents a standard symplectic structure on $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ and calculates the sign of the required cup product to be negative. Lemma 6 shows that this product has to be negative for any symplectic structure on $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ or $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, and this is a rather special result for $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ given $2 \leq m \leq 9$. In general, the sign $K_\omega \cdot \omega$ can be used as a smooth invariant on a symplectic manifold only when we know the manifold in question is minimal, and this is called *the Kodaira dimension.*

Lemma 5. *For every* $k > 0$, $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ *admits a symplectic structure* ω *that satisfies* $PD(\omega) = ah - b_1 e_1 - ... - b_k e_k$ *for some positive rational numbers* $a, b_1, ..., b_k$. *For fixed* $a > 0$, b_i*'s can be chosen to be arbitrarily small. The induced canonical class* $K := K_\omega$ *satisfies* $PD(K) = -3h + e_1 + ... + e_k$ *and for small enough* $b_i's$, *we have* $K \cdot \omega < 0$.

Proof. In $\mathbb{CP}^2$, the dual of the cohomology class of ω is ah for some $a > 0$ and we can choose it to be rational - this is because the symplectic area of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is a positive number a and it can be normalized to be rational (we could normalize it so that $a = 1$, but keep "a" to see its importance). The proof of this lemma follows from [14, section 7.1], and more precisely from Theorem 7.1.21 on the existence and properties of the symplectic blow up. Namely, part (v) of that theorem implies that after the blow up, the cohomology class of the symplectic form changes as $\omega_{\tilde{M}} = \omega_M - \pi\lambda^2 PD(e)$. Here e denotes the homology class of the exceptional curve and λ is the radius of the ball removed in the process of the symplectic blow up as explained in [14]. Choosing the ball in Darboux's chart to be as small as needed and $\pi\lambda^2$ rational, and repeating the procedure k times, gives us $PD(\omega) = ah - b_1 e_1 - ... - b_k e_k$ as required.

Formula (7.1.31) in [14] shows the canonical class of the blow up $\tilde{M}$ to be $c_1(T^*\tilde{M}) = c_1(T^*M) + PD(e)$. From the previous and $PD(K_{\mathbb{CP}^2}) = -3h$, we get $PD(K) = -3h + e_1 + ... + e_k$. Finally, $K \cdot \omega = -3a + b_1 + ... + b_k$ is negative for b_i's small enough. $\square$

Lemma 6. *For any symplectic structure $\overline{\omega}$ on $M = \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ for $2 \leq m \leq 9$:*

$$K_{\overline{\omega}} \cdot \overline{\omega} < 0$$

Proof. This result essentially follows from Lemma 4, as $\overline{\omega}$ has to be deformation equivalent to the standard symplectic structure ω, meaning that up to diffeomorphism, there is a path of symplectic forms on M connecting them.

So there is a symplectomorphism $\psi : (M, \omega_M) \longrightarrow (\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}, \omega)$ such that there is a path of symplectic forms ω_t connecting $\omega_0 = \overline{\omega}$ and $\omega_1 = \omega_M$. Naturality of Chern classes gives $K_{\omega_M} = \psi^*(K)$ so $K_{\omega_M} \cdot \omega_M = \psi^*(K) \cdot \psi^*(\omega) = \psi^*(K \cdot \omega) = K \cdot \omega$ so symplectomorphism does not change this product.

Assume that $K_{\overline{\omega}} \cdot \overline{\omega} \geq 0$. Firstly, the canonical class $K_{\overline{\omega}}$ does not change by deformation so $PD(K_{\overline{\omega}}) = -3h + e_1 + ... + e_m$. Now $PD(\overline{\omega}) = a_0 h + a_1 e_1 + ... + a_m e_m$ for some numbers $a_i \in \mathbb{R}$. However, as $\overline{\omega}$ is symplectic, we must have $\overline{\omega} \cdot \overline{\omega} > 0$ so $a_0^2 > \sum_{i=1}^m a_i^2$. Having $K_{\overline{\omega}} \cdot \overline{\omega} = -3a_0 - a_1 - ... - a_m \geq 0$, we get $3a_0 \leq -(\sum_{i=1}^m a_i)$. If $a_0 \leq 0$, from the path of symplectic forms with $PD(\omega_t) = a_0^t h + a_1^t e_1 + ... + a_m^t e_m$, we would have a continuous function a_0^t connecting $a_0^0 = a_0 \leq 0$ and $a_0^1 > 0$ (as $a > 0$ for symplectomorphic ω). Then there would be τ for which $a_0^\tau = 0$ and thus $\omega_\tau \cdot \omega_\tau \leq 0$, which is not possible. Therefore, $a_0 > 0$ and from earlier we have $0 < 3a_0 \leq -(\sum_{i=1}^m a_i)$ so:

