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BOUNDS FOR SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS AND ULAM STABILITY

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Abstract. We obtain Gronwall type bounds for the solutions of a linear system of differential equations. As applications we get results on Ulam stability for linear differential equations and linear systems of differential equations.

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1. INTRODUCTION

Let $I=[a,b)$, $a\in\mathbb{R}$, $b\in\mathbb{R}\cup\{+\infty\}$, $a<b$, be an interval, $\mathbb{K}$ one of the fields $\mathbb{R}$ or $\mathbb{C}$, $A=(a_{ij})_{1\le i,j\le n}$, $B=(b_1,\dots,b_n)^T$, $a_{ij},b_i\in C(I,\mathbb{K})$, $1\le i,\ j\le n$. We consider linear systems of differential equations of the form

$$y^{'}(x)=A(x)y(x)+B(x),\quad x\in I, \tag{1.1}$$

where the unknown is $y\in C^1(I,\mathbb{K}^n)$, $y=(y_1,\dots,y_n)^T$.

We are looking for bounds of solutions for the equation (1.1) and study its Ulam stability. In what follows we suppose that $\mathbb{K}^n$ is endowed with the Euclidean norm $\|\cdot\|$ defined as usual by

$$\|u\|=\sqrt{\langle u,u\rangle},\qquad u\in\mathbb{K}^n,$$

and the inner product $\langle,\rangle$ given by

$$\langle u,v\rangle=\sum_{k=1}^{n}u_k\cdot\overline{v_k}$$

for $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)$ in $\mathbb{K}^n$.

In this paper we present a new approach to Ulam stability of linear differential equations. We obtain bounds of solutions for a linear system of differential equations by using a result on boundedness of solutions for a Bernoulli type differential inequality.

Let $X = (x_{ij})_{1\le i,j\le n}$ be a matrix with entries in $\mathbb{K}$. The induced matrix norm $\sigma(X)$ is the spectral norm, defined as the square root of the largest eigenvalue of the matrix X^*X, where X^* denotes the conjugate transposed of X.

The Frobenius norm of X, denoted by $\|X\|_F$, is given by

$$\|X\|_F = \left(\sum_{i,j=1}^{n} |x_{ij}|^2\right)^{\frac{1}{2}}$$

and is an upper bound of $\sigma(X)$.

Recall that the equation (1.1) is called stable in Ulam sense if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in C^1(I,\mathbb{K}^n)$ satisfying

$$\left\|y'(x) - A(x)y(x) - B(x)\right\| \le \varepsilon, \qquad x \in I, \tag{1.2}$$

there exists a solution $z \in C^1(I,\mathbb{K}^n)$ of (1.1) such that

$$\|y(x) - z(x)\| \le \delta, \qquad x \in I.$$

If in the previous definition ε and δ are replaced by some functions $\varphi, \psi : I \to \mathbb{R}_+$, the equation (1.1) is called generalized Ulam stable.

A function $y \in C^1(I,\mathbb{K}^n)$ which satisfies (1.2) is called approximate solution of (1.1). So, generally we say that equation (1.1) is stable (or generalized stable) in Ulam sense if for every approximate solution of the equation there exists an exact solution close to it. For more details on Ulam stability see [2–4, 8].

The first result on Ulam stability of differential equations was given by M. Obłoza, who investigated also the relations between Ulam and Lyapunov stability [9, 10]. T. Miura, S. Miyajima and S.E. Takahasi [7] and S.E. Takahasi, H. Takagi, T. Miura and S. Miyajima [14] obtained results on Ulam stability for the first order linear differential operators and the higher order linear differential operators with constant coefficients. Some extensions of the results mentioned above were obtained by D. Popa and I. Raşa for the linear differential operators of order n with constant coefficients [11–13]. Recall also the results obtained by S.M. Jung on the stability of various differential equations [5, 6].

It should be mentioned that the results presented above concern linear differential equations and operators with constant coefficients or particular forms of linear differential operators with nonconstant coefficients.

In this paper we obtain an Ulam stability result for a linear system of differential equations with nonconstant coefficients such that the difference of the approximate and the exact solution is controlled by a function depending only on the norm of the matrix of the system. We present several examples and applications of this general result.

2. Bounds of solutions and Ulam stability

Let $\mathbb{R}_+$ denote the interval $[0,+\infty)$. The following result concerning bounds of solutions of a Bernoulli type differential inequality will be useful for the main result of this paper.

