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Generalisation of the rainbow neighbourhood number and k-jump colouring of a graph

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Abstract

In this paper, the notions of rainbow neighbourhood and rainbow neighbourhood number of a graph are generalised and further to these generalisations, the notion of a proper k-jump colouring of a graph is also introduced. The generalisations follow from the understanding that a closed k-neighbourhood of a vertex $v \in V(G)$ denoted, $N_k[v]$ is the set, $N_k[v] = \{u : d(v,u) \leq k, k \in \mathbb{N}$ and $k \leq diam(G)\}$. If the closed k-neighbourhood $N_k[v]$ contains at least one of each colour of the chromatic colour set, we say that v yields a k-rainbow neighbourhood.

Keywords: k-rainbow neighbourhood, k-rainbow neighbourhood number, k-jump colouring.

MSC: 05C15, 05C38, 05C75, 05C85.

1. Introduction

For general notation and concepts in graphs and digraphs we refer to [1, 2, 8]. Unless mentioned otherwise all graphs G are simple, connected and finite graphs. For corresponding results of disconnected graphs, see [3].

Recall that a *vertex colouring* of a graph G is an assignment $\varphi : V(G) \mapsto \mathcal{C}$, where $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$ is a set of distinct colours. A vertex colouring is said to be a *proper vertex colouring* of a graph G if no two distinct adjacent vertices have the same colour. The cardinality of a minimum set of colours in a proper vertex colouring of G is called the *chromatic number* of G and is denoted $\chi(G)$. A colouring of G consisting of exactly $\chi(G)$ colours may be called a χ*-colouring* or a *chromatic colouring* of G.

When the cardinality of the set of colours $\mathcal{C}$ is bound by conditions such as minimum, maximum or others and since $c(V(G)) = \mathcal{C}$, it can be agreed that $c(G)$ means $c(V(G))$ and hence $c(G) \Rightarrow \mathcal{C}$ and $|c(G)| = |\mathcal{C}|$.

Index labelling the elements of a graph such as the vertices say, $v_1, v_2, v_3, \ldots, v_n$ or written as v_i; $1 \leq i \leq n$ or as v_i; $i = 1, 2, 3, \ldots, n$, is called a *minimum parameter indexing*. Similarly, a *minimum parameter colouring* of a graph G is a proper colouring of G which consists of the colours c_i; $1 \leq i \leq \ell$. The set of vertices of G having the colour c_i is said to be the *colour class* of c_i in G and is denoted by $\mathcal{C}_i$. Unless stated otherwise, we consider minimum parameter colouring throughout this paper.

Note that the closed neighbourhood $N[v]$ of a vertex $v \in V(G)$ which contains at least one vertex from each colour class of G in the chromatic colouring, is called *a rainbow neighbourhood* (see [4–7] for further results on rainbow neighbourhoods of different graphs). The number of vertices in G which yield rainbow neighbourhoods, denoted by $r_\chi(G)$, is called the *rainbow neighbourhood number* of G corresponding to the chromatic colouring. Note that $r^-_\chi(G)$ and $r^+_\chi(G)$ respectively denote the minimum value and maximum value of $r_\chi(G)$ over all minimum proper colourings (see [4]).

Rainbow neighbourhood convention ([4]): The rainbow neighbourhood convention is a colouring protocol as described below:

Let X_1 be a maximal independent set in G. Let $G_1 = G - X_1$. Let X_2 be a maximal independent set in G_1 and $G_2 = G_1 - X_2$. Proceed like this, until after a finite number of iterations, say k, the induced graph $\langle X_k \rangle$ is a trivial or empty graph. Clearly, we have $|X_1| \geq |X_2| \geq \ldots |X_{k-1}| \geq |X_k|$. Now, consider a set $\mathbb{C} = \{c_1, c_2, \ldots, c_k\}$ of k colours and we assign the colour c_i to all vertices in X_i for $1 \leq i \leq k$.

Unless mentioned otherwise the rainbow neighbourhood convention together with a minimum parameter colouring will be used for all graph colourings.

2. *k*-rainbow neighbourhood number of a graph

In this section, we generalise the notion of a rainbow neighbourhood of a graph. A *closed k-neighbourhood* of a vertex $v \in V(G)$, denoted by $N_k[v]$, is the set, $N_k[v] = \{u : d(v,u) \leq k, k \in \mathbb{N}\}$ (Note that $k \leq diam(G)$).