$$9a_0^2 \leq (\textstyle\sum_{i=1}^m a_i)^2 \leq m(\textstyle\sum_{i=1}^m a_i^2) \leq 9(\textstyle\sum_{i=1}^m a_i^2) < 9a_0^2$$

provides the required contradiction using the Cauchy–Schwarz inequality. $\square$

Proposition 2. *X is not diffeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$.*

Proof. As mentioned, the strategy is to calculate the cup product of the symplectic class and a compatible canonical class for X, and see that the sign of this product is positive, which proves exoticness of X using Lemma 6.

Let ω denote the symplectic form on $\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}$ provided by Lemma 5, whose Poincaré dual is equal to:

$$PD(\omega) = ah - b_1 e_1 - ... - b_{16} e_{16}$$

and let K denote the corresponding canonical class:

$$PD(K) = -3h + e_1 + ... + e_{16}$$

From the previous two we have:

$$K \cdot \omega = -3a + b_1 + ... + b_{16}$$

The symplectic structure ω_X on X obtained after the rational blow down, was defined earlier in section 4, and it has a compatible symplectic canonical class K_X coming from a generic almost complex structure compatible with ω_X.

To be able to calculate $K_X \cdot \omega_X$, we will decompose the cohomology classes K and ω. Denoting again $V = \mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} - intP$, we have decompositions $\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} = V \cup P$ and $X = V \cup B$.

As a first step, note that the boundary Seifered fibered 3-manifold $\partial P = -\partial B$ is a *rational homology sphere* because $\frac{3}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{18}{25} \neq 0$ (see section 1.2.3 in [20]). To prove it directly, we can calculate $H_1(\partial P; \mathbb{Z})$ from Lemma 1 and see that it is a finite group, which then implies $H^*(\partial P; \mathbb{Q}) = H^*(S^3; \mathbb{Q})$.

From the Mayer-Vietoris sequences for decompositions $\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2} = V \cup P$ and $X = V \cup B$, we get exact sequences:

$$H^1(\partial P; \mathbb{Q}) \longrightarrow H^2(\mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}; \mathbb{Q}) \longrightarrow H^2(V; \mathbb{Q}) \oplus H^2(P; \mathbb{Q}) \longrightarrow H^2(\partial P; \mathbb{Q})$$

$$H^1(\partial B; \mathbb{Q}) \longrightarrow H^2(X; \mathbb{Q}) \longrightarrow H^2(V; \mathbb{Q}) \oplus H^2(B; \mathbb{Q}) \longrightarrow H^2(\partial B; \mathbb{Q})$$

The triviality in $\mathbb{Q}$-cohomology gives $H^1(\partial P; \mathbb{Q}) = 0 = H^2(\partial P; \mathbb{Q})$ and $H^1(\partial B; \mathbb{Q}) = 0 = H^2(\partial B; \mathbb{Q})$, so both middle arrows are isomorphisms. From the first sequence, we can decompose the cohomology classes:

$$K = K|_V + K|_P \text{ and } \omega = \omega|_V + \omega|_P$$

As B is a rational homology 4-ball, $H^2(B; \mathbb{Q}) = 0$ so the second sequence gives that classes K_X and ω_X satisfy:

$$K_X = K_X|_V = \phi_V^*(K|_V) \text{ and } \omega_X = \omega_X|_V = \phi_V^*(\omega|_V)$$

where ϕ_V is the symplectomorphism from the beginning of section 4. So:

$$K_X \cdot \omega_X = \phi_V^*(K|_V) \cdot \phi_V^*(\omega|_V) = \phi_V^*(K|_V \cdot \omega|_V) = K|_V \cdot \omega|_V = K \cdot \omega - K|_P \cdot \omega|_P$$

$$\boxed{K_X \cdot \omega_X = K \cdot \omega - K|_P \cdot \omega|_P}$$

The intersection matrix M of the plumbing P is defined by the intersections $[u_i \cdot u_j]$ as in Figure 5:

$$M = \begin{bmatrix} -4 & 1 & & & & & & \\ 1 & -3 & 1 & 1 & 1 & & & \\ & 1 & -4 & & & & & \\ & 1 & & -2 & & & & \\ & 1 & & & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -3 & 1 \\ & & & & & & 1 & -4 \end{bmatrix}$$

