

CRITICAL ELLIPTIC EQUATIONS ON NON-COMPACT FINSLER MANIFOLDS

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Abstract: In the present paper we deal with a quasilinear elliptic equation involving a critical Sobolev exponent on non-compact Finsler manifolds, i.e. on Randers spaces. Under very general assumptions on the perturbation we prove the existence of a non-trivial solution. The approach is based on the direct methods of calculus of variations. One of the key step is to prove that the energy functional associated with the problem is weakly lower semicontinuous on small balls of the Sobolev space, which is provided by a general inequality. At the end, we prove Hardy-type inequalities on Finsler manifolds as an application of this inequality.

Keywords: Critical problems; non-compact Finsler manifolds; Randers spaces; Compact embedding; Hardy-type inequalities.

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1. INTRODUCTION AND MAIN RESULT

Since the pioneering work of Brezis and Nirenberg ([BN83]) a lot of attention has been paid to the following problem:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u, & \text{in } \Omega \subset \mathbb{R}^d, \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (\mathcal{P})$$

where Ω is a bounded domain of \mathbb{R}^d , and $p^* = \frac{pd}{d-p}$ is the critical Sobolev exponent.

In [BN83] the authors studied the case $p = 2$ and they proved that if $\lambda_1(\Omega)$ is the first eigenvalue for $-\Delta$ with Dirichlet boundary conditions, then, if $N \geq 4$ for every $\lambda \in (0, \lambda_1(\Omega))$ there exists a positive solution, if $N = 3$ and Ω is a ball, then, a solution exists if and only if $\lambda \in (\frac{\lambda_1(\Omega)}{4}, \lambda_1(\Omega))$. The proof is based on a *local Palais Smale* condition and, accordingly, on the construction of minimax levels for the energy functional associated with the problem (\mathcal{P}) in suitable intervals.

Zou in [Zou12] proved that when

$$d \geq d(p) := [p^2] - [[p^2] - p^2],$$

then, problem (\mathcal{P}) has a positive solution if and only if $\lambda \in (0, \lambda_1)$ (λ_1 being the first eigenvalue for $-\Delta_p$ with Dirichlet boundary conditions). When $d < d(p)$, and Ω is a ball, then (\mathcal{P}) has no solution if $\lambda \geq \lambda_1$, has a solution for $\lambda \in (\lambda^*, \lambda_1)$, no solution for $\lambda \in (0, \lambda_*)$ (for convenient $0 < \lambda_* \leq \lambda^* < \lambda_1$).

In the literature most of the papers dealing with critical elliptic equations, prove a *local Palais-Smale condition*, which relies on the well known *concentration compactness principle*, see P.L. Lions in [Lio82], which is one of the most powerful tools in the case of lack of compactness. In this connection we mention the result of Chabrowski ([Cha95]) where the author applies concentration compactness principle to a non-homogeneous problem with non-constant coefficients.

Recently in [FF15] the authors proposed an alternative method to the problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + g(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (\mathcal{P})$$

where Ω is a bounded domain, by employing the direct methods of calculus of variations. Indeed, they proved that the energy functional \mathcal{E} associated with the problem is locally sequentially weakly lower semicontinuous, and with direct, simple arguments they prove that \mathcal{E} has a local minimum, which is a weak solution of the problem (\mathcal{P}) . Later, the idea of this paper was successfully applied in different context, see [MMB17, MBV20].

In the light of the above papers, the aim of the present manuscript is to study critical elliptic equation on the d -dimensional non-compact Finsler manifolds, i.e. we consider the following problem

$$\begin{cases} -\Delta_{p,F} u + |u|^{p^*-2}u = \lambda |u|^{p^*-2}u + \mu \alpha(x)h(u), & \text{in } M, \\ u \in W_F^{1,p}(M) \end{cases} \quad (\mathcal{P}_{\lambda,\mu})$$

where M is a d -dimensional Randers space endowed with the Finsler metric defined as

$$F(x, y) = \sqrt{g_x(y, y)} + \beta_x(y), \quad (x, y) \in TM, \quad (1.1)$$

where g is a Riemannian metric and β_x is a 1-form on M . $\Delta_{p,F}$ is the *Finsler p -Laplace operator*, $h : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $p^* = \frac{pd}{d-p}$ is the critical Sobolev exponent.

It is not difficult to see that the above equation is closely related to the well-known *Yamabe problem*; that is, for any smooth compact Riemannian manifold (M, g) of dimension $d \geq 3$, there exists a conformal metric \tilde{g} to g with constant scalar curvature, see Hebey [Heb99]. This problem can be transposed into an elliptic equation which involves the critical Sobolev exponent,

$$-\Delta_g u + c(d)R_g u = Ku^{2^*-1}, \quad \text{on } M, \quad (1.2)$$

where Δ_g is the Laplace-Beltrami operator associated with g , R_g is the scalar curvature of g , $c(d) = \frac{d-2}{4(d-1)}$, and K is a constant satisfying $K = c(d)R_{\tilde{g}}$, where $R_{\tilde{g}}$ is the scalar curvature of \tilde{g} . The literature of such elliptic equations related to Yamabe's problem is quite rich, see [Aub76, Aub82, DHR04] and reference therein.

Yau [Yau82] and later Kazdan [Kaz85] (see also [Aub01]) suggested the study of (1.2) in a non-compact Riemannian manifold. In this case the situation changes dramatically, see Jin [Jin88]; it is possible to exhibit examples of complete non-compact manifolds for which the Yamabe problem does not have any solution. Thus as one expects, the curvature plays a crucial role in the study of Yamabe problem, in this connection we mention the work of Aviles and McOwen [AM88] the authors have established some existence result

for the Yamabe equation in the case of non-compact complete Riemannian manifold with non-positive scalar curvature.

In the case of non-compact Finsler manifolds the study of critical elliptic equation is much more delicate. First, it turns out that if M is a non-compact Finsler-Hadamard manifold with infinite reversibility constant, then the Sobolev space $W_F^{1,2}(M)$ is not necessary a vector space see Farkas, Kristály, Varga [FKV15]. On the other hand by Farkas, Kristály, Mester [FKM20], on non-compact Finsler manifolds the continuous Morrey-type embedding do not necessarily hold, thus one can expect similar pathological phenomena in the case of the embedding $W_F^{1,p}(M) \hookrightarrow L^q(M)$, $q \in (p, p^*)$.

Thus beside the aforementioned issues, the main obstacle in dealing with existence and multiplicity results for quasilinear problems with critical nonlinearity is represented the weakly lower semicontinuity of the energy functional associated with the problem $(\mathcal{P}_{\lambda,\mu})$. Thus

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_M (F^{*p}(x, Du) + |u|^p) dV_F(x) - \frac{\lambda}{p^*} \int_M |u|^{p^*} dV_F(x) - \mu \int_M \alpha(x) H(u) dV_F(x),$$

is not sequentially weakly lower semicontinuous in $W_F^{1,p}(M)$ and does not satisfy the well known Palais Smale condition. However, $E_{\lambda,\mu}$ is of class C^1 in $W_F^{1,p}(M)$ and its critical points turn out to be the weak solutions of problem $(\mathcal{P}_{\lambda,\mu})$.

In the sequel we state our main result, in order to do that, we need some notations and assumptions. Let (M, F) be a d -dimensional Randers space endowed with the Finsler metric (1.1), such that (M, g) is a complete non-compact Riemannian manifold of bounded geometry, i.e. with Ricci curvature bounded from below, i.e. $\text{Ric}_{(M,g)} \geq k(d-1)$, having positive injectivity radius. For further use, let $\|\beta\|_g(x) := \sqrt{g_x^*(\beta_x, \beta_x)}$ for every $x \in M$, where g^* is the co-metric of g .

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.

(H): For each $s \in \mathbb{R}$, put $H(s) = \int_0^s h(t) dt$, and we further assume that:

(h₁) there exist $q \in (1, p)$ and $c > 0$ such that $|h(t)| \leq C_h (|t|^{q-1} + |t|^{p-1})$, $\forall t \in \mathbb{R}$;

(h₂) $\liminf_{s \rightarrow 0} \frac{H(s)}{|s|^p} = +\infty$.

On the function α we assume that:

(α): $\alpha : [0, +\infty) \rightarrow \mathbb{R}$ be a non-increasing, non-negative continuous function which depends on $d_F(x_0, \cdot)$, for $x_0 \in M$, i.e. $\alpha(x) = \alpha_0(d_F(x_0, x))$ and assume that

$$\alpha_0(s) \sinh \left(k \frac{s}{1-a} \right)^{d-1} \sim \frac{1}{s^\gamma},$$

for some $\gamma > 1$, whenever $s \rightarrow \infty$.