Definition 2.1. If the closed k-neighbourhood $N_k[v]$; $v \in V(G)$ contains at least one of each colour from the chromatic colour class, we say that v yields a k-rainbow neighbourhood.

In this context, a rainbow neighbourhood defined in [5] is indeed a 1-rainbow neighbourhood.

Definition 2.2. For a chromatic colouring of a graph G, the number of distinct vertices which yield a k-rainbow neighbourhood is called the *k-rainbow neighbourhood number* of G and is denoted by $r_{\chi,k}(G)$.

Definition 2.3. The *k^--rainbow neighbourhood number* of a graph G, denoted by $r^-_{\chi,k}(G)$, is defined as the minimum number of distinct vertices which yield a k-rainbow neighbourhood. That is,

$$r^-_{\chi,k}(G) = \min\{r_{\chi,k}(G) : \text{over all chromatic colourings of } G\}.$$

Definition 2.4. The *k^+-rainbow neighbourhood number* of a graph G, denoted by $r^+_{\chi,k}(G)$, is defined as the maximum number of distinct vertices which yield a k-rainbow neighbourhood. That is,

$$r^+_{\chi,k}(G) = \max\{r_{\chi,k}(G) : \text{over all chromatic colourings of } G\}.$$

Note that $r^-_{\chi,k}(G)$ necessarily corresponds to a chromatic colouring in accordance with the rainbow neighbourhood convention. Note that if vertex v yields a k-rainbow neighbourhood it does not imply that v yields a $(k-1)$-rainbow neighbourhood. The aforesaid is true because $N_{(k-1)}[v] \subseteq N_k[v]$ and hence for any colouring, $|N_k[v]| \geq |N_{(k-1)}[v]|$. However, all vertices yield a $diam(G)$-rainbow neighbourhood. Hence, for a graph G of order n we have, $r_{\chi,diam(G)}(G) = n$. Also, if the vertex v yields a 1-rainbow neighbourhood, it yields a k-rainbow neighbourhood, where $2 \leq k \leq diam(G)$.

We now present a fundamental recursive lemma.

Lemma 2.5. *If the vertex $v \in V(G)$ yields a t-rainbow neighbourhood in graph G, it yields a k-rainbow neighbourhood for $t+1 \leq k \leq diam(G)$.*

Proof. Because $N_t[v] \subseteq N_k[v]$, $t+1 \leq k \leq diam(G)$, the result immediately follows by mathematical induction. □

Lemma 2.5 implies that $r_{\chi,k}(G) \geq r_\chi(G)$, because for a vertex v that yields a rainbow neighbourhood all $u \in N[v]$ yields a 2-rainbow neighbourhood if $N_2[u]$ exists. For now our interest lies in understanding the invariant $r_{\chi,2}(G)$ and determining $r^-_{\chi,2}(G)$.

Proposition 2.6. *The minimum* 2*-rainbow neighbourhood number for the following graphs, all of order n are:*

(i) For 2*-colourable graphs* G, $r^-_{\chi,2}(G) = n$.

(ii) For cycle C_3, $r^-_{\chi,2}(C_3) = 3$, *for* C_n, n *is odd and* $n \geq 5$ *we have:* $r^-_{\chi,2}(C_n) = 5$.

(iii) For wheels $W_n = K_1 + C_n$, $n \geq 3$:

$$r^-_{\chi,2}(W_n) = \begin{cases} 4, & \text{if } n = 3; \\ 6, & \text{if } n \geq 5,\ n \text{ is odd}; \\ n+1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. (i) Because $r^-_{\chi}(G) = n$ for 2-colourable graphs and $r^-_{\chi}(G) = r^-_{\chi,1}(G)$ it follows that, $r^-_{\chi,2}(G) = n$.

(ii) The first part, which states that $r^-_{\chi,2}(C_3) = 3$, is straight forward. Furthermore, because $r^-_{\chi,2}(G)$ corresponds to a chromatic colouring in accordance with the rainbow neighbourhood convention, such chromatic colouring of a cycle C_n, n is odd and $n \geq 5$ permits a single vertex to have colour c_3. The result follows immediately from the aforesaid.

(iii) Part (1) and Part(2) of (iii) are direct consequence of (ii). Furthermore, since an even cycle is 2-colourable, result (i) read together with the fact that the central vertex is adjacent to all cycle vertices implies that, $r^-_{\chi,2}(C_n) = n + 1$ if n is even. □

The results for many other cycle related graphs such as sun graphs, sunlet graphs, helm graphs and so on, can be derived easily through similar reasoning.