Lemma 1. *Let $f,g \in C(I,\mathbb{R}_+)$ and $u \in C^1(I,\mathbb{R}_+)$ be such that*

$$u^{'}(x) \le 2f(x)u(x) + 2g(x)\cdot\sqrt{u(x)}, \quad x \in I.$$

Then

$$\sqrt{u(x)} \le \sqrt{u(a)}e^{F(x)-F(a)} + e^{F(x)}\int_a^x g(t)e^{-F(t)}dt, \quad x \in I, \tag{2.1}$$

where F is an antiderivative of f, i.e., $F^{'} = f$ on I.

Proof. Suppose first that $u(x) > 0$ for all $x \in I$ and let $z(x) = \sqrt{u(x)}$, $x \in I$. Then $z^{'}(x) = \frac{u^{'}(x)}{2\sqrt{u(x)}}$, $x \in I$, and

$$z^{'}(x) - f(x)z(x) \le g(x), \quad x \in I.$$

A multiplication by $e^{-F(x)}$ leads to

$$\left(z(x)e^{-F(x)}\right)^{'} \le g(x)e^{-F(x)}, \quad x \in I.$$

By integration on $[a,x]$, $x \in I$, we get

$$z(x)e^{-F(x)} - z(a)e^{-F(a)} \le \int_a^x g(t)e^{-F(t)}dt,$$

or

$$z(x) \le z(a)e^{F(x)-F(a)} + e^{F(x)}\int_a^x g(t)e^{-F(t)}dt, \quad x \in I,$$

which leads to (2.1). Now let $u(x) \ge 0$ for $x \in I$. Let $n \in \mathbb{N}$, and consider the function

$$u_n(x) = u(x) + \frac{1}{n}, \qquad x \in I.$$

Since $u_n^{'}(x) = u^{'}(x)$, $x \in I$, we get

$$u_n^{'}(x) \le 2f(x)u_n(x) + 2g(x)\sqrt{u_n(x)}, \qquad x \in I.$$

According to the first part of this proof it follows that

$$\sqrt{u_n(x)} \le \sqrt{u_n(a)}e^{F(x)-F(a)} + e^{F(x)}\cdot\int_a^x g(t)e^{-F(t)}dt, \qquad x \in I.$$

Letting $n \to \infty$ in the previous relation we get (2.1).

The lemma is proved. □

Theorem 1. *Suppose that $\sigma(A(t))$ is locally integrable on I, and let $F(x) = \int_a^x \sigma(A(t))dt$, $x \in I$. Then for every $y \in C^1(I, \mathbb{K}^n)$ satisfying (1.1) the following relation holds:*

$$\|y(x)\| \leq \|y(a)\| e^{F(x)} + e^{F(x)} \cdot \int_a^x \|B(t)\| e^{-F(t)} dt, \quad x \in I.$$

Proof. Let $y \in C^1(I, \mathbb{K}^n)$ be such that

$$y'(x) = A(x)y(x) + B(x), \quad x \in I.$$

A scalar multiplication of the previous relation by $y(x)$ leads to

$$\left\langle y'(x), y(x) \right\rangle = \langle A(x)y(x), y(x) \rangle + \langle B(x), y(x) \rangle, \quad x \in I.$$

It follows that

$$\begin{aligned} \left| \left\langle y'(x), y(x) \right\rangle \right| &\leq |\langle A(x)y(x), y(x) \rangle| + |\langle B(x), y(x) \rangle| \\ &\leq \|A(x)y(x)\| \cdot \|y(x)\| + \|B(x)\| \cdot \|y(x)\| \\ &\leq \sigma(A(x)) \cdot \|y(x)\|^2 + \|B(x)\| \cdot \|y(x)\|, \quad x \in I. \end{aligned}$$

Taking account of

$$\left(\|y(x)\|^2 \right)' = 2\Re \left\langle y'(x), y(x) \right\rangle, \quad x \in I,$$

it follows that the function u defined by

$$u(x) = \|y(x)\|^2, \quad x \in I,$$

satisfies the Bernoulli type differential inequality

$$u'(x) \leq 2\sigma(A(x)) \cdot u(x) + 2\|B(x)\| \cdot \sqrt{u(x)}, \quad x \in I.$$

Now the conclusion is a simple consequence of Lemma 1. □

Remark 1. It is obvious that Theorem 1 holds with

$$F(x) = \int_a^x M(t)dt, \quad x \in I,$$

where $M(x)$ is any locally integrable upper bound of $\sigma(A(x))$ on I.