12

Let $\{\gamma_i\}_{i=1}^8$ be the basis of $H^2(P;\mathbb{Q})$ which is dual to the basis $\{u_i\}_{i=1}^8$, meaning $\gamma_i(u_j) = \delta_{ij}$. Then the intersections $[\gamma_i \cdot \gamma_j]$ are given by $[M^{-1}]_{ij}$:

$$M^{-1} = -\frac{1}{512} \cdot \begin{bmatrix} 153 & 100 & 25 & 50 & 72 & 44 & 16 & 4 \\ 100 & 400 & 100 & 200 & 288 & 176 & 64 & 16 \\ 25 & 100 & 153 & 50 & 72 & 44 & 16 & 4 \\ 50 & 200 & 50 & 356 & 144 & 88 & 32 & 8 \\ 72 & 288 & 72 & 144 & 576 & 352 & 128 & 32 \\ 44 & 176 & 44 & 88 & 352 & 528 & 192 & 48 \\ 16 & 64 & 16 & 32 & 128 & 192 & 256 & 64 \\ 4 & 16 & 4 & 8 & 32 & 48 & 64 & 144 \end{bmatrix}$$

From $K|_P = \sum_{i=1}^8 (K|_P(u_i))\gamma_i$, and $K|_P(u_i) = K(u_i) = PD(K) \cdot u_i$, we have $K|_P = \sum_{i=1}^8 (PD(K) \cdot u_i)\gamma_i$. Taking the values of u_i's from Table 2:

$$K|_P = 2\gamma_1 + \gamma_2 + 2\gamma_3 + \gamma_7 + 2\gamma_8$$

Analogously, we get $\omega|_P = \sum_{i=1}^8 (PD(\omega) \cdot u_i)\gamma_i$:

$$\omega|_P = (a - b_4 - b_5 - b_6 - b_7 - b_{13})\gamma_1 + (a - b_1 - b_2 - b_3 - b_{14})\gamma_2 + (a - b_7 - b_9 - b_{11} - b_{15} - b_{16})\gamma_3 + (b_2 - b_{12})\gamma_4 + (a - b_4 - b_8 - b_9)\gamma_5 + (b_4 - b_{13})\gamma_6 + (2a - b_1 - b_2 - b_4 - b_7 - b_9 - b_{10} - b_{12})\gamma_7 + (2a - b_1 - b_3 - b_5 - b_6 - b_8 - b_9 - b_{10} - b_{11})\gamma_8$$

After calculating $K|_P \cdot \omega|_P$, we use $K_X \cdot \omega_X = K \cdot \omega - K|_P \cdot \omega|_P$ to get:

$$K_X \cdot \omega_X = 5.625a - 2.5b_1 - 0.875b_2 - 1.5b_3 - 1.1875b_4 - 0.6875b_5 - 0.6875b_6 - 1.875b_7 - 1.25b_8 - 3.1875b_9 - 0.75b_{10} - 0.6875b_{11} - 0.875b_{12} - 1.1875b_{13} - 0.75b_{14} + 0.0625b_{15} + 0.0625b_{16}$$

We have $K_X \cdot \omega_X > 0$ because a is positive and we can choose b_i's to be arbitrarily small. If X was diffeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$, Lemma 6 would imply $K_X \cdot \omega_X < 0$ so this concludes that X is exotic. $\qquad\square$

5   Exotic $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ via a graph from class $\mathcal{C}^4$

In this section we construct a different plumbing from the one in section 4, again starting with the construction in section 3. We keep the notation of some auxiliary objects as in the previous sections to simplify the exposition. Apart from the construction of the plumbing, all calculations are similar so we only emphasize the differences.