2.1. *k*-rainbow neighbourhood number of certain graph operations

Generally, graph operations are distinguished between *operations on a graph* G such as the complement graph, the line graph, the total graph, the power graph and so on. It results in a new graph or a derivative graph of the given graph G. Then there are those which are *operations between* graphs G and H. In this subsection the join and the corona of graphs G and H will be considered.

Theorem 2.7. *Let two graphs* G *and* H *of order* n_1, n_2 *respectively. Let* $G + H$ *and* $G \circ H$ *be the join and the corona of* G *and* H. *Then,*

(i) $r^-_{\chi,2}(G + H) = n_1 + n_2$.

(ii) (a) $r^-_{\chi,2}(G \circ H) = n_1 \cdot n_2$, *if* $\chi(H) \geq \chi(G) - 1$; *else,*

(b) $r^-_{\chi,2}(G \circ H) = r^-_{\chi,2}(G)$.

Proof. (i) Since, for any two vertices $v, u \in V(G + H)$ the distance is, $d(v, u) \leq 2$, the result is immediate.

(ii)(a): For $\chi(H) \geq \chi(G) - 1$ each $v \in V(G)$ yields a rainbow neighbourhood. Also for $u \in V(H)$, $d(v, u) \leq 2$, and therefore, the result is immediate.

(b): For the second part it is clear that all the vertices $v \in V(G)$ that yield a 2-rainbow neighbourhood in G will yield a 2-rainbow neighbourhood in $G \circ H$. Therefore, $r^-_{\chi,2}(G \circ H) \geq r^-_{\chi,2}(G)$.

It also follows that no vertex $w \in V(H)$ can yield a 2-rainbow neighbourhood in $G \circ H$. To show the aforesaid, assume that the vertex $w \in V(H)$ of the t-th copy of H joined to $v \in V(G)$ is a vertex yielding a 2-rainbow neighbourhood in $G \circ H$. It means that vertex w has at least one 2-reach neighbour for each colour c_i, $1 \leq i \leq \chi(H) < \chi(G) - 1$ as well as the neighbour v with, without loss of generality the colour $c(v) = c_{\chi(H)+1}$. Since, $c_{\chi(H)+1}$ can at best be the colour $c_{\chi(G)-1}$, the colour $c_{\chi(G)} \notin N[w]$ in $r^-_{\chi,2}(G \circ H)$ which is a contradiction. Therefore, $r^-_{\chi,2}(G \circ H) = r^-_{\chi,2}(G)$. □

3. On k-jump colouring

In this section, we introduce the main concept of study and the main results of this paper.

A path of length k also called a *k-path* is a path on $k+1$ vertices. Similarly, a cycle of length (or circumference) k, also called a *k-cycle* is a cycle on k vertices. If a graph G has $diam(G) = \ell$, then clearly it is possible for each vertex $v \in V(G)$ to find a vertex u which is at maximum distance $d(v,u) = \ell' \leq \ell$ and hence furthest away from v in G. We say u is a *ℓ'-jump away* from v. Consider a graph G for which $X \cup Y = V(G)$ and for which the vertices in set $X \subseteq V(G)$ are uncoloured and the vertices in set $Y \subseteq V(G)$ are coloured. We say G is partially coloured.

Definition 3.1. Consider a partially coloured graph G and let the set of uncoloured vertices be $X \subseteq V(G)$. A *k-jump colouring* in G with respect to v is the colouring in G such that of vertex $v \in X$ together with all vertices $u \in X$ for which $d(v,u) = k$ have the same colour.

The rainbow neighbourhood convention can naturally be extended to vertices at distance k. The derivative is called the rainbow k-neighbourhood convention. It is also clear that since G is finite, that colouring v say, c_1 and then colouring all vertices u_i at jump ℓ' from v also c_1 followed by repeating the colouring procedure for all ℓ'-jumps from vertices u_i and so on will exhaust in finite number of iterations and either, colour all vertices in G the colour c_1 or result in some vertices remaining uncoloured. It means that no vertex which remains uncoloured is at distance ℓ from any vertex coloured c_1. The aforesaid implies that the procedure is possible for a k-jump, $k \leq \ell$. For a graph G with $diam(G) = \ell$ and $0 \leq k \leq \ell$, consider the k-jump colouring procedure (k-JCP) as explained below:

k-JCP for a graph

Step-0: Let $\mathcal{V}_0 = \emptyset$.
Step-1: For $0 \leq k \leq diam(G)$, choose an arbitrary vertex $v_1 \in V(G)$. Let $\mathcal{V}_1 = \mathcal{V}_0 \cup \{v_1\}$ and colour v_1 and all uncoloured vertices $u_{1,i} \in V(G)$ at distance k