Remark 2. If A is a constant matrix with entries in $\mathbb{K}$, $\lambda = \sigma(A)$, $\lambda > 0$, then for every solution $y \in C^1(I, \mathbb{K}^n)$ of (1.1) the following estimation holds

$$\|y(x)\| \leq e^{\lambda(x-a)} \left(\|y(a)\| + \frac{1}{\lambda}(1 - e^{-\lambda(x-a)}) \cdot \sup_{x \in I} \|B(x)\| \right).$$

Proof. $F(x) = \lambda(x-a)$, $x \in I$, therefore in view of Theorem 1 we get

$$\|y(x)\| \leq e^{\lambda(x-a)} \left(\|y(a)\| + \int_a^x \|B(t)\| e^{-\lambda(t-a)} dt \right)$$

$$\leq e^{\lambda(x-a)}\left(\|y(a)\|+\sup_{x\in I}\|B(x)\|\cdot\int_a^x e^{-\lambda(t-a)}dt\right)$$
$$= e^{\lambda(x-a)}\left(\|y(a)\|+\frac{1}{\lambda}(1-e^{-\lambda(x-a)})\sup_{x\in I}\|B(x)\|\right), \quad x\in I.$$

□

The next result concerns generalized Ulam stability of the equation (1.1).

Theorem 2. *Let $\phi : I \to \mathbb{R}_+$ be a continuous function, $F(x) = \int_a^x M(t)dt$, where $M(x)$ is any locally integrable upper bound of $\sigma(A(x))$ on I.*
Then for every $y \in C^1(I,\mathbb{K}^n)$ satisfying

$$\left\|y'(x)-A(x)y(x)-B(x)\right\| \leq \phi(x), \quad x\in I, \tag{2.2}$$

there exists a unique solution $z \in C^1(I,\mathbb{K}^n)$ of (1.1) such that

$$\|y(x)-z(x)\| \leq e^{F(x)}\int_a^x \phi(t)e^{-F(t)}dt, \quad x\in I. \tag{2.3}$$

Proof. Existence. Let $y \in C^1(I,\mathbb{K}^n)$ satisfying (2.2) and define

$$h(x) := y'(x)-A(x)y(x)-B(x), \quad x\in I.$$

Let z be the unique solution of (1.1) with $z(a) = y(a)$. Then the function $w : I \to \mathbb{R}$ given by

$$w(x) = y(x)-z(x), \quad x\in I,$$

satisfies

$$w'(x) = A(x)w(x)+h(x),$$
$$w(a) = 0.$$

Taking account of Theorem 1 and Remark 1 it follows that

$$\|w(x)\| \leq e^{F(x)}\int_a^x \|h(t)\|e^{-F(t)}dt$$
$$\leq e^{F(x)}\int_a^x \phi(t)e^{-F(t)}dt, \quad x\in I.$$

Uniqueness. Suppose that for an $y \in C^1(I,\mathbb{K}^n)$ satisfying (2.2) there exist two solutions $z_1, z_2 \in C^1(I,\mathbb{R})$ of (1.1) such that

$$\|y(x)-z_k(x)\| \leq e^{F(x)}\int_a^x \phi(t)e^{-F(t)}dt, \quad x\in I, \quad k=1,2.$$

Then

$$\|z_1(x)-z_2(x)\| \leq \|z_1(x)-y(x)\|+\|y(x)-z_2(x)\|$$
$$\leq 2e^{F(x)}\int_a^x \phi(t)e^{-F(t)}dt, \quad x\in I.$$

It follows $z_1(a) = z_2(a)$ and, taking account of the existence and uniqueness theorem for the Cauchy problem of (1.1), we get

$$z_1(x) = z_2(x), \quad x \in I.$$

The theorem is proved. □

Corollary 1. *Let* $\varepsilon > 0$*. Then for every* $y \in C^1(I, \mathbb{K}^n)$ *satisfying the relation*

$$\left\| y'(x) - A(x)y(x) - B(x) \right\| \le \varepsilon \sigma(A(x)), \qquad x \in I,$$

there exists a unique solution $z \in C^1(I, \mathbb{K}^n)$ *of (1.1) such that*

$$\|y(x) - z(x)\| \le \varepsilon (e^{\int_a^x \sigma(A(t))dt} - 1), \qquad x \in I.$$