Starting from the incidence graph in Figure 3, in Figure 6 we highlight nodes and edges which will form the required subgraph from $\mathcal{C}^4$.
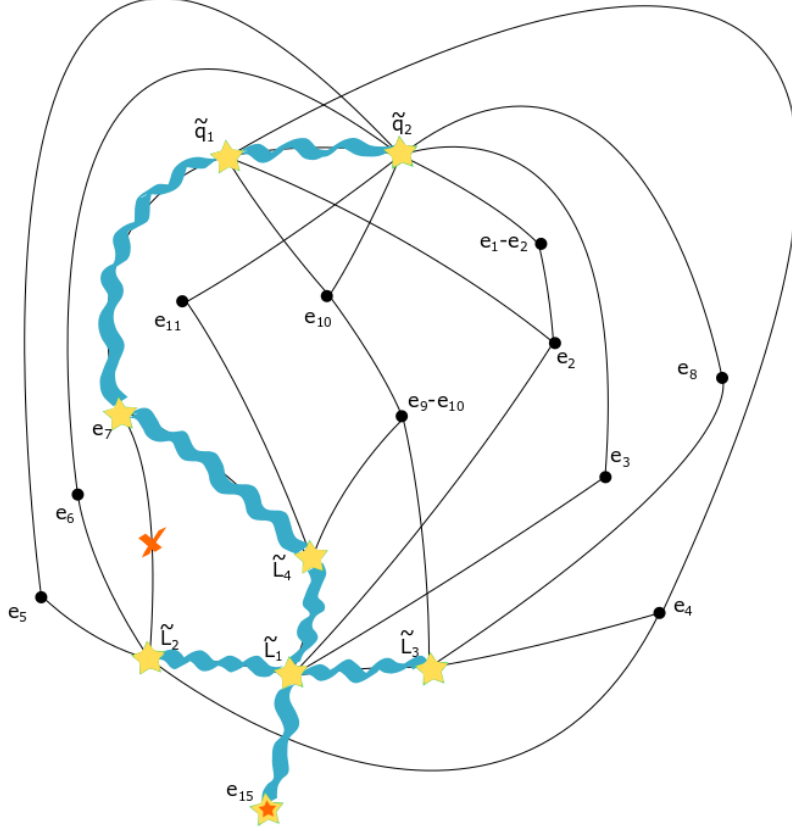
Figure 6: Yellow stars are vertices and blue curly lines are edges which form a subgraph from class $\mathcal{C}^4$ presented in Figure 7. Note that e_{15} is a new vertex compared to the starting Figure 3, marked with a smaller orange star because it comes from a new blow up. To arrive to an embedding, orange X shows which intersection to blow up. Some additional blow ups used for adjusting the self-intersections to match the vertex markings in Figure 7 are not visible here but are described in the main text.

We first blow up the intersection between e_7 and $\widetilde{L_2}$ and denote the exceptional curve by e_{12}. This way the proper transform of $\widetilde{L_2}$ gets self-intersection -4. With two further blow ups of different generic points of $\widetilde{L_2}$, we transform it into a curve of self-intersection -6, getting curves e_{13} and e_{14} in the process. Then blow up a generic point of the curve $\widetilde{L_1}$ getting e_{15}, and setting the self-intersection of the proper transform of $\widetilde{L_1}$ to -3. Now blow up a generic point of e_{15}, allowing its self-intersection to drop to -2, and name the exceptional

curve e_{16}. Lastly, blow up a generic point of $\widetilde{L_3}$ dropping its self-intersection to -3 via the curve e_{17}.

Denote the classes by $v_1 = e_{15} - e_{16}$, $v_2 = \widetilde{L_1} - e_{15}$, $v_3 = \widetilde{L_3} - e_{17}$, $v_4 = \widetilde{L_2} - e_{12} - e_{13} - e_{14}$, $v_5 = \widetilde{L_4}$, $v_6 = e_7 - e_{12}$, $v_7 = \widetilde{q_1}$ and $v_8 = \widetilde{q_2}$. These curves form the plumbing Q embedded in $\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2}$, and its graph is presented in Figure 7. Therefore, the homology classes of spheres in the plumbing Q are:

$$
\begin{aligned}
v_1 &= e_{15} - e_{16} \\
v_2 &= h - e_1 - e_2 - e_3 - e_{15} \\
v_3 &= h - e_4 - e_8 - e_9 - e_{17} \\
v_4 &= h - e_4 - e_5 - e_6 - e_7 - e_{12} - e_{13} - e_{14} \\
v_5 &= h - e_7 - e_9 - e_{11} \\
v_6 &= e_7 - e_{12} \\
v_7 &= 2h - e_1 - e_2 - e_4 - e_7 - e_9 - e_{10} \\
v_8 &= 2h - e_1 - e_3 - e_5 - e_6 - e_8 - e_9 - e_{10} - e_{11}
\end{aligned}
$$