(k-jump) from v_1 if such vertices exist, the colour c_1. Repeat the procedure for all vertices $u_{1,i}$ to obtain all vertices $w_{1,i}$ to be coloured c_1 and so on. When this procedure is exhausted proceed to Step 2.
Step 2: If any uncoloured vertices exist, choose an arbitrary vertex v_2. Let $\mathcal{V}_2 = \mathcal{V}_1 \cup \{v_2\}$ and colour v_2 and all uncoloured vertices $u_{2,i}$ at distance k (k-jump) from v_2 if such vertices exist, the colour c_2. Repeat the procedure similar to that in Step 1 for all vertices $u_{2,i}$ to obtain all vertices $w_{2,i}$ to be coloured c_2, if such vertices exist and so on. When this procedure is exhausted proceed to Step 3.
Step-3: If possible proceed iteratively through the arbitrary choice of an uncoloured v_3 and update $\mathcal{V}_3 = \mathcal{V}_2 \cup \{v_3\}$ and colour corresponding k-jump vertices c_3, and so on, until the graph has a k-jump colouring which might not be proper.
Step-4: When this iterative procedure is exhausted, delete all edges between vertices u and v for which $c(u) = c(v)$.

On conclusion of Step-4, a proper colouring is obtained. Call the concluding set of vertices say, $\mathcal{V}_i$, a *k-string*. Note that it means that the graph permits a maximum of i colours in respect of the k-string $\mathcal{V}_i$. For the corresponding set of colours $\mathcal{C}$, we call the mapping $f_{\mathcal{V}_\rangle} : V(G) \mapsto \mathcal{C}$, a *k-jump colouring* of G in respect of $\mathcal{V}_i$. The *k-jump colouring number* of G, with respect to the rainbow k-neighbourhood convention, is defined to be, $\chi_{J(k)}(G) = j = |\mathcal{V}_j| = \max\{|\mathcal{V}_i| : f_{\mathcal{V}_i}(G); \text{a } k\text{-jump colouring of } G \text{ in respect of } \mathcal{V}_i\}$. It is easy to verify that $\chi_{J(2)}(C_9) = 1$, $\chi_{J(3)}(C_9) = 3$ and $\chi_{J(4)}(C_9) = 1$. Hence, in general there is no relation between $\chi_{J(k)}(G)$ and k per se. Also, there is no relation between the chromatic number $\chi(G)$ and the jump colouring number, $\chi_{J(k)}(G)$.

For $k = 0$ we have the jump string $\mathcal{V}_n = V(G)$ and $c(v) \neq c(u) \Leftrightarrow v \neq u$. It is called the *Type I primitive jump colouring*. For $k = 1$ the we have the k-string, $\mathcal{V}_1 = \{v\}$, $v \in V(G)$, $c(G) = c_1$. It is called the *Type II primitive jump colouring* which returns a null graph in Step 4 of the k-JCP.

Further throughout this section the bounds for a k-jump colouring, $2 \leq k \leq diam(G)$ will apply. A complete graph K_n, $n \geq 3$ only permits a k-jump colouring for $k = 0, 1$ and the 1-jump colouring always returns a null graph. It is easy to verify that a path P_n, $n \geq 3$ has $\chi_{J(k)}(P_n) = k$, $1 \leq k \leq n-1$. Because acyclic graphs are bipartite and hence 2-colourable, such graphs permit a 2-jump colouring without the deletion of any edges. It implies that the 2-jump colouring returns a chromatic 2-colouring. For 2-colourable graphs G, $\chi_{J(2)}(G) = \chi(G)$. It is easy to see that a 2-jump colouring returns a null graph for an odd cycle graph, meaning that all vertices are coloured c_1. We say that an odd cycle permits a *Type II primitive jump colouring* or *returns a null graph* in respect of a 2-jump colouring. We are now in a position to state and prove two of the main results of this study.

Theorem 3.2. *A non-trivial graph G returns a null graph in respect of a 2-jump colouring if and only if G contains an odd cycle (not necessarily an induced odd cycle).*

Proof. Say that for an odd cycle $C_m \subseteq G$ and $u, v \in V(C_m)$, $m \leq n$, a 2-path from u to v, if it exists, is *within* C_m. Similarly, say that a 2-path from u to v, $u \notin V(C_m)$, $v \in V(C_m)$ if it exists, is *into* C_m. Also, say that a 2-path from u to v,

$u \in V(C_m)$, $v \notin V(C_m)$ if it exists, is out of C_m. Consider a graph which contains an odd cycle, C_m, $m \leq n$. Here are two sub-cases to be considered.