Proof. Let $M(x) = \sigma(A(x))$ and $\phi(x) = \varepsilon \sigma(A(x))$, $x \in I$, in Theorem 2. Then

$$\begin{aligned} e^{F(x)} \int_a^x \phi(t) e^{-F(t)} dt &= \varepsilon e^{F(x)} \int_a^x F'(t) e^{-F(t)} dt \\ &= \varepsilon (e^{F(x)} - 1) = \varepsilon (e^{\int_a^x \sigma(A(t))dt} - 1), \qquad x \in I, \end{aligned}$$

and the result holds in view of Theorem 2. □

Remark 3. Concerning the upper bound in (2.3), it is easy to see that

$$\int_a^x \phi(t) dt \le e^{F(x)} \int_a^x \phi(t) e^{-F(t)} dt \le e^{F(x) - F(a)} \int_a^x \phi(t) dt.$$

3. Applications

In this section we give some applications of the previous results to the Ulam stability of linear differential equations with constant coefficients and linear differential equations of order two with non-constant coefficients.

Theorem 3. *Let* A *be a constant matrix with entries in* $\mathbb{K}$*, with* $\lambda = \sigma(A) > 0$ *and* $\varepsilon > 0$*. Then for every* $y \in C^1(I, \mathbb{K}^n)$ *satisfying*

$$\left\| y'(x) - Ay(x) - B(x) \right\| \le \varepsilon, \quad x \in I, \tag{3.1}$$

there exists a unique $z \in C^1(I, \mathbb{K}^n)$ *satisfying (1.1) such that*

$$\|y(x) - z(x)\| \le \frac{\varepsilon}{\lambda} (e^{\lambda(x-a)} - 1), \quad x \in I. \tag{3.2}$$

Proof. Let $M(x) = \sigma(A)$, $\phi(x) = \varepsilon$, $x \in I$ in Theorem 2. Then $F(x) = \lambda(x - a)$, $x \in I$. Therefore, for every $y \in C^1(I, \mathbb{K}^n)$ satisfying (3.1) there exists a unique solution z of (1.1) such that

$$\|y(x) - z(x)\| \le e^{\lambda(x-a)} \cdot \int_a^x \varepsilon e^{-\lambda(t-a)} dt = \frac{\varepsilon}{\lambda} \left(e^{\lambda(x-a)} - 1 \right), \quad x \in I.$$

□

We give in what follows a result on Ulam stability for a linear differential equation of order two with non-constant coefficients. More precisely, we consider the equation

$$y''(x)+f(x)y'(x)+g(x)y(x)=0, \quad x\in I, \tag{3.3}$$

where $f,g\in C(I,\mathbb{K})$ and $y\in C^2(I,\mathbb{K})$.

Theorem 4. *Let $\varphi : I\to\mathbb{R}_+$ be a continuous function. Then for every $y\in C^2(I,\mathbb{K})$ satisfying*

$$\left|y''(x)+f(x)y'(x)+g(x)y(x)\right|\le\varphi(x), \quad x\in I,$$

there exists a unique solution $z\in C^2(I,\mathbb{K})$ of (3.3) with the property

$$|y(x)-z(x)|\le e^{F(x)}\int_a^x\varphi(t)e^{-F(t)}dt, \quad x\in I,$$

where $F(x)=\int_a^x\sqrt{|f(t)|^2+|g(t)|^2+1}dt, x\in I$.

Proof. Let $y\in C^2(I,\mathbb{K})$ satisfying (3.3) and define

$$y''(x)+f(x)y'(x)+g(x)y(x)=h(x), \quad x\in I.$$

Let also $y_1=y$, $y_2=y'$, $u=(y_1\ y_2)^T$, $B=(0\ h)^T$.

$$A(x)=\begin{pmatrix}0 & 1\\ -g(x) & -f(x)\end{pmatrix}, \quad x\in I.$$

Then

$$u'(x)=A(x)u(x)+B(x), \quad x\in I. \tag{3.4}$$

We have

$$A^*(x)A(x)=\begin{pmatrix}|g(x)|^2 & \overline{g(x)}f(x)\\ \overline{f(x)}g(x) & 1+|f(x)|^2\end{pmatrix}.$$

The eigenvalues $\lambda_1(x)$, $\lambda_2(x)$ of $A^*(x)A(x)$ are the roots of the polynomial

$$P(\lambda)=\lambda^2-(|f(x)|^2+|g(x)|^2+1)\lambda+|g(x)|^2,$$

therefore

$$\sigma(A(x))=\max\left\{\sqrt{\lambda_1(x)},\sqrt{\lambda_2(x)}\right\}.$$

On the other hand $P(0)\ge 0$, $P(1)\le 0$, $P(|f(x)|^2+|g(x)|^2+1)\ge 0$, so it follows that

$$M(x)=\sqrt{|f(x)|^2+|g(x)|^2+1}$$

is an upper bound of $\sigma(A(x))$.