Theorem 1.1. *Let (M, F) be a d -dimensional Randers space endowed with the Finsler metric (1.1) such that $\sup_{x \in M} \|\beta\|_g(x) < 1$ and (M, g) is a complete non-compact Riemannian manifold of bounded geometry. Let G be a coercive, compact connected subgroup of $\text{Isom}_F(M)$ such that $\text{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function verifying **(H)**, and let $\alpha : [0, +\infty) \rightarrow \mathbb{R}$ be a continuous function which satisfies assumption **(α)**. Then for every $\mu > 0$, there exists $\lambda_* > 0$ such that for every $\lambda < \lambda_*$ the problem $(\mathcal{P}_{\lambda,\mu})$ has a non-zero G -invariant weak solution.*

The organization of the paper is the following. After presenting some preliminary results in Riemannian geometry, In Section 2 we present some preliminary results and notions from Riemann and Finsler geometry, while Section 3 is devoted to the proof of Theorem 1.1. At the end, in Section 4, we shall present some concluding remarks.

2. MATHEMATICAL BACKGROUND

2.1. Elements from Riemannian geometry. Let (M, g) be a homogeneous complete non-compact Riemannian manifold with $\dim M = d$. Let $T_x M$ be the tangent space at $x \in M$, $TM = \bigcup_{x \in M} T_x M$ be the tangent bundle, and $d_g : M \times M \rightarrow [0, +\infty)$ be the distance function associated to the Riemannian metric g . Let $B_g(x, \rho) = \{y \in M : d_g(x, y) < \rho\}$ be the open metric ball with center x and radius $\rho > 0$. If dv_g is the canonical volume element on (M, g) , the volume of a bounded open set $\Omega \subset M$ is $\text{Vol}_g(\Omega) = \int_{\Omega} dv_g = \mathcal{H}^d(\Omega)$. If $d\sigma_g$ denotes the $(d-1)$ -dimensional Riemannian measure induced on $\partial\Omega$ by g , then

$$\text{Area}_g(\partial\Omega) = \int_{\partial\Omega} d\sigma_g = \mathcal{H}^{d-1}(\partial\Omega)$$

stands for the area of $\partial\Omega$ with respect to the metric g .

For every $p > 1$, the norm of $L^p(M)$ is given by

$$\|u\|_{L^p(M)} = \left(\int_M |u|^p dv_g \right)^{1/p}.$$

Let $u : M \rightarrow \mathbb{R}$ be a function of class C^1 . If (x^i) denotes the local coordinate system on a coordinate neighbourhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x^i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij} u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has the eikonal equation

$$|\nabla_g d_g(x_0, \cdot)| = 1 \text{ a.e. on } M. \quad (2.1)$$

When no confusion arises, if $X, Y \in T_x M$, we simply write $|X|$ and $\langle X, Y \rangle$ instead of the norm $|X|_x$ and inner product $g_x(X, Y) = \langle X, Y \rangle_x$, respectively.

The $L^p(M)$ norm of $\nabla_g u : M \rightarrow TM$ is given by

$$\|\nabla_g u\|_{L^p(M)} = \left(\int_M |\nabla_g u|^p dv_g \right)^{\frac{1}{p}}.$$

The space $W_g^{1,p}(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|_{W_g^{1,p}(M)}^p = \|u\|_{L^p(M)}^p + \|\nabla_g u\|_{L^p(M)}^p.$$

2.2. Elements from Finsler geometry. Let M be a connected n -dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_x M$ is its tangent bundle, the pair (M, F) is a Finsler manifold if the continuous function $F : TM \rightarrow [0, \infty)$ satisfies the conditions:

- $F \in C^\infty(TM \setminus \{0\})$;
- $F(x, ty) = tF(x, y)$ for all $t \geq 0$ and $(x, y) \in TM$;
- $g_{ij}(x, y) := [\frac{1}{2}F^2(x, y)]_{y^i y^j}$ is positive definite for all $(x, y) \in TM \setminus \{0\}$.

If $F(x, ty) = |t|F(x, y)$ for all $t \in \mathbb{R}$ and $(x, y) \in TM$, we say that the Finsler manifold (M, F) is reversible. Clearly, the Randers metric F is symmetric, i.e. $F(x, -y) = F(x, y)$ for every $(x, y) \in TM$, if and only if $\beta = 0$ (which means that $(M, F) = (M, g)$ is the original Riemannian manifold).

Let $\pi^* TM$ be the pull-back bundle of the tangent bundle TM generated by the natural projection $\pi : TM \setminus \{0\} \rightarrow M$, see Bao, Chern and Shen [BCS00, p. 28]. The vectors of the pull-back bundle $\pi^* TM$ are denoted by $(v; w)$ with $(x, y) = v \in TM \setminus \{0\}$ and $w \in T_x M$. For simplicity, let $\partial_i|_v = (v; \partial/\partial x^i|_x)$ be the natural local basis for $\pi^* TM$, where $v \in T_x M$. One can introduce on $\pi^* TM$ the *fundamental tensor* \mathfrak{g} by

$$\mathfrak{g}_v := \mathfrak{g}(\partial_i|_v, \partial_j|_v) = \mathfrak{g}_{ij}(x, y) \quad (2.2)$$

where $v = y^i(\partial/\partial x^i)|_x$.

Let $u, v \in T_x M$ be two non-collinear vectors and $\mathcal{S} = \text{span}\{u, v\} \subset T_x M$. By means of the curvature tensor R , the *flag curvature* of the flag $\{\mathcal{S}, v\}$ is defined by

$$\mathbf{K}(\mathcal{S}; v) = \frac{\mathbf{g}_v(R(U, V)V, U)}{\mathbf{g}_v(V, V)\mathbf{g}_v(U, U) - \mathbf{g}_v(U, V)^2}, \quad (2.3)$$

where $U = (v; u), V = (v; v) \in \pi^* TM$. If (M, F) is Riemannian, the flag curvature reduces to the well known sectional curvature. If $\mathbf{K}(\mathcal{S}; v) \leq 0$ for every choice of U and V , we say that (M, F) has *non-positive flag curvature*, and we denote by $\mathbf{K} \leq 0$. (M, F) is a *Finsler-Hadamard manifold* if it is simply connected, forward complete with $\mathbf{K} \leq 0$.

Let $\{e_i\}_{i=1, \dots, n}$ be a basis for $T_x M$ and $\mathbf{g}_{ij}^v = \mathbf{g}_v(e_i, e_j)$. The *mean distortion* $\zeta : TM \setminus \{0\} \rightarrow (0, \infty)$ is defined by $\zeta(v) = \frac{\sqrt{\det(\mathbf{g}_{ij}^v)}}{\sigma_F}$. The *mean covariation* $\mathbf{S} : TM \setminus \{0\} \rightarrow \mathbb{R}$ is defined by

$$\mathbf{S}(x, v) = \frac{d}{dt}(\ln \zeta(\dot{\sigma}_v(t)))|_{t=0},$$

where σ_v is the geodesic such that $\sigma_v(0) = x$ and $\dot{\sigma}_v(0) = v$. We say that (M, F) has *vanishing mean covariation* if $\mathbf{S}(x, v) = 0$ for every $(x, v) \in TM$, and we denote by $\mathbf{S} = 0$.

Let $\sigma : [0, r] \rightarrow M$ be a piecewise C^∞ curve. The value $L_F(\sigma) = \int_0^r F(\sigma(t), \dot{\sigma}(t)) dt$ denotes the *integral length* of σ . For $x_1, x_2 \in M$, denote by $\Lambda(x_1, x_2)$ the set of all piecewise C^∞ curves $\sigma : [0, r] \rightarrow M$ such that $\sigma(0) = x_1$ and $\sigma(r) = x_2$. Define the *distance function* $d_F : M \times M \rightarrow [0, \infty)$ by

$$d_F(x_1, x_2) = \inf_{\sigma \in \Lambda(x_1, x_2)} L_F(\sigma). \quad (2.4)$$

One clearly has that $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, and that d_F verifies the triangle inequality.

The *Hausdorff volume form* dV_F on the Randers space (M, F) is given by

$$dV_F(x) = (1 - \|\beta\|_g^2(x))^{\frac{d+1}{2}} dv_g, \quad (2.5)$$

where dv_g denotes the canonical Riemannian volume form induced by g on M .