(a) Assume that G has odd cycle C_m and the arbitrary vertex $v_1 \notin V(C_m)$. For any vertex $u \in V(C_m)$ a vu-path exists because G is connected. If the vu-path is odd then $c(v_1) = c(u) = c_1$. Without loss of generality, 2-jump colour the cycle to exhaustion, followed by 2-jump colouring the vu-path. It follows that $c(V(vu\text{-path}) \cup V(C_m))) = c_1$.

(b) If the vu-path is even then a vertex w which is adjacent to u exists and which does not lie on the vu-path. Extend to the vw-path which is odd and 2-jump colour similar to (a). It follows that $c(v_1) = c(w) = c_1$. Without loss of generality, 2-jump colour the cycle to exhaustion, followed by 2-jump colouring the vu-path. It follows that $c(V(vu - path) \cup V(C_m))) = c_1$.

Invoking the sub-cases (a), (b) together, the result follows by mathematical induction.

If a non-trivial graph G returns a null graph with respect to a 2-jump colouring, the result follows by logical deduction in that, from say v_j, the 2-jump colouring iteration must be along a combination of paths or even cycles (not necessarily induced even cycles). □

The proof of Theorem 3.2 makes a generalized result for cycles possible. Note that for the discussion of cycles and chorded cycles and certain cycle related graphs the bounds on k are relaxed for convenience to, $2 \leq k \leq n$. For graphs in general a similar relaxation is possible by substituting modulo bounds on $diam(G)$.

Theorem 3.3. *Let $k \geq 3$. A cycle C_n, returns a null graph in respect of a k-jump colouring if and only if $n \neq t \cdot k$ where $t \in \mathbb{N}$.*

Proof. For a cycle C_n, $n \geq 3$ and by relaxed convention, $2 \leq k \leq n$, all paths from vertices u to v are within C_n. Also, for any n-path from u to v we have $u = v$. Similarly, for any k for which n is divisible by k, a $(k \cdot \frac{n}{k})$-path from u to v implies $u = v$. Therefore, for any k for which n is not divisible by k, Step 1 will exhaust all vertices with colouring c_1. Hence, the result. □

The following two corollaries are direct consequences of Theorem 3.3.

Corollary 3.4. *For $k_1, k_2, k_3, \ldots, k_s$ and $k_i \geq 3$, let the least common multiple, $LCM(k_1, k_2, k_3, \ldots, k_s) = \ell$. A cycle C_n, returns a null graph in respect of a k_i-jump colouring if and only if $n \neq t \cdot \ell$ where $t \in \mathbb{N}$.*

Corollary 3.5. *For $k_1, k_2, k_3, \ldots, k_s$ and $k_i \geq 3$, let the least common multiple, $LCM(k_1, k_2, k_3, \ldots, k_s) = \ell$. A cycle C_n, has $\chi_{J(k_i)}(C_n) = 1$ or k_i in respect of a k_i-jump colouring.*

It is observed that cycles has the extremal edge deletion properties i.e. either all edges are deleted for a k-jump colouring or no edges are deleted.

3.1. Investigating chorded cycles, slings graphs and *p*-sling graphs

From Corollary 3.4 a general result for chorded cycles follows.

Theorem 3.6. *For $k_1, k_2, k_3, \dots, k_s$ and $k_i \geq 3$ let the least common multiple, $LCM(k_1, k_2, k_3, \dots, k_s) = \ell$. A chorded cycle $C_n^{\circledast}$, $n \geq 4$ returns a null graph in respect of a k_i-jump colouring.*

Proof. From Corollary 3.4, a cycle C_{m_1} and C_{m_2} must both have $m_1 = t_1 \cdot \ell$, $t_1 \in \mathbb{N}$ and $m_2 = t_2 \cdot \ell$, $t_2 \in \mathbb{N}$ for each to permit a k_i-jump colouring, $1 \leq i \leq s$. Obtain a chorded cycle $C_n^{\circledast}$ by merging two edges, one each from C_{m_1} and C_{m_2}. It is easy to verify that $n = m_1 + m_2$ is not divisible by at least one k_i, $1 \leq i \leq s$. From Corollary 3.3 it then follows that $C_n^{\circledast}$ will return a null graph. Though immediate induction the resut follows for any chorded graph $C_n^{\circledast}$, $n \geq 4$. □

An immediate consequence of Theorem 3.6 is that Theorem 3.2 cannot be generalized for k-jump colouring for $k \geq 3$. Hence, for k-jump colouring, $k \geq 3$ only graphs with edge-disjoint holes (induced cycles) can be investigated.