Remark that $M(x)$ is in fact the Frobenius norm of the matrix $A(x)$.

From (3.4) we get

$$\left\|u'(x)-A(x)u(x)\right\|=\|B(x)\|=|h(x)|\le\varphi(x), \quad x\in I,$$

so according to Theorem 2 and Remark 1 it follows that there exists $w = (w_1\ w_2)^T \in C^1(I, \mathbb{K}^2)$ such that

$$w'(x) = A(x)w(x), \quad x \in I, \tag{3.5}$$
$$\|u(x) - w(x)\| \le e^{F(x)} \cdot \int_a^x \varphi(t)e^{-F(t)}dt, \quad x \in I.$$

Relation (3.5) leads to $w_1''(x) + f(x)w_1'(x) + g(x)w(x) = 0, x \in I$. Let $z(x) = w_1(x)$, $x \in I$. Then

$$\begin{aligned} |y(x) - z(x)| &\le \|u(x) - w(x)\| \\ &\le e^{F(x)} \int_a^x \varphi(t)e^{-F(t)}dt, \quad x \in I. \end{aligned}$$

□

Remark 4. In the setting of Theorem 4, let $f(x) = 0$, $x \in I$. Then

$$\lambda_1(x) = |g(x)|^2, \quad \lambda_2(x) = 1, \quad x \in I.$$

Consequently,

$$\sigma(A(x)) = \max\{|g(x)|, 1\}.$$

If, in addition, $|g(x)| \le 1$, then

$$\sigma(A(x)) = 1, \quad x \in I.$$

In such a situation, the conclusion of Theorem 4 holds with

$$F(x) = \int_a^x \max\{|g(t)|, 1\}dt,$$

respectively with

$$F(x) = x - a, \quad x \in I.$$

We present now some applications of Theorem 4.

Example 1. Let $\varepsilon > 0$ and $y \in C^2(I, \mathbb{K})$ satisfying

$$\left|y''(x) + (\sin x)y'(x) + (\cos x)y(x)\right| \le \varepsilon, \qquad x \in I.$$

Then there exists a $z \in C^2(I, \mathbb{K})$ with properties

$$z''(x) + (\sin x)z'(x) + (\cos x)z(x) = 0, \qquad x \in I$$

and

$$|y(x) - z(x)| \le \frac{\varepsilon}{\sqrt{2}}(e^{\sqrt{2}(x-a)} - 1), \qquad x \in I.$$

Proof. $F(x) = \int_a^x \sqrt{\sin^2 t + \cos^2 t + 1}dt = \sqrt{2}(x-a), x \in I$, and the result is a simple consequence of Theorem 4. In fact, also in this case it is easy to find $\lambda_{1,2}(x) = 1 \pm \sin x$, so that

$$\sigma(A(x)) = \sqrt{1 + |\sin x|}, \quad x \in I.$$

□

In the case of the linear differential equation of second order with constant coefficients we have the next general stability result.

Example 2. Let $p,q \in \mathbb{K}$ and $\varepsilon > 0$. For every $y \in C^2(I,\mathbb{K})$ satisfying

$$\left| y''(x) + py'(x) + qy(x) \right| \leq \varepsilon, \qquad x \in I,$$

there exists a unique $z \in C^2(I,\mathbb{K})$ with properties

$$z''(x) + pz'(x) + qz(x) = 0, \qquad x \in I$$

and

$$|y(x) - z(x)| \leq \frac{\varepsilon}{\sqrt{|p|^2 + |q|^2 + 1}} \left(e^{(x-a)\sqrt{|p|^2+|q|^2+1}} - 1 \right), \qquad x \in I.$$

Proof. Theorem 4 is applied taking account of

$$F(x) = \sqrt{|p|^2 + |q|^2 + 1}(x-a), \qquad x \in I.$$

□

In many concrete cases it is difficult to estimate the difference between the approximate solution and the exact solution in Theorem 4 because of the impossibility to calculate the integrals from those relations.

In practice one can use lower bounds or upper bounds of the function F.