Table 3: Homology classes of spheres of the plumbing Q
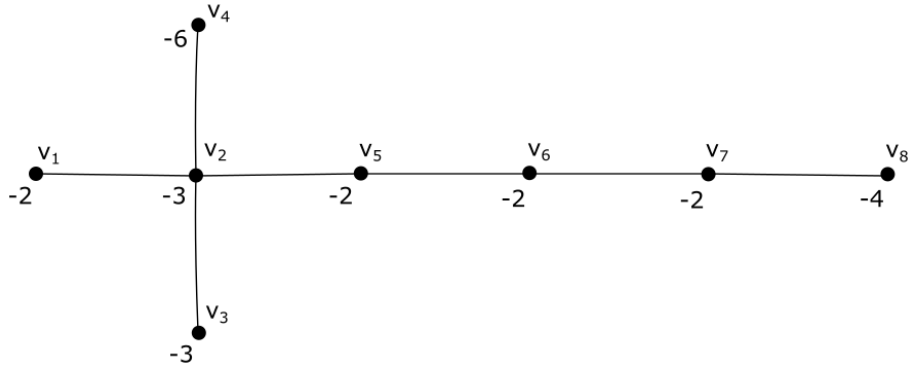
Figure 7: Plumbing graph Q from class $\mathcal{C}^4$

We can rationally blow down Q by [4] and get the manifold:

$$Y = (\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2} - intQ) \cup D$$

where D is a different rational homology ball than the one from section 4. Details are very similar to the ones in the previous section and we only emphasize the differences, showing this time:

Theorem 2. Y *is homeomorphic but not diffeomorphic to* $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$.

Proof. Propositions 3 and 4 together will complete the proof. $\qquad\square$

5.1 The topology of Y

In this example, the boundary ∂Q is a Seifert fibered 3-manifold [15] with Seifert ivariant $\{0; (1,3), (6,1), (3,1), (2,1), (13,10)\}$. Analagously to Lemma 1, by [10] we have:

Lemma 7. *$\pi_1(\partial Q)$ has a presentation given by generators $q_0, q_1, q_2, q_3, q_4, h$ and relations:*

- $q_0 q_1 q_2 q_3 q_4 = 1$

- $[h, q_i] = 1$ *for all* $i = 0, 1, 2, 3, 4$

- $q_0 h^3 = 1$, $q_1^6 h = 1$, $q_2^3 h = 1$, $q_3^2 h = 1$, $q_4^{13} h^{10} = 1$

Furthermore, the classes of q_1, q_2 and q_3 can be chosen to be normal circles to spheres v_4, v_3 and v_1 respectively.

Lemma 8. *$i_*(\pi_1(\partial Q))$ is trivial.*

Proof. In this case, compared to the previous section, it is easier to deduce the triviality of $i_*(\pi_1(\partial Q))$, as we made a lot of generic blow ups. More precisely, each of the three leaves of the plumbing graph Q in Figure 7, that is v_4, v_3 and v_1, is intersecting a different exceptional sphere otherwise disjoint from the plumbing. As in the proof of Lemma 2, the normal circles can be contracted in the complement of Q, so we can deduce $i_*(q_1) = 1$, $i_*(q_2) = 1$ and $i_*(q_3) = 1$. From $q_1^6 h = 1$, we get $i_*(h) = 1$ and then $q_0 h^3 = 1$ implies $i_*(q_0) = 1$. The first relation of Lemma 7 now gives $i_*(q_4) = 1$ and concludes that $i_*(\pi_1(\partial Q))$ is a trivial group. $\square$

Lemma 9. *Y is simply connected.*

Proof. Using Lemma 8 instead of Lemma 2, the proof is analogous to the proof of Lemma 3. $\square$

Proposition 3. *Y is homeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$.*

Proof. As before we have:

$$\chi(Y) = \chi(\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2}) - \chi(Q) + \chi(D) = 20 - 9 + 1 = 12$$

$$\sigma(Y) = \sigma(\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2}) - \sigma(Q) + \sigma(D) = -16 - (-8) = -8$$

Y has an odd intersection form by Rohlkin's theorem [19] and thus, all the invariants match the ones of $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. From Lemma 9, these 4-manifolds are both simply connected, so by Freedman's theorem we get that they must be homeomorphic. $\square$

5.2 Exoticness of Y

Proposition 4. Y *is not diffeomorphic to* $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$.