Consider a cycle C_n, $n \geq 3$ and a path P_{m+1}, $m \geq 1$ (also called a m-path). The graph obtained by merging an end vertex of the path with a vertex of C_n is called a *sling graph* and is denoted by $S_{n,m+1}$. We begin with an important lemma.

Lemma 3.7. *Let the vertices of a m-path be labeled, $v_1, v_2, v_3, \dots, v_{m+1}$. For the cycle C_n, $n = t \cdot \ell$, $t = 1, 2, \dots,$ and $\ell = LCM(1, 2, 3, \dots, m)$, construct the sling graph $S_{n,m+1}$ by merging v_1 with a vertex on C_n. For $2 \leq k \leq m$ initiate (Step 1 of the k-JCP) a k-jump colouring from vertex v_{k+1}. The sling graph $S_{n,m+1}$ permits such k-jump colouring.*

Proof. Initiating a k-jump colouring from vertex v_{k+1} in accordance with the conditions set, clearly colours vertex v_1 to be, $c(v_1) = c_1$. Proceeding along the cycle without returning a null graph follows from Corollary 3.4. □

A p-sling graph has paths, P_{m_i+1}, $1 \leq i \leq p$, each linked to a common cycle in accordance to the construction of a sling graph. It is denoted, $S^{1 \leq i \leq p}_{n,m_i+1}$. In this sense a sling graph is a 1-sling graph.

Assume without loss of generality that $m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_p$. Label the vertices of the respective paths to be, $v_{i,1}, v_{i,2}, v_{i,3}, \dots, v_{i,m_i}$, $1 \leq i \leq p$. The next lemma generalizes Lemma 3.7.

Lemma 3.8. *For a cycle C_n, $n = t \cdot \ell$, $t \in \mathbb{N}$, and $\ell = LCM(1, 2, 3, \dots, m_p)$, construct the p-sling graph $S^{1 \leq i \leq p}_{n,m_i+1}$ by merging $v_{i,1}$ with some vertex on C_n. For $2 \leq k \leq m_p$ initiate (Step-1 of the k-JCP), a k-jump colouring from any vertex $v_{i,k+1}$. The p-sling graph $S^{1 \leq i \leq p}_{n,m_i+1}$ permits such k-jump colouring if all paths P_{m_j+1}, $j \neq i$ are merged with some vertex on C_n which is coloured c_1.*

Proof. Note that ℓ is divisible by m_i, $1 \leq i \leq p$. The result follows trivially from Lemma 3.7 by induction on the number of paths. □

A trivial illustration of Lemma 3.8 is the observation that a thorny cycle $C_n^\star$, n is even, permits a 2-jump colouring.

Theorem 3.9. *If a graph G which permits a k-jump colouring then $v \in V(G)$ yields a $(k-1)$-rainbow neighbourhood.*

Proof. Consider any vertex v and any $(k-1)$-path P_k leading from v. Label the vertices on P_k to be, $v_1, v_2, v_3, \dots, v_k$. Since for any pair of distinct vertices say, v_i, v_j the distance, $d(v_i, v_j) \leq k-1$ it follows that $c(v_i) \neq c(v_j)$. Therefore, all $c(P_k) = \mathcal{C}$. Hence, the result. □

Theorem 3.9 implies that if G permits a k-jump colouring, then $r^-_{\chi,(k-1)}(G) = |V(G)|$.

Theorem 3.10. *For $2 \leq k \leq diam(G)$, the k-jump colouring of G returns a null graph if G contains a cycle C_m (not necessarily induced) of length, $m \neq t \cdot k; t = 1, 2, 3 \dots$.*

Proof. The result follows by similar reasoning to that found in the proof of Theorem 3.2. □

3.2. On acyclic graphs

With some understanding of the importance of path, cycles and chorded cycles two general results can be stated. We begin with two important lemmas.

Lemma 3.11. *If an acyclic graph G with $diam(G) = \ell$, permits a k-jump colouring for $2 \leq k \leq diam(G)$ such colouring is unique (up to isomorphism).*

Proof. Note that for an acyclic graph a path from v to v in G exists and is unique. Hence, Theorem 3.9 read together with with any injective mapping $f : \mathcal{C} \mapsto \mathcal{C}$ implies up to isomorphism that the k-jump colouring is unique. □

Lemma 3.11 implies that a k-jump colouring may initiate from any $v \in V(G)$.