For every $(x, \alpha) \in T^*M$, the *polar transform* (or, co-metric) of F from (1.1) is

$$F^*(x, \alpha) = \sup_{y \in T_x M \setminus \{0\}} \frac{\alpha(y)}{F(x, y)} = \frac{\sqrt{g_x^{*2}(\alpha, \beta) + (1 - \|\beta\|_g^2(x))\|\alpha\|_g^2(x)} - g_x^*(\alpha, \beta)}{1 - \|\beta\|_g^2(x)}. \quad (2.6)$$

Let $u : M \rightarrow \mathbb{R}$ be a differentiable function in the distributional sense. The *gradient* of u is defined by

$$\nabla_F u(x) = J^*(x, Du(x)), \quad (2.7)$$

where $Du(x) \in T_x^* M$ denotes the (distributional) *derivative* of u at $x \in M$ and J^* is the Legendre transform given by

$$J^*(x, y) := \frac{\partial}{\partial y} \left(\frac{1}{2} F^{*2}(x, y) \right).$$

In local coordinates, one has

$$Du(x) = \sum_{i=1}^n \frac{\partial u}{\partial x^i}(x) dx^i, \quad \nabla_F u(x) = \sum_{i,j=1}^n h_{ij}^*(x, Du(x)) \frac{\partial u}{\partial x^i}(x) \frac{\partial}{\partial x^j}. \quad (2.8)$$

In general, note that $u \mapsto \nabla_F u$ is not linear. If $x_0 \in M$ is fixed, then due to [OS09], for a.e. $x \in M$ one has

$$F^*(x, Dd_F(x_0, x)) = F(x, \nabla_F d_F(x_0, x)) = Dd_F(x_0, x)(\nabla_F d_F(x_0, x)) = 1 \quad (2.9)$$

Let $p > 1$. The norm of $L^p(M)$ is given by

$$\|u\|_p = \left(\int_M |u|^p dV_F(x) \right)^{1/p}.$$

Let X be a vector field on M . In a local coordinate system (x^i) the *divergence* is defined by $\operatorname{div}(X) = \frac{1}{\sigma_F} \frac{\partial}{\partial x^i} (\sigma_F X^i)$, where

$$\sigma_F(x) = \frac{\omega_d}{\operatorname{Vol}(\{y = (y^i) : F(x, y^i \frac{\partial}{\partial x^i}) < 1\})}.$$

The Finsler p -Laplace operator is defined by

$$\Delta_{F,p} u = \operatorname{div}(F^{*p-2}(Du) \cdot \nabla_F u),$$

while the Green theorem reads as: for every $v \in C_0^\infty(M)$,

$$\int_M v \Delta_{F,p} u dV_F(x) = - \int_M F^{*p-2}(Du) Dv(\nabla_F u) dV_F(x), \quad (2.10)$$

see [OS09] and [She01] for $p = 2$. Note that in general $\Delta_{F,p}(-u) \neq -\Delta_{F,p}u$. When $(M, F) = (M, g)$, the Finsler-Laplace operator is the usual Laplace-Beltrami operator,

We introduce the Sobolev space associated with (M, F) , namely let

$$W_F^{1,p}(M) = \left\{ u \in W_{\operatorname{loc}}^{1,p}(M) : \int_M F^{*p}(x, Du(x)) dV_F(x) < +\infty \right\}$$

be the closure of $C^\infty(M)$ with respect to the (asymmetric) norm

$$\|u\|_F = \left(\int_M F^{*p}(x, Du(x)) dV_F(x) + \int_M |u(x)|^p dV_F(x) \right)^{\frac{1}{p}}.$$

We notice that the *reversibility constant* associated with F (see (1.1)) is given by

$$r_F = \sup_{x \in M} r_F(x) \quad \text{where} \quad r_F(x) = \sup_{y \in T_x M \setminus \{0\}} \frac{F(x, y)}{F(x, -y)} = \frac{1 + \|\beta\|_g(x)}{1 - \|\beta\|_g(x)}, \quad (2.11)$$

see [Rad04, ZY13]. Note that $r_F \geq 1$ (possibly, $r_F = +\infty$), and $r_F = 1$ if and only if (M, F) is Riemannian. Analogously, the *uniformity constant* of F is defined by the number

$$l_F = \inf_{x \in M} l_F(x) \quad \text{where} \quad l_F(x) = \inf_{y, v, w \in T_x M \setminus \{0\}} \frac{g_{(x,v)}(y, y)}{g_{(x,w)}(y, y)} = \left(\frac{1 - \|\beta\|_g(x)}{1 + \|\beta\|_g(x)} \right)^2, \quad (2.12)$$

and measures how far F and F^* are from Riemannian structures, see [Egl97]. Note that $l_F \in [0, 1]$, and $l_F = 1$ if and only if (M, F) is Riemannian, i.e. $\beta = 0$. The definition of l_F in turn shows that

$$F^{*2}(x, t\xi + (1-t)\beta) \leq tF^{*2}(x, \xi) + (1-t)F^{*2}(x, \beta) - l_F t(1-t)F^{*2}(x, \beta - \xi) \quad (2.13)$$

for all $x \in M$, $\xi, \beta \in T_x^* M$ and $t \in [0, 1]$.

3. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.1. First we deal with the lower semicontinuity of the energy functional

$$E_{\lambda, \mu}(u) = \mathcal{F}_\mu(u) - \lambda \mathcal{K}(u),$$

where

$$\mathcal{F}_\mu(u) = \frac{1}{p} \int_M (F^{*p}(x, Du(x)) + |u|^p) dV_F(x) - \frac{\mu}{p^*} \int_M |u|^{p^*} dV_F(x),$$

and

$$\mathcal{K}(u) = \int_M \alpha(x) H(u(x)) dV_F(x).$$

First we prove the Finslerian version of the following inequality (see Lemma 3.1): if $p \geq 2$ then for $a, b \in \mathbb{R}^N$, we have that (see [Lin90, Lemma 4.2]):

$$|b|^p \geq |a|^p + p(|a|^{p-2}a, b - a) + 2^{1-p}|a - b|^p \quad (3.1)$$

Note that, Lemma 3.1 is indispensable for the lower semicontinuity of \mathcal{F}_μ , see Proposition 3.1.

Lemma 3.1. *Let (M, F) be a Finsler manifold, then we have the following inequality:*

$$p(\beta - \xi) (F^{*p-2}(x, \xi) J^*(x, \xi)) + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} F^{*p}(x, \beta - \xi) + F^{*p}(x, \xi) \leq F^{*p}(x, \beta), \quad \forall \xi, \beta \in T_x^* M.$$

Proof. From (2.13) with the choice $t = \frac{1}{2}$, one has that

$$F^{*2} \left(x, \frac{\xi + \beta}{2} \right) \leq \frac{F^{*2}(x, \xi) + F^{*2}(x, \beta)}{2} - \frac{l_F}{4} F^{*2}(x, \beta - \xi). \quad (3.2)$$

Now, it is clear that if $p \geq 2$, then $p \mapsto (a^p + b^p)^{\frac{1}{p}}$ is non-increasing, moreover by Hölder inequality, one has that

$$(a^p + b^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-\frac{1}{2}} (a^2 + b^2)^{\frac{1}{2}} \leq (a^2 + b^2)^{\frac{1}{2}}.$$

Thus, applying this inequality, one has that

$$\begin{aligned} F^{*p} \left(x, \frac{\xi + \beta}{2} \right) + \frac{l_F^{\frac{p}{2}}}{2^p} F^{*p}(x, \beta - \xi) &\leq 2^{\frac{1}{p}-\frac{1}{2}} \cdot \left(F^{*2} \left(x, \frac{\xi + \beta}{2} \right) + \frac{l_F}{4} F^{*2}(x, \beta - \xi) \right)^{\frac{p}{2}} \\ &\stackrel{(3.2)}{\leq} \left(\frac{F^{*2}(x, \xi) + F^{*2}(x, \beta)}{2} \right)^{\frac{p}{2}}. \end{aligned}$$

On the other hand, by convexity we have that

$$\left(\frac{F^{*2}(x, \xi) + F^{*2}(x, \beta)}{2} \right)^{\frac{p}{2}} \leq \frac{1}{2} F^{*p}(x, \xi) + \frac{1}{2} F^{*p}(x, \beta).$$

On the other hand, by convexity

$$F^{*p} \left(x, \frac{\xi + \beta}{2} \right) \geq F^{*p}(x, \xi) + \frac{p}{2}(\beta - \xi) (F^{*p-2}(x, \xi) J^*(x, \xi)),$$

thus

$$F^{*p} \left(x, \frac{\xi + \beta}{2} \right) + \frac{l_F^{\frac{p}{2}}}{2^p} F^{*p}(x, \beta - \xi) \geq F^{*p}(x, \xi) + \frac{p}{2}(\beta - \xi) (F^{*p-2}(x, \xi) J^*(x, \xi)) + \frac{l_F^{\frac{p}{2}}}{2^p} F^{*p}(x, \beta - \xi).$$

Putting together, one has that

$$\frac{p}{2}(\beta - \xi) (F^{*p-2}(x, \xi) J^*(x, \xi)) + \frac{l_F^{\frac{p}{2}}}{2^p} F^{*p}(x, \beta - \xi) + \frac{1}{2} F^{*p}(x, \xi) \leq \frac{1}{2} F^{*p}(x, \beta),$$

or for every $\xi, \beta \in T_x^* M$ the following inequality hold true

$$p(\beta - \xi) (F^{*p-2}(x, \xi) J^*(x, \xi)) + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} F^{*p}(x, \beta - \xi) + F^{*p}(x, \xi) \leq F^{*p}(x, \beta), \quad (3.3)$$

which proves the Lemma. \square

Remark 3.1. *In the case of Minkowski spaces the inequality (3.3) reads as*

$$F^{*p}(\beta) \geq F^{*p}(\xi) + p F^{*p-1}(\xi) \langle \nabla F^*(\xi), \beta - \xi \rangle + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} F^{*p}(\beta - \xi),$$

where

$$l_F = \min \left\{ \left\langle \nabla \left(\frac{F^2}{2} \right) (x) y, y \right\rangle : F(x) = F(y) = 1 \right\}.$$

It is worth to mention that this result generalizes the lower semicontinuity of functionals involving critical Sobolev exponent in both Euclidean and Riemannian cases.