Proof. The proof is essentially the same as the proof of Proposition 2. Start by introducing a symplectic form on $\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2}$ using Lemma 5:

$$PD(\omega) = ah - b_1 e_1 - ... - b_{17} e_{17}$$

This time, let K be the standard canonical class of $\mathbb{CP}^2 \# 17\overline{\mathbb{CP}^2}$:

$$PD(K) = -3h + e_1 + ... + e_{17}$$

From these two we have:

$$K \cdot \omega = -3a + b_1 + ... + b_{17}$$

The intersection matrix of the plumbing Q is $[v_i \cdot v_j]$:

$$N = \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -3 & 1 & 1 & 1 & & & \\ & 1 & -3 & & & & & \\ & 1 & & -6 & & & & \\ & 1 & & & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -4 \end{bmatrix}$$

The intersection matrix of the basis $\{\gamma_i\}_{i=1}^8$ dual to $\{v_i\}_{i=1}^8$ is:

$$N^{-1} = -\frac{1}{576} \cdot \begin{bmatrix} 405 & 234 & 78 & 39 & 180 & 126 & 72 & 18 \\ 234 & 468 & 156 & 78 & 360 & 252 & 144 & 36 \\ 78 & 156 & 244 & 26 & 120 & 84 & 48 & 12 \\ 39 & 78 & 26 & 109 & 60 & 42 & 24 & 6 \\ 180 & 360 & 120 & 60 & 720 & 504 & 288 & 72 \\ 126 & 252 & 84 & 42 & 504 & 756 & 432 & 108 \\ 72 & 144 & 48 & 24 & 288 & 432 & 576 & 144 \\ 18 & 36 & 12 & 6 & 72 & 108 & 144 & 180 \end{bmatrix}$$

To calculate $K_Y \cdot \omega_Y$, we can aquire $K|_Q$ and $\omega|_Q$ decomposing the second cohomology classes as before. Again, this is possible because the boundary manifold ∂Q is Seifert fibered and $\frac{3}{1} + \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{10}{13} \neq 0$, so it is a rational homology sphere (see [20]). $K|_Q = \sum_{i=1}^8 (PD(K) \cdot v_i)\gamma_i$ so using the values of $PD(K)$ and v_i's from Table 3:

$$K|_Q = \gamma_2 + \gamma_3 + 4\gamma_4 + 2\gamma_8$$

A similar formula $\omega|_Q = \sum_{i=1}^{8}(PD(\omega)\cdot v_i)\gamma_i$ gives:

$\omega|_Q = (b_{15} - b_{16})\gamma_1 + (a - b_1 - b_2 - b_3 - b_{15})\gamma_2 + (a - b_4 - b_8 - b_9 - b_{17})\gamma_3 + (a - b_4 - b_5 - b_6 - b_7 - b_{12} - b_{13} - b_{14})\gamma_4 + (a - b_7 - b_9 - b_{11})\gamma_5 + (b_7 - b_{12})\gamma_6 + (2a - b_1 - b_2 - b_4 - b_7 - b_9 - b_{10})\gamma_7 + (2a - b_1 - b_3 - b_5 - b_6 - b_8 - b_9 - b_{10} - b_{11})\gamma_8$

And once again, from $K_Y \cdot \omega_Y = K \cdot \omega - K|_Q \cdot \omega|_Q$:

$K_Y \cdot \omega_Y = 5.625a - 2.5b_1 - 1.75b_2 - 1.5b_3 - 1.875b_4 - 0.708\overline{3}b_5 - 0.708\overline{3}b_6 - 1.208\overline{3}b_7 - 0.\overline{6}b_8 - 3.1\overline{6}b_9 - 0.69791\overline{6}b_{10} - 1.25b_{11} - 1.208\overline{3}b_{12} + 0.041\overline{6}b_{13} + 0.041\overline{6}b_{14} + 0.125b_{15} + 0.125b_{16} + 0.08\overline{3}b_{17}$

$K_Y \cdot \omega_Y > 0$ because a is positive and b_i's can be arbitrarily small. By Lemma 6, this is impossible unless Y is exotic. $\qquad\square$

Remark. *Finding interesting configurations of lines and quadrics could produce even smaller exotic 4-manifolds via suitable rational blowdowns, so this is one upcoming challenge. It seems that the exoticness proof will remain true if enough curves from the initial configuration are used in the plumbing, so it would only remain to take care of simple connectedness.*

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