Lemma 3.12. *If an acyclic graph G is k-jump colourable, $2 \leq k \leq diam(G)$ then G is tk-jump colourable for $2 \leq tk \leq diam(G)$.*

Proof. Let G be k-jump colourable, $2 \leq k \leq diam(G)$. Note that for an acyclic graph G a path from v to v in G exists and is unique. Consider a vertices v, u, w such that $d(v, u) = k$ and $d(u, w) = (t-1)k$. Clearly $c(v) = c(u) = c(w)$. Hence, in a $t\cdot$-jump colouring, $c(v) = c(w) \neq c(u)$. The aforesaid holds for all vu-paths and all uw-paths in G. Therefore, the result follows through immediate induction. □

Theorem 3.13. *An acyclic graph G with $diam(G) = \ell$, permits a k-jump colouring for $k = 2, 3, 4, \dots, \ell$.*

Proof. If G is acyclic the result for $k = 2, 3, 5, 7, \ldots, p \leq diam(G)$, p is prime follows by the same reasoning as for $d(v, u) = k$ and $d(u, w) = (t-1)k$ in the proof of Lemma 3.12. For the multiples of the corresponding prime jumps, the result is a direct consequence of Lemma 3.12. □

We can now state and prove results for the elementary graph operations, join and corona. First, the result for the corona $P_n \circ H$ will be stated.

Remark 3.14. Heuristic reasoning suggests that in Step i of the k-JCP the vertex v_i should be such that an uncoloured vertex u at maximum distance from v_i (furthest away) exists. So for such v_1 such u always exists at distance $d(v_1, u) = diam(G)$.

Theorem 3.15. *The join $G+H$ of two graphs G and H returns a Type II primitive jump colouring.*

Proof. Since $diam(G + H) = 2$ we only consider $k = 2$. Without loss of generality consider vertices $v, u \in V(G)$ and vertex $w \in V(H)$. Since $d(v, u) \geq 2$ in G there exists a cycle from v to u to w to v in $G+H$ with length (circumference) at least 4. If the cycle length is odd the result follows from Theorem 3.2. If the cycle length is even then since there exists a vertex v' adjacent to v on a vu-path in G, there exists an odd cycle from v' to u to w to v' in $G + H$. Similarly the result follows from Theorem 3.2. □

For the corona of graphs some special graph classes will be discussed.

Proposition 3.16. *(i) For a path P_n, $n \geq 4$ and graph H of order m, the corona $P_n \circ G$ is k-jump colourable, if $2 \leq k \leq n + 1$ and $k \neq 3$. A 3-jump colouring returns a Type-II trivial jump colouring.*
(ii) For P_n, $n = 1, 2, 3$, 2-jump colourings are returned.

Proof. (i) Consider any path P_n, $n \geq 4$ and any graph H of order m. Two sub-cases must be considered.

(a) Let $k = 3$. In accordance with the rainbow k-neighbourhood convention and without loss of generality begin Step 1 of the k-JCP by selecting any $u \in V(H_1)$. The first iteration results in $c(u) = c(v_3) = c(V(H_2) = c_1$. The second iteration results in $c(v_4) = c(V(H_3) = c_1$ followed by, $c(v_1) = c_1$. Immediate iterative exhaustion shows that a Type II trivial jump colouring returns.

(b) Begin by considering the case of maximum k-jump. Clearly $diam(P_n \circ H) = n + 1$. Let the path vertices be $v_1, v_2, v_3, \ldots, v_n$ and the corresponding corona'd copies of H be labeled $H_1, H_2, H_3, \ldots, H_n$. In accordance with the rainbow k-neighbourhood convention and without loss of generality begin Step 1 of the k-JCP by selecting any $u \in V(H_1)$. Step 1 results in $c(u) = c(V(H_n) = c_1$. Similarly, Step 2 results in $c(V(H_1) = c_1$. Hereafter, for $1 \leq i, j \leq n$ and $2 \leq j' \leq n - 2$, all pairs of vertices v_i, v_j, $v_i u_{j'}$, $u_{j'} \in V(H_{j'}$ and pair $u_{i'} u_{j'}$ all distances are at most, $n - 1$. Hence, k-JCP results in each vertex in $\{v_i : 1 \leq i \leq n\} \bigcup\limits_{j=2}^{n-1} V(H_j)$ to be distinctly coloured. The result follows for $k = n + 1$. By immediate inverse induction the result follows for $k \neq 3$.