Proposition 3.1. *For every $\mu > 0$ and for every $0 < \rho < \rho^* \equiv \left(\frac{1}{\mu} \frac{p^*}{p} \frac{l_F^{\frac{p}{2}}}{2^{p-1} \kappa_{p^*}^{p^*}} \right)^{\frac{1}{p^*-p}}$, the restriction of \mathcal{F}_μ to $B_\rho := \{u \in W_F^{1,p}(M) : \|u\|_F \leq \rho\}$ is sequentially weakly lower semicontinuous.*

Proof. Let $u_n \rightharpoonup u$ in $W_F^{1,p}(M)$, thus we have that

$$\begin{aligned} \frac{1}{p} \int_M F^{*p}(x, Du_n(x)) dV_F(x) - \frac{1}{p} \int_M F^{*p}(x, Du(x)) dV_F(x) &\geq \\ &\geq p \int_M (Du_n(x) - Du(x)) (F^{*p-2}(x, Du(x)) J^*(x, Du(x))) dV_F(x) \\ &\quad + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \int_M F^{*p}(x, Du_n(x) - Du(x)) dV_F(x). \end{aligned}$$

On the other hand, by (3.1)

$$\begin{aligned} \int_M |u_n|^p dV_F(x) - \int_M |u|^p dV_F(x) &\geq p \int_M |u|^{p-2} u (u_n - u) dV_F(x) + \frac{1}{2^{p-1}} \int_M |u_n - u|^p dV_F(x) \\ &\geq p \int_M |u|^{p-2} u (u_n - u) dV_F(x) + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \int_M |u_n - u|^p dV_F(x). \end{aligned}$$

Summing up

$$\begin{aligned} \|u_n\|_F^p - \|u\|_F^p &\geq p \int_M (Du_n(x) - Du(x)) (F^{*p-2}(x, Du(x)) J^*(x, Du(x))) dV_F(x) + \\ &\quad + p \int_M |u|^{p-2} u (u_n - u) dV_F(x) + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \|u_n - u\|_F^p. \end{aligned}$$

By the well know Brezis-Lieb Lemma one has

$$\liminf_{n \rightarrow \infty} \left(\int_M |u_n|^{p^*} dV_F(x) - \int_M |u|^{p^*} dV_F(x) \right) = \liminf_{n \rightarrow \infty} \int_M |u_n - u|^{p^*} dV_F(x).$$

Putting all together, we have that

$$\begin{aligned} \liminf_{n \rightarrow \infty} (\mathcal{F}_\mu(u_n) - \mathcal{F}_\mu(u)) &\geq \liminf_{n \rightarrow \infty} \left(\frac{1}{p} \cdot \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \|u_n - u\|_F^p - \frac{\mu}{p^*} \int_M |u_n - u|^{p^*} dV_F(x) \right) \\ &\geq \liminf_{n \rightarrow \infty} \|u_n - u\|_F^p \left(\frac{1}{p} \cdot \frac{l_F^{\frac{p}{2}}}{2^{p-1}} - \frac{\mu}{p^*} \kappa_{p^*}^{p^*} \rho^{p^*-p} \right) \geq 0, \end{aligned}$$

which concludes the proof. \square

Before we prove the main result, we have to prove the lower semicontinuity of the functional \mathcal{H} . It turns out that, for proving such essential property one needs compact embedding of the Sobolev space to the Lebesgue space, therefore in the sequel we focus on the (compact) embedding $W_F^{1,p}(M) \hookrightarrow L^q(M)$.

Even in \mathbb{R}^d it is well known that the compactness of the embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ need not hold, see [AF03]. On the other hand, it was proved by Berestycki–Lions (see [BL80] and [Lio82], and also [Str77, Wil96]) that if $p \leq d$ then the embedding $W_{\text{rad}}^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ is compact whenever $p < q < p^*$, where $W_{\text{rad}}^{1,p}(\mathbb{R}^d)$ stands for the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^d)$.

The Berestycki–Lions-type theorem has been established on Riemannian manifolds by [HV97], see also [Heb99, Theorems 9.5 & 9.6]. More precisely, if G is a compact subgroup

of the group of global isometries of the complete Riemannian manifold (M, g) then (under some additional assumptions on the geometry of (M, g) and some assumptions on the orbits under the action of G), the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ turns out to be compact, where $W_G^{1,p}(M)$ denotes the set of G -invariant functions of the Sobolev space $W^{1,p}(M)$. Such compactness results have been extended to non-compact metric measure spaces as well, see [G18], and generalized to Lebesgue–Sobolev spaces $W_G^{1,p(\cdot)}(M)$ in the setting of complete Riemannian manifolds, see [GGP16].

Recently, in [ST13] the authors proved that, if (M, g) is a d -dimensional homogeneous Hadamard manifold and G is a compact connected subgroup of the group of global isometries of (M, g) such that $\text{Fix}_M(G)$ is a singleton, then the subspace of the G -invariant functions of $W_g^{1,p}(M)$ is compactly embedded into $L^q(M)$ whenever $p \leq d$ and $p < q < p^*$. For further use we use the following definition

Definition 3.1. *We say that a continuous action of a group G on a complete Riemannian manifold M is coercive if for every $t > 0$, the set*

$$O_t = \{x \in M : \text{diam} Gx < t\}$$

is bounded.

It was also pointed out that, see [ST13, Tin20], the condition of a single fixed point is restrictive, because on manifolds where Sobolev embeddings hold, one has a necessary and sufficient condition in terms of coercivity condition on the group G . According to this, for proving a compact embedding result for Sobolev spaces defined on Randers spaces, we use the following result, see [ST20, Tin20]:

Theorem 3.1 (Theorem 7.10.12, [Tin20]). *Let G be a compact, connected group of isometries of a d -dimensional non-compact connected Riemannian manifold M of bounded geometry. Let $1 < p < d$ and $p < q < p^*$. Then the subspace $W_G^{1,p}(M)$ is compactly embedded into $L^q(M)$ if and only if G is coercive.*

Remark 3.2. *Based on [FKM20], one could expect an alternative proof of the aforementioned embedding. Indeed, if we assume that there exists $\kappa = \kappa(G, d) > 0$ such that for every $y \in M$ with $d_g(x_0, y) \geq 1$, one has*

$$\mathcal{H}^l(\mathcal{O}_G^y) \geq \kappa \cdot d_g(x_0, y),$$

where $l = l(y) = \dim \mathcal{O}_G^y \geq 1$, where $\mathcal{O}_G^y = \{\tau(x) : \tau \in G\}$, then $W_G^{1,p}(M)$ is compactly embedded into $L^q(M)$. For the proof, see [FKM20, Theorem 4.1] and [G18].

As before, $W_{F,G}^{1,p}(M)$ will stand for the subspace of G -invariant functions of $W_F^{1,p}(M)$, where G is a subgroup of $\text{Isom}_F(M)$, such that G is coercive. In this case we have the following result:

Theorem 3.2. *Let (M, F) be a d -dimensional Randers space endowed with the Finsler metric (1.1), such that (M, g) is a d -dimensional non-compact connected Riemannian manifold of bounded geometry, and let G be a coercive compact connected subgroup of $\text{Isom}_F(M)$. If $1 < p < d$, $q \in (p, p^*)$ and $\sup_{x \in M} \|\beta\|_g(x) < 1$, then the embedding $W_F^{1,p}(M) \hookrightarrow L^q(M)$ is continuous, while the embedding $W_{F,G}^{1,p}(M) \hookrightarrow L^q(M)$ is compact.*

Proof. The volume form on (M, F) is given by (2.5), thus one has that

$$(1 - a^2)^{\frac{d+1}{2}} dv_g \leq dV_F(x) \leq dv_g. \quad (3.4)$$

Next, by using the definition of the polar transform of F , see (2.6), we get that

$$F^*(x, \xi) \leq \frac{\sqrt{\|\xi\|_g^2(x) \cdot \|\beta\|_g^2(x) + (1 - \|\beta\|_g^2(x))\|\xi\|_g^2(x) + \|\xi\|_g(x) \cdot \|\beta\|_g(x)}}{1 - \|\beta\|_g^2(x)}$$

$$= \frac{\|\xi\|_g(x)(1 + \|\beta\|_g(x))}{1 - \|\beta\|_g^2(x)} = \frac{\|\xi\|_g(x)}{1 - \|\beta\|_g(x)} \leq \frac{\|\xi\|_g(x)}{1 - a}. \quad (3.5)$$

On the other hand,

$$\begin{aligned} F^*(x, \xi) &= \frac{\|\xi\|_g^2(x)}{\sqrt{g_x^{*2}(\xi, \beta) + (1 - \|\beta\|_g^2(x))\|\xi\|_g^2(x) + g_x^*(\xi, \beta)}} \\ &\geq \frac{\|\xi\|_g^2(x)}{\sqrt{\|\xi\|_g^2(x) \cdot \|\beta\|_g^2(x) + (1 - \|\beta\|_g^2(x))\|\xi\|_g^2(x) + \|\xi\|_g(x) \cdot \|\beta\|_g(x)}} \\ &= \frac{\|\xi\|_g(x)}{1 + \|\beta\|_g(x)} \geq \frac{\|\xi\|_g(x)}{1 + a}. \end{aligned} \quad (3.6)$$