Example 3. We will exemplify this in the case of Hermite's differential equation. Hermite equation is defined as

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \qquad n \in \mathbb{N}, \qquad x \in \mathbb{R}$$

and its particular solution is Hermite's polynomial H_n defined by

$$H_n(x) = (-1)^n e^{x^2} (e^{-x^2})^{(n)}, \qquad x \in \mathbb{R}.$$

Let $I = [0,+\infty)$, $\varepsilon > 0$ and $y \in C^2(I,\mathbb{R})$ such that

$$\left| y''(x) - 2xy'(x) + 2ny(x) \right| \leq \varepsilon, \qquad x \in I. \tag{3.6}$$

Then there exists a unique $z \in C^2(I,\mathbb{R})$ with the properties

$$z''(x) - 2xz'(x) + 2nz(x) = 0, \qquad x \in I, \tag{3.7}$$

and

$$\begin{aligned} |y(x) - z(x)| &\leq \varepsilon e^{F(x)} \cdot \int_0^x e^{-F(t)} dt \\ &\leq \frac{\sqrt{\pi}\varepsilon}{2} e^{x^2+(2n+1)x}, \qquad x \in I. \end{aligned}$$

Proof. Let

$$F(x)=\int_0^x \sqrt{4t^2+4n^2+1}dt$$
$$=\frac{x\sqrt{4x^2+4n^2+1}}{2}+\frac{4n^2+1}{4}\ln\frac{2x+\sqrt{4x^2+4n^2+1}}{\sqrt{4n^2+1}}, \qquad x\in I.$$

Taking $\varphi(x)=\varepsilon$, $x\in I$, in Theorem 4, it follows that for an $y\in C^2(I,\mathbb{R})$ satisfying (3.6) there exists a unique z satisfying (3.7) such that

$$|y(x)-z(x)|\leq \varepsilon e^{F(x)}\int_0^x e^{-F(t)}dt.$$

On the other hand taking account of the relation

$$2t\leq\sqrt{4t^2+4n^2+1}\leq 2t+2n+1, \qquad t\in I$$

we get by integration on $[0,x]$

$$x^2\leq F(x)\leq x^2+(2n+1)x, \qquad x\in I.$$

Finally we get

$$e^{F(x)}\cdot\int_0^x e^{-F(t)}dt\leq e^{x^2+(2n+1)x}\cdot\int_0^x e^{-t^2}dt$$
$$\leq e^{x^2+(2n+1)x}\cdot\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}e^{x^2+(2n+1)x}, \qquad x\in I.$$

□

Clearly the result obtained in Theorem 4 can be extended to a linear differential equation of order n as follows.

Consider the linear differential equation

$$y^{(n)}(x)+a_1(x)y^{(n-1)}(x)+\ldots+a_n(x)y(x)=0, \qquad x\in I, \tag{3.8}$$

where $a_1,\ldots,a_n\in C(I,\mathbb{K})$. Then we obtain:

Theorem 5. *Let $\varphi: I\to\mathbb{R}_+$ be a continuous function. Then for every $y\in C^n(I,\mathbb{K})$ satisfying*

$$\left|y^{(n)}(x)+a_1(x)y^{(n-1)}(x)+\ldots+a_n(x)y(x)\right|\leq\varphi(x), \qquad x\in I,$$

there exists a unique solution $z\in C^n(I,\mathbb{K})$ of the equation (3.8) with the property

$$|y(x)-z(x)|\leq e^{F(x)}\cdot\int_a^x\varphi(t)e^{-F(t)}dt, \qquad x\in I,$$

where

$$F(x)=\int_a^x\sqrt{\sum_{k=1}^n|a_k(t)|^2+n-1}dt, \qquad x\in I.$$

Proof. Analogous with the proof of Theorem 4. □

4. CONCLUSIONS

Ulam stability of differential equations is a topic intensively studied in the last years. In this respect, results were obtained regarding Ulam stability of the first order linear differential equations [5–7] and the linear differential equations of higher orders with constant coefficients [3, 12]. Results on the best Ulam constant for the linear differential equations of orders one and two are given in [1, 14]. We also mention that there are some results on Ulam stability for the linear differential equation with nonconstant coefficients (see [2, 3, 13]).

The main contribution of the present paper is Theorem 2, establishing the Ulam stability for a system of differential equations with nonconstant coefficients. It can be applied in order to investigate the Ulam stability for the linear differential equations with nonconstant coefficients.

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