(ii)(a) For $k = 2$, and applying k-JCP to $P_1 \circ H_1$ returns a 2-jump colouring. $P_2 \circ H$ returns a 2-jump colouring. Also, $P_3 \circ H$ returns a 2-colouring.

(b) For $k = 3$, and applying k-JCP to $P_2 \circ H$ returns a 2-jump colouring with 3 colours needed. $P_3 \circ H$ returns a 2-jump colouring with all vertices except v_2 coloured c_1 and $c(v_2) = c_2$. □

Theorem 3.17. *Consider a cycle C_n, $n \geq 3$. For all graphs H, of order m the k-colourability of the corona, $C_n \circ H$ is equivalent to the k-colourability of the thorny graph $C_N^\star$ with m thorns (pendant vertices) attached to each vertex, $v \in V(C_n)$.*

Proof. The adjacency properties of H are irrelevant in $C_n \circ H$ in that for $v, u \in V(H)$ the distance reduces to $d(v, u) \leq 2$. So for the direct application of Lemma 3.8, $C_n \circ H$ can be treated as if, equivalent to a thorny cycle. □

3.3. On modified k-jump colouring

Consider a cycle C_n, $n \geq 3$ which for some $2 \leq k \leq n-1$ is not k-jump colourable. Certainly P_n is k-jump colourable. Now allocate any colour $c_i \in \mathcal{V}_k$, $c_i \neq c(v_n)$ or a new colour c_{k+1} to vertex v_n in accordance to a proper colouring. If colour c_{k+1} is needed, then update, $\mathcal{V}_{k+1} = \mathcal{V}_k \cup \{c_{k+1}\}$. The $(k+1)$-string colouring of C_n is called a *modified k-jump colouring* of C_n. Now similarly for P_n which has been k-jump coloured, it is possible to recolour a vertex v_i with $c_j \in \mathcal{V}_k$ or with c_{k+1} to add the edge v_iv_j. From Theorem 3.12 it follows that for a graph G and $2 \leq k \leq diam(G)$, any spanning tree T of G is k-jump colourable. Therefore it is possible to obtain a modified k-jump colouring of G by iteratively applying the colouring principles set out. Clearly the modified modified k-jump colouring obtained in respect of a particular spanning tree is minimal. The minimum colours in a modified k-jump colouring over all distinct spanning trees is the *optimal modified k-jump colouring* of G.

Theorem 3.18. *For any graph G and $2 \leq k \leq diam(G)$, an optimal modified k-jump colouring exists.*

Proof. For any graph G and any spanning tree T we have, $diam(G) \leq diam(T)$. Hence, $2 \leq k \leq diam(G) \Rightarrow 2 \leq k \leq diam(T)$. Therefore, from Theorem 3.15, it follows that all possible distinct spanning trees are k-jump colourable and therefore permits a corresponding modified k-jump colouring. By the principle of well-ordering of integers a minimum number of colours exists over all minimal modified k-jump colourings of G. □

4. Conclusion

In this paper, we introduced the notion of the k-rainbow neighbourhood number of a graph G. There is a wide scope for determining the minimum and maximum k-rainbow neighbourhood numbers for many other classes of graphs. In terms of graph operations on and between graphs, investigations in respect of the complement of a graph, the line graph, the jump graph, the total graph etc. seem to be promising. Studies in this area on graph products such as the Cartesian product, the tensor product, the strong product and the lexicographical product of various graph classes also seem to be worthy research directions.

In this article, we also introduced a new notion of a k-jump colouring of graphs. Further studies on various aspects of k-jump colouring remains open. Note from Proposition 3.16 that for the $(n+1)$-jump colouring, where $n \geq 4$, $\chi_{J(n+1)}(P_n \circ H) = (n+1)+m(n-2)$. Determining the values of $\chi_{J(k)}(P_n \circ H)$, $0 \leq k \leq diam(P_n \circ H)$ is another open problem in this area.

Complexity analysis with respect to the optimal modified k-jump colouring of a graph G is considered to be worthy research. There are good algorithms to find the spanning trees such as Prim's algorithm for edge weighted graphs and Kruskal's algorithm. It is also well-known that the number of distinct spanning trees of a graph denoted by, $t(G)$ can be calculated by using the Kirchhoff matrix-tree theorem.

All the above mentioned facts show that there is a wide scope for further investigations in this direction.

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