Combining (3.4), (3.5) and (3.6), one gets that

$$\frac{(1 - a^2)^{\frac{d+1}{2}}}{(1 + a)^p} \|u\|_{W_g^{1,p}(M)}^p \leq \|u\|_F^p \leq \frac{1}{(1 - a)^p} \|u\|_{W_g^{1,p}(M)}^p. \quad (3.7)$$

Thus, by the continuous embedding on the Riemannian manifold, we have that

$$\|u\|_F^p \geq \frac{(1 - a^2)^{\frac{d+1}{2}}}{(1 + a)^p} \kappa_q^p \|u\|_q^q.$$

For the compact embedding, let G be a compact connected subgroup of $\text{Isom}_F(M)$. According to [Den12, Proposition 7.1], G is a closed subgroup of the isometry group of the Riemannian manifold (M, g) . Now let $\{u_n\}$ be a bounded sequence in $W_{F,G}^{1,p}(M)$. From (3.7), it follows that $\{u_n\}$ is bounded in $W_G^{1,p}(M)$, thus by Theorem 3.1, there exists a subsequence $\{u_{n_k}\}$ which converges strongly to a function u in $L^q(M)$, thus based on (3.4) the proof is complete. \square

Example 3.1 (see [FKM20]). Let $d \geq 2$, and p be a fixed number, $p < q < p^*$, and let $B^d(1) = \{x \in \mathbb{R}^d : |x| < 1\}$ be the d -dimensional unit ball, $d \geq 2$. Consider the function $F : B^d(1) \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$F(x, y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} + \frac{\langle x, y \rangle}{1 - |x|^2}.$$

The pair $(B^d(1), F)$ is the usual Finslerian Funk model, see [CS12] and [She01]. We have that $d_F(0, x) = -\ln(1 - |x|)$, $x \in B^d(1)$. Now, let $t = \frac{p+q}{2}$, and consider the function $u : B^d(1) \rightarrow \mathbb{R}$ defined by $u(x) = \frac{|x|}{(1 - |x|)^{\frac{1}{t}}} = e^{\frac{d_F(0,x)}{t}} (1 - e^{-d_F(0,x)})$. It is clear that

$$Du(x) = \frac{1}{t} e^{\frac{d_F(0,x)}{t}} [1 + (t - 1)e^{-d_F(0,x)}] Dd_F(0, x).$$

Thus, by $F^*(x, Dd_F(0, x)) = 1$ and $dV_F(x) = dx$, we have that

$$\|u\|_{W_F^{1,p}}^p \leq \omega_{d-1} \mathbf{B}\left(p + d, 1 - \frac{p}{t}\right) \left[1 + 2^{p-1} \left(1 - \frac{1}{t}\right)^p\right] + \omega_{d-1} 2^{p-1} \mathbf{B}\left(d, 1 - \frac{p}{t}\right), \quad (3.8)$$

where $\mathbf{B}(x, y)$ is the Euler-Beta function. On the other hand, the L^q norm of the function u is given by $\|u\|_{L^q}^q = \omega_{d-1} \mathbf{B}\left(q + d, 1 - \frac{q}{t}\right)$. Therefore, it is clear that $\|u\|_{W_F^{1,p}} < \infty$, and $\|u\|_{L^q} = +\infty$, therefore $W_F^{1,p}(B^d(1)) \not\hookrightarrow L^q(B^d(1))$.

From the assumption (α) one can easily observe that $\alpha \in L^\infty(M)$. On the other hand, by the layer cake representation it follows that

$$\int_M \alpha(x) dV_F(x) = \int_M \alpha_0(d_F(x_0, x)) dV_F(x)$$

$$\begin{aligned}
&= \int_0^\infty \text{Vol}_F(\{x \in M : \alpha_0(d_F(x_0, x)) > t\}) dt \quad [\text{chang. of var. } t = \alpha_0(z)] \\
&= \int_\infty^0 \text{Vol}_F(B_F(x_0, z)) \alpha'_0(z) dz = \int_0^\infty \text{Vol}_F(B_F(x_0, z)) (-\alpha'_0(z)) dz.
\end{aligned}$$

Since, (M, F) is a Randers space, and $a := \sup_x \|\beta\|_g(x) < 1$ one can observe that

$$(1 - a)d_g(x, y) \leq d_F(x, y) \leq (1 + a)d_g(x, y),$$

thus

$$B_g\left(x_0, \frac{z}{1+a}\right) \subseteq B_F(x_0, z) \subseteq B_g\left(x_0, \frac{z}{1-a}\right). \quad (3.9)$$

On account of this inclusion and the Bishop-Gromov inequality (see [Cha06]) and from the assumption (α) , one can see that

$$\begin{aligned}
\int_0^\infty \text{Vol}_F(B_F(x_0, z)) (-\alpha'_0(z)) dz &\leq \int_0^\infty \text{Vol}_g\left(B_g\left(x_0, \frac{z}{1-a}\right)\right) (-\alpha'_0(z)) dz \\
&= \int_0^\infty \text{Area}_g\left(B_g\left(x_0, \frac{z}{1-a}\right)\right) \alpha_0(z) dz \\
&\leq c \int_0^\infty \sinh\left(k \frac{z}{1-a}\right)^{d-1} \alpha_0(z) dz < +\infty,
\end{aligned}$$

thus $\alpha \in L^1(M)$.

In the sequel we prove that the energy is G -invariant

Lemma 3.2. *Let G be a compact connected subgroup of $\text{Isom}_F(M)$ with $\text{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$. Then $E_{\lambda, \mu}$ is G -invariant, i.e., for every $\tau \in G$ and $u \in W_F^{1,p}(M)$ one has $E_{\lambda, \mu}(\tau u) = E_{\lambda, \mu}(u)$.*

Proof. The proof is based on [FKM20, Lemma 5.1]. First we focus on the G -invariance of the functional $u \mapsto \int_M F^{*p}(x, Du(x)) dV_F(x)$. Since $\tau \in G$, we have that (see [DH02])

$$F(\xi(x), d\tau_x(X)) = F(x, X), \quad \forall x \in M, X \in T_x M. \quad (3.10)$$

Since $(\tau u)(x) = u(\tau^{-1}(x))$, by the chain rule, one has

$$\begin{aligned}
\int_M F^{*p}(x, D(\tau u)(x)) dV_F(x) &= \int_M F^{*p}(x, D(u(\tau^{-1}(x)))) dV_F(x) \\
&= \int_M F^{*p}(x, D(u(\tau^{-1}(x))) d\tau_x^{-1}) dV_F(x) \quad (\tau^{-1}(x) = y) \\
&= \int_M F^{*p}(\tau(y), D(u(y)) d\tau_{\tau(y)}^{-1}) dV_F(\tau(y)). \quad (3.11)
\end{aligned}$$

Since $\tau \in G$, $dV_F(\tau(y)) = dV_F(y)$. On the other hand, by the definition of the polar transform (2.6) and relation (3.10), we have

$$\begin{aligned}
F^*(\tau(y), D(u(y)) d\tau_{\tau(y)}^{-1}) &= \sup_{\sigma \in T_{\tau(y)} M \setminus \{0\}} \frac{D(u(y)) d\tau_{\tau(y)}^{-1}(\sigma)}{F(\tau(y), \sigma)} \quad (\sigma := d\tau_y(z), z \in T_y M) \\
&= \sup_{z \in T_y M \setminus \{0\}} \frac{Du(y) d\tau_{\tau(y)}^{-1}(d\tau_y(z))}{F(\tau(y), d\tau_y(z))} \\
&= \sup_{z \in T_y M \setminus \{0\}} \frac{Du(y)(z)}{F(y, z)} = F^*(y, Du(y)). \quad (3.12)
\end{aligned}$$

Combining (3.11) and (3.12), we get the desired G -invariance of the functional $u \mapsto$

$$\int_M F^{*p}(x, Du(x)) dV_F(x).$$

Since $\tau \in G$ and $\alpha \in L^1(M) \cap L^\infty(M)$ is a non-zero, non-negative function which depends on $d_F(x_0, \cdot)$ and $\text{Fix}_M(G) = \{x_0\}$, it turns out that for every $u \in W_F^{1,p}(M)$, we have $\mathcal{K}(\tau u) = \mathcal{K}(u)$. Indeed

$$\begin{aligned} \mathcal{K}(\tau u) &= \int_M \alpha(x) H(\tau u) dV_F(x) = \int_M \alpha_0(d_F(x_0, x)) H(u(\tau^{-1}(x))) dV_F(x) \\ &= \int_M \alpha_0(d_F(x_0, \tau(x))) H(u(x)) dV_F(x) \\ &= \int_M \alpha_0(d_F(x_0, x)) H(u(x)) dV_F(x) = \mathcal{K}(u), \end{aligned}$$

which concludes the proof. \square

For further use, we restrict the energy functional to the space $W_{F,G}^{1,p}(M)$. For simplicity, in the following we denote

$$\mathcal{E}_{\lambda,\mu} = E_{\lambda,\mu}|_{W_{F,G}^{1,p}(M)}, \quad \text{and} \quad \mathcal{K}_G = \mathcal{K}|_{W_{F,G}^{1,p}(M)}.$$

The principle of symmetric criticality of Palais (see Kristály, Rădulescu and Varga [KRV10, Theorem 1.50]) and Lemma 3.2 imply that the critical points of $\mathcal{E}_{\lambda,\mu} = E_{\lambda,\mu}|_{W_{F,G}^{1,p}(M)}$ are also critical points of the original functional E_λ .

Lemma 3.3. *The functional $\mathcal{K}_G : W_{F,G}^{1,p}(M) \rightarrow \mathbb{R}$ is weakly lower semicontinuous on $W_{F,G}^{1,p}(M)$.*

Proof. Consider $\{u_n\}$ a sequence in $W_{F,G}^{1,p}(M)$ which converges weakly to $u \in W_{F,G}^{1,p}(M)$, and suppose that

$$\mathcal{K}_G(u_n) \not\rightarrow \mathcal{K}_G(u), \quad \text{as } n \rightarrow \infty.$$

Thus, there exist $\varepsilon > 0$ and a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that

$$0 < \varepsilon \leq |\mathcal{K}_G(u_n) - \mathcal{K}_G(u)|, \quad \text{for every } n \in \mathbb{N}.$$

Thus, by the mean value theorem, there exists $\theta_n \in (0, 1)$ such that

$$\begin{aligned} \varepsilon &\leq |\mathcal{K}_G(u_n) - \mathcal{K}_G(u)| \leq \int_M \alpha(x) |H(u_n(x)) - H(u(x))| dV_F(x) \\ &\leq \int_M \alpha(x) |h(u + \theta_n(u_n - u))| \cdot |u_n - u| dV_F(x) \\ &\leq \int_M \alpha(x) |u_n - u| (c_1 |u + \theta_n(u_n - u)|^{r-1} + c_2 |u + \theta_n(u_n - u)|^{q-1}) dV_F(x) \\ &\leq \int_M \alpha(x) (c_1 |u_n|^{r-1} |u_n - u| + c_2 |u_n|^{q-1} |u_n - u|) dV_F(x). \end{aligned} \tag{3.13}$$

For further use, let $a, b > 0$ be two real numbers, such that

$$\begin{cases} p < \frac{a}{a-r+1} < p^*, & \text{or } \frac{p^*(r-1)}{p^*-1} < a < \frac{p(r-1)}{p-1}, \\ p < \frac{b}{b-q+1} < p^*, & \text{or } \frac{p^*(q-1)}{p^*-1} < b < \frac{p(q-1)}{p-1}. \end{cases}$$

In this case, by Hölder inequality we have that

$$c_1 \int_M \alpha(x) |u_n|^{r-1} |u_n - u| dV_F(x) \leq c_1 \|\alpha\|_\infty \|u_n - u\|_{\frac{a}{a-r+1}} \|u_n\|_a^{r-1},$$

and

$$c_2 \int_M \alpha(x) |u_n|^{q-1} |u_n - u| dV_F(x) \leq c_2 \|\alpha\|_\infty \|u_n - u\|_{\frac{b}{b-q+1}} \|u_n\|_b^{q-1}.$$

Combining the above two inequalities with Theorem 3.2, we get that

$$\int_M \alpha(x) |H(u_n(x)) - H(u(x))| dV_F(x) \rightarrow 0,$$

which is a contradiction in the light of (3.13), which concludes the proof of the lemma. \square

According to Lemma 3.1 and Lemma 3.3 the functional $\mathcal{E}_{\lambda,\mu}$ is weakly lower semicontinuous on small ball of the Sobolev space $W_{F,G}^{1,p}(M)$. Now, we are in the position to prove our main result.

Proof of Theorem 1.1. For any $r > 0$ denote

$$B_r = \{u \in W_{F,G}^{1,p} : \|u\|_F \leq r\},$$

and define $\mathcal{J}_{\mu,\lambda} : W_{F,G}^{1,p} \rightarrow \mathbb{R}$, by

$$\mathcal{J}_{\mu,\lambda}(u) = \frac{\mu}{p^*} \int_M |u|^{p^*} dV_F + \lambda \int_M \alpha(x) H(u) dV_F.$$

For any $\lambda, \mu, \rho > 0$ define

$$\varphi_{\mu,\lambda}(\rho) := \inf_{\|u\| < \rho} \frac{\sup_{B_\rho} \mathcal{J}_{\mu,\lambda} - \mathcal{J}_{\mu,\lambda}(u)}{\rho^p - \|u\|_F^p} \quad \text{and} \quad \psi_{\mu,\lambda}(\rho) := \sup_{B_\rho} \mathcal{J}_{\mu,\lambda}. \quad (3.14)$$

We claim that, under our assumptions, there exist $\lambda, \mu, \rho > 0$ such that

$$\varphi_{\mu,\lambda}(\rho) < \frac{1}{p}, \quad (3.15)$$

that is, there exist $\lambda, \mu, \rho > 0$ such that

$$\inf_{\sigma < \rho} \frac{\psi_{\mu,\lambda}(\rho) - \psi_{\mu,\lambda}(\sigma)}{\rho^p - \sigma^p} < \frac{1}{p}.$$

Notice that by putting $\sigma = \rho - \varepsilon$, for some $\varepsilon > 0$

$$\begin{aligned} \frac{\psi_{\mu,\lambda}(\rho) - \psi_{\mu,\lambda}(\sigma)}{\rho^p - \sigma^p} &= \frac{\psi_{\mu,\lambda}(\rho) - \psi_{\mu,\lambda}(\rho - \varepsilon)}{\rho^p - (\rho - \varepsilon)^p} \\ &= \frac{\psi_{\mu,\lambda}(\rho) - \psi_{\mu,\lambda}(\rho - \varepsilon)}{\varepsilon} \cdot \frac{-\frac{\varepsilon}{\rho}}{\rho^{p-1} \left[\left(1 - \frac{\varepsilon}{\rho}\right)^p - 1 \right]}, \end{aligned}$$

so that (3.15) is fulfilled if there exist $\mu, \rho > 0$ such that

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{\psi_{\mu}(\rho) - \psi_{\mu}(\rho - \varepsilon)}{\varepsilon} < \rho^{p-1}. \quad (3.16)$$

$$\begin{aligned} \frac{\psi_{\mu,\lambda}(\rho) - \psi_{\mu,\lambda}(\rho - \varepsilon)}{\varepsilon} &\leq \frac{1}{\varepsilon} \sup_{\|v\|_F \leq 1} \int_M \left| \int_{(\rho-\varepsilon)v(x)}^{\rho v(x)} \mu |t|^{p^*-1} + \lambda \alpha(x) |h(t)| dt \right| dV_F \\ &\leq \frac{\kappa_{p^*}^{p^*} \mu}{p^*} \left| \frac{\rho^{p^*} - (\rho - \varepsilon)^{p^*}}{\varepsilon} \right| + \lambda c_1 \frac{\kappa_{p^*}^q \|\alpha\|_{\frac{p^*}{p^*-q}}}{q} \cdot \left| \frac{\rho^q - (\rho - \varepsilon)^q}{\varepsilon} \right| \\ &\quad + \lambda c_2 \frac{\kappa_{p^*}^r \|\alpha\|_{\frac{p^*}{p^*-r}}}{r} \left| \frac{\rho^r - (\rho - \varepsilon)^r}{\varepsilon} \right|. \end{aligned}$$

Thus

$$\limsup_{\varepsilon \rightarrow 0} \frac{\psi_{\mu}(\rho) - \psi_{\mu}(\rho - \varepsilon)}{\varepsilon} \leq \mu \kappa_{p^*}^{p^*} \rho^{p^*-1} + \lambda c_1 \kappa_{p^*}^q \|\alpha\|_{\frac{p^*}{p^*-q}} \rho^{q-1} + \lambda c_2 \kappa_{p^*}^r \|\alpha\|_{\frac{p^*}{p^*-r}} \rho^{r-1}. \quad (3.17)$$

Let $\rho_0 > 0$ such that

$$\frac{\rho_0^{p-1} - \mu \kappa_{p^*}^{p^*} \rho_0^{p^*-1}}{c_1 \kappa_{p^*}^q \|\alpha\|_{\frac{p^*}{p^*-q}} \rho_0^{q-1} + c_2 \kappa_{p^*}^r \|\alpha\|_{\frac{p^*}{p^*-r}} \rho_0^{r-1}} = \max_{t>0} \frac{t^{p-1} - \mu \kappa_{p^*}^{p^*} t^{p^*-1}}{c_1 \kappa_{p^*}^q \|\alpha\|_{\frac{p^*}{p^*-q}} t^{q-1} + c_2 \kappa_{p^*}^r \|\alpha\|_{\frac{p^*}{p^*-r}} t^{r-1}},$$

and consider $\rho_\mu := \min\{\rho_0, \rho^*\}$. Thus if

$$0 < \lambda < \lambda_* := \frac{\rho_\mu^{p-1} - \mu \kappa_{p^*}^{p^*} \rho_\mu^{p^*-1}}{c_1 \kappa_{p^*}^q \|\alpha\|_{\frac{p^*}{p^*-q}} \rho_\mu^{q-1} + c_2 \kappa_{p^*}^r \|\alpha\|_{\frac{p^*}{p^*-r}} \rho_\mu^{r-1}},$$

therefore, based on (3.17) there exists (λ, ρ) such that

$$\limsup_{\varepsilon \rightarrow 0} \frac{\psi_\mu(\rho) - \psi_\mu(\rho - \varepsilon)}{\varepsilon} < \rho_\mu^{p-1},$$

so that (3.15) is fulfilled.

Condition (3.15) implies the existence of $u_0 \in W_{F,G}^{1,p}$ with $\|u_0\|_F < \rho_\mu$ such that

$$\mathcal{E}_\mu(u_0) < \frac{1}{p} \rho_\mu^p - \mathcal{J}_\mu(u) \quad (3.18)$$

for every $u \in B_{\rho_\mu}$. Since the energy $\mathcal{E}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous in B_{ρ_μ} , its restriction to the ball has a global minimum u_* . If $\|u_*\| = \rho_\mu$, then, from (3.18)

$$\mathcal{E}_{\lambda,\mu}(u_*) = \frac{1}{p} \rho_\mu^p - \mathcal{J}_\mu(u_*) > \mathcal{E}_{\lambda,\mu}(u_0),$$

a contradiction. It follows that u_* is a local minimum for $\mathcal{E}_{\lambda,\mu}$ with $\|u_*\|_F < \rho_\mu$, hence in particular, a weak solution of problem $(\mathcal{P}_{\lambda,\mu})$.

In the sequel we prove that u_0 is not identically zero. In order to prove that our solution is non-trivial, we show that there exists a function for which the energy is negative.

To this end, observe that from the assumption on the function α , it is clear that, there exists $R > 0$ such that $\alpha_R := \operatorname{ess\,inf}_{d_F(x_0,x) \leq R} \alpha(x) > 0$. Let $\zeta > 0$ such that $\zeta < R \frac{1-a}{1+a}$. Now we can define the following function:

$$u_{R,\zeta} = \begin{cases} 0, & x \in M \setminus B_F(x_0, R) \\ \frac{1}{R-\zeta}(R - d_F(x_0, x)), & x \in B_F(x_0, R) \setminus B_F(x_0, \zeta) \\ 1, & x \in B_F(x_0, \zeta) \end{cases}$$

Due to (h_2) , it is clear that there exists a sequence θ_j , with $\theta_j \rightarrow 0$ as $j \rightarrow +\infty$, such that

$$c\theta_j^p \leq H(\theta_j), \quad \forall c > 0, \quad (3.19)$$

and j large enough.

Consider the following function $u_1 := \theta_j u_{R,\zeta}$. In the sequel we are going to estimate $\mathcal{E}_{\mu,\lambda}(u_1)$.

First of all, recall that $r_F > 0$ is the reversibility constant on (M, F) , see (2.11), i.e. by the eikonal identity (2.9) we have that $\frac{1}{r_F} \leq F^*(x, -Dd_F(x_0, x)) \leq r_F$. Therefore,

$$\begin{aligned} \int_M F^{*p}(x, Du_1(x)) \, dV_F(x) &= \int_M F^{*p}(x, \theta_j Du_{R,\zeta}(x)) \, dV_F(x) \\ &= \theta_j^p \left(\frac{1}{R-\zeta} \right)^p \int_{B_F(x_0, R) \setminus B_F(x_0, \zeta)} F^{*p}(x, -Dd_F(x_0, x)(x)) \, dV_F(x) \\ &\leq \theta_j^p \left(\frac{1}{R-\zeta} \right)^p r_F^p \operatorname{Vol}_F(B_F(x_0, R)), \end{aligned}$$

and

$$\int_M |u_1|^p dV_F(x) = \theta_j^p \int_M |u_{R,\zeta}|^p dV_F(x) \leq \theta_j^p \text{Vol}_F(B_F(x_0, R)).$$

On the other hand

$$\int_M |u_1|^{p^*} dV_F(x) = \theta_j^p \int_M |u_{R,\zeta}|^{p^*} dV_F(x) \theta_j^{p^*} \text{Vol}_F(B_F(x_0, \zeta)).$$

Finally, by (3.19) we have that

$$\int_M \alpha(x) H(u_1) dV_F(x) \geq c \theta_j^p \int_{B_F(x_0, \zeta)} \alpha(x) dV_F(x) \geq c \theta_j^p \alpha_R \text{Vol}_F(B_F(x_0, \zeta)).$$

Putting all together, one has that

$$\begin{aligned} \mathcal{E}_{\mu, \lambda}(u_1) &\leq \left(\frac{1}{p} \left(\frac{1}{R - \zeta} \right)^p r_F^p \text{Vol}_F(B_F(x_0, R)) + \frac{1}{p} \text{Vol}_F(B_F(x_0, R)) - c \alpha_R \text{Vol}_F(B_F(x_0, \zeta)) \right) \theta_j^p \\ &\quad - \frac{\theta_j^{p^*}}{p^*} \text{Vol}_F(B_F(x_0, \zeta)). \end{aligned}$$

By choosing $c > 0$ large enough, one can easily see that 0 is not a local minimizer of the energy functional, and hence $u_0 \neq 0$. Which concludes the proof. \square

4. FINAL REMARKS

4.1. Critical elliptic equations on Hadamard type Randers spaces. Through this section, assume that (M, F) is a d -dimensional Randers space endowed with the Finsler metric (1.1) such that $\sup_{x \in M} \|\beta\|_g(x) < 1$ and g is a Riemannian metric where (M, g) is a Hadamard manifold. Let G be a coercive, compact connected subgroup of $\text{Isom}_F(M)$ such that $x_0 \in \text{Fix}_M(G)$ for some $x_0 \in M$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function verifying (H), and let $\alpha \in L^1(M) \cap L^\infty(M) \setminus \{0\}$ be a non-negative and radially symmetric function w.r.t $x_0 \in M$, i.e. there exists $\alpha_0 : [0, +\infty) \rightarrow \mathbb{R}$ such that $\alpha(x) = \alpha_0(d_F(x_0, x))$, $\forall x \in M$. In this case a similar result as Theorem 1.1 can be obtained. Note that, such result is an extension of [MBV20]. Note that if the sectional curvature of (M, g) is non/positive and the compact connected group G of isometries fixes some point, then G is coercive if and only if G has no other fixed point.

Moreover, if with sectional curvature of (M, g) bounded above by $-\mathbf{k}^2$, $\mathbf{k} > 0$, for example one can consider the following Funk-type metric (see [KR15]): let $B^d(1) = \{x \in \mathbb{R}^d : |x| < 1\}$ be the d -dimensional unit ball, $d \geq 2$, and for every $a \in [0, 1)$, the Funk-type metric $F_a : B^d(1) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$F_a(x, y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} + a \frac{\langle x, y \rangle}{1 - |x|^2}, \quad x \in B^n(1), \quad y \in T_x B^d(1) = \mathbb{R}^d. \quad (4.1)$$

In fact, for $a = 0$, the manifold $(B^d(1), F_0)$ reduces to the well known Riemannian Klein model; with constant sectional curvature $-\frac{1}{4}$.

Then for every $\mu > 0$, there exists $\lambda_* > 0$ such that for every $\lambda < \lambda_*$ the problem

$$\begin{cases} -\Delta_{p,F} u = \lambda |u|^{p^*-2} u + \mu \alpha(x) h(u), & \text{in } M, \\ u \in W_F^{1,p}(M) \end{cases}$$

has a non-zero G -invariant weak solution.

The proof of the above statement is similar to the proof of Theorem 1.1. The key point is a McKean-type inequality (see for instance [YH14, Theorem 0.6]), we have that

$$\lambda_{1,g} := \inf_{u \in W_g^{1,p}(M)} \frac{\int_M |\nabla_g u|^p dv_g}{\int_M |u|^p dv_g} \geq \left(\frac{(d-1)\mathbf{k}}{p} \right)^p,$$

therefore,

$$\int_M |\nabla_g u|^p dv_g \geq \frac{(d-1)^p \mathbf{k}^p}{p^p + (d-1)^p \mathbf{k}^p} \|u\|_{W_g^{1,p}(M)}^p, \quad u \in W_g^{1,p}(M).$$

Using (3.4), (3.6), denoting by $\mathbf{C}_{\mathbf{k},p} := \frac{(1-a^2)^{(d+1)/2}}{(1+a)^p} \frac{(d-1)^p \mathbf{k}^p}{p^p + (d-1)^p \mathbf{k}^p}$, we obtain that for every u

$$\left(\frac{1}{1-a}\right)^p \|u\|_{W_g^{1,p}(M)}^p \geq \int_M F^{*p}(x, Du(x)) dV_F(x) \geq \mathbf{C}_{\mathbf{k},p} \|u\|_{W_g^{1,p}(M)}^p.$$

4.2. Hardy inequalities on Finsler–Hadamard manifolds. One can notice that using the inequality described in Lemma 3.1, one can deduce a Hardy type inequality on general Finsler manifolds. Such inequalities allows us to consider elliptic equations involving critical Sobolev exponent and a Hardy-type singularities, see [OSV20, BFP20, FP16].

Theorem 4.1. *Let (M, F) be an d -dimensional ($d \geq 3$) Finsler–Hadamard manifold with $\mathbf{S} = 0$, and let $x_0 \in M$ be fixed. If $d + a - p > 0$, $b + p > 0$ and $p > 1$ then $\forall u \in C_0^\infty(M)$*

$$\int_M d_F(x_0, x)^a |u|^b F^{*p}(x, -D|u|) dV_F(x) \geq \left(\frac{a + d - p}{b + p}\right)^p \int_M \frac{|u|^{p+b}}{d_F^{p-\alpha}(x_0, x)} dV_F(x). \quad (4.2)$$

Proof. To prove this inequality, consider $u \in C_0^\infty(M)$, and denote by $\gamma = \frac{d+a-p}{b+p}$. We consider the function $v(x) = d_F(x_0, x)^\gamma u(x)$. Therefore, and one has

$$D(|u|)(x) = -\gamma d_F(x_0, x)^{-\gamma-1} |v| Dd_F(x_0, x) + d_F(x_0, x)^{-\gamma} D(|v|)(x).$$

Applying the inequality (which is a consequence of Lemma 3.1)

$$p(\beta - \xi) (F^{*p-2}(x, \xi) J^*(x, \xi)) + F^{*p}(x, \xi) \leq F^{*p}(x, \beta), \quad \forall \xi, \beta \in T_x^* M. \quad (4.3)$$

with the choices $\beta = -D|u|$ and $\xi = \gamma d_F(x_0, x)^{-\gamma-1} |v| Dd_F(x_0, x)$, respectively, one can deduce that

$$\begin{aligned} F^{*p}(x, -D(|u|)(x)) &\geq F^{*p}(x, \gamma d_F(x_0, x)^{-\gamma-1} |v(x)| Dd_F(x_0, x)) \\ &\quad - p\gamma^{p-1} d_F(x_0, x)^{-p\gamma-p+1} |v|^{p-1} D(|v|)(\nabla d_F(x_0, x)). \end{aligned}$$

Multiplying the above inequality by $d_F^a(x_0, x) |u|^b$ and integrating it we obtain

$$\int_M d_F^a(x_0, x) |u|^b F^{*p}(x, -D|u|) dV_F(x) \geq \left(\frac{a + d - p}{b + p}\right)^p \int_M \frac{|u|^{p+b}}{d_F^{p-\alpha}(x_0, x)} dV_F(x) + R_0.$$

where

$$R_0 = -p\gamma^{p-1} \int_M d_F(x_0, x)^{-p\gamma-p+1+a-b\gamma} |v(x)|^{b+p-1} D(|v|)(x) (\nabla d_F(x_0, x)) dV_F(x).$$

Since $\mathbf{S} = 0$ and $\mathbf{K} \leq 0$, we have the Laplace-comparison principle (see [WX07])

$$d_F(x_0, x) \Delta d_F(x_0, x) \geq d - 1 \text{ for a.e. } x \in M.$$

Consequently, by (2.10) and the latter estimate one has

$$\begin{aligned} R_0 &= -\frac{p\gamma^{p-1}}{b+p} \int_M D(|v|^{b+p})(x) (d_F(x_0, x)^{-p\gamma-p+1+a-b\gamma} \nabla d_F(x_0, x)) dV_F(x) \\ &= \frac{p\gamma^{p-1}}{b+p} \int_M |v(x)|^{b+p} \operatorname{div} (d_F(x_0, x)^{-p\gamma-p+1+a-b\gamma} \nabla d_F(x_0, x)) dV_F(x) \\ &= \frac{p\gamma^{p-1}}{b+p} \int_M |v(x)|^{b+p} d_F(x_0, x)^{-p\gamma-p+a-b\gamma} (-d + 1 + d_F(x_0, x) \Delta d_F(x_0, x)) dV_F(x) \geq 0, \end{aligned}$$

which completes the proof of the inequality. Note that, one can prove that the constant $\left(\frac{a + d - p}{b + p}\right)^p$ is optimal and never achieved, for the argument see [FKV15]. \square

Brézis and Vazquez first discovered improved version of the Hardy inequality (see [BV97]). Later, lot of attention was paid for the improvements of Brézis Vazquez inequalities, see [WW03] or [ACR02]. In the sequel we shortly present a Wang-Willem type inequality as an application of Lemma 3.1. For simplicity we focus on Minkowski spaces:

Theorem 4.2. *Let $F : \mathbb{R}^d \rightarrow [0, +\infty)$ be a positively homogeneous Minkowski norm, $\Omega \subset \mathbb{R}^d$ be an open bounded domain, $0 \in \Omega$, and $R > \sup_{x \in \Omega} F^*(-x)$. If $d + a - p > 0$, then for every $u \in C_0^\infty(\Omega)$ we have*

$$\begin{aligned} \int_{\Omega} F^{*a}(-x) F^p(\nabla|u|) dx &\geq \left(\frac{d+a-p}{p} \right)^p \int_{\Omega} F^{*a-p}(-x) |u|^p dx \\ &\quad + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \left(\frac{p-1}{p} \right)^p \int_{\Omega} F^{*a-p}(-x) \frac{|u|^p}{\ln \left(\frac{R}{F^*(-x)} \right)^p} dx, \end{aligned}$$

Proof. As in the Proof of Theorem 4.1, let $\gamma = \frac{d+a-p}{p}$, $u \in C_0^\infty(\Omega)$, and consider $v = F^{*\gamma}(-x)u$. Using Lemma 3.1, one can prove the following inequality (in a similar way as above):

$$\int_{\Omega} F^{*a}(-x) F^p(\nabla|u|) dx \geq \gamma^p \int_{\Omega} F^{*a-p}(-x) |u|^p dx + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \int_{\Omega} F^{*a-p\gamma}(-x) F^p(\nabla|v|) dx.$$

Now, consider the following function $w(x) = \ln \left(\frac{R}{F^*(-x)} \right)^{\frac{1-p}{p}} v(x)$. Then

$$\nabla|v| = \frac{p-1}{p} \ln \left(\frac{R}{F^*(-x)} \right)^{-\frac{1}{p}} \frac{\nabla F^*(-x)}{F^*(-x)} |w| + \ln \left(\frac{R}{F^*(-x)} \right)^{\frac{p-1}{p}} \nabla|w|.$$

Now, applying inequality (4.3), we obtain

$$\begin{aligned} F^p(\nabla|u|) &\geq \left(\frac{p-1}{p} \right)^p \ln \left(\frac{R}{F^*(-x)} \right)^{-1} \frac{|w|^p}{F^{*p}(-x)} \\ &\quad + p \left(\frac{p-1}{p} \right)^{p-1} \frac{|w|^{p-1}}{F^{*p-1}(-x)} \langle \nabla(\nabla F^*(-x)), \nabla|w| \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\Omega} F^{*a-p\gamma}(-x) F^p(\nabla|v|) dx &\geq \left(\frac{p-1}{p} \right)^p \int_{\Omega} F^{*a-p\gamma}(-x) \ln \left(\frac{R}{F^*(-x)} \right)^{-1} \frac{|w|^p}{F^{*p}(-x)} dx \\ &\quad - \left(\frac{p-1}{p} \right)^{p-1} \int_{\Omega} F^{*-d}(-x) \langle x, \nabla|w|^p \rangle dx. \end{aligned}$$

On the other hand, by [FK09, Theorem 3.1] we obtain

$$\begin{aligned} - \left(\frac{p-1}{p} \right)^{p-1} \int_{\Omega} F^{*-d}(-x) \langle x, \nabla|w|^p \rangle dx &= \left(\frac{p-1}{p} \right)^{p-1} \int_{\Omega} |w|^p \operatorname{div} (F^{*-d}(-x)x) dx \\ &= \left(\frac{p-1}{p} \right)^{p-1} \int_{\Omega} |w|^p \Delta_F(F^{*2-d}(-x)) dx = 0. \end{aligned}$$

Putting all together

$$\begin{aligned} \int_{\Omega} F^{*a}(-x) F^p(\nabla|u|) dx &\geq \gamma^p \int_{\Omega} F^{*a-p}(-x) |u|^p dx \\ &\quad + \frac{l_F^{\frac{p}{2}}}{2^{p-1}} \left(\frac{p-1}{p} \right)^p \int_{\Omega} F^{*a-p}(-x) \frac{|u|^p}{\ln \left(\frac{R}{F^*(-x)} \right)^p} dx. \end{aligned}$$

□

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