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INEQUALITIES OF CHEBYSHEV-PÓLYA-SZEGÖ TYPE VIA GENERALIZED PROPORTIONAL FRACTIONAL INTEGRAL OPERATORS

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Abstract. This study is an example of a solid connection between fractional analysis and inequality theory, and includes new inequalities of the Pólya-Szegö-Chebyshev type obtained with the help of Generalized Proportional Fractional integral operators. The results have been performed by using Generalized Proportional Fractional integral operators, some classical inequalities such as AM-GM inequality, Cauchy-Schwarz inequality and Taylor series expansion of exponential function. The findings give new approaches to some types of inequalities that have involving the product of two functions in inequality theory.

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1. INTRODUCTION AND PRELIMINARIES

Inequalities are a concept that contributes to the solution of many problems with their applications in different disciplines such as engineering, physics, statistics and economics as well as being used in many branches of mathematics. With the help of convex, differentiable, integrable, continuous, limited, synchronous functions or some other specially defined functions, many different types of inequality have been proved in the inequality theory field. These inequalities were brought to the literature under different names and later became functional with their applications. We will start by expressing Chebyshev inequality, one of the major inequality types of inequality theory (see [2]):

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \quad (1.1)$$

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where f and g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$(f(x)-f(y))(g(x)-g(y))\geq 0$$

for any $x,y\in[a,b]$, then the Chebyshev inequality states that $T(f,g)\geq 0$.

Chebyshev inequality has been proven for synchronous functions, and has been the focus of researchers and many different versions have been obtained. We encourage interested readers to review the following articles. Chebyshev inequality has been proven for synchronous functions and has been the focus of researchers and many different versions have been obtained. We encourage interested readers to review the following articles [3, 14, 21] and [15].

Another interesting inequality is the Pólya-Szegö inequality, which gives boundaries for two functions that can be integrated and their product. This inequality is given as follows: (see [18])

$$\frac{\int_a^b f^2(x)\,dx\int_a^b g^2(x)\,dx}{\left(\int_a^b f(x)\,g(x)\,dx\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2$$

This inequality is very useful in proving Grüss and Chebyshev type inequalities. With the help of this inequality, a Chebyshev-Grüss type inequality is expressed by Dragomir and Diamond as follows in [7]:

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}_+$ be two integrable functions so that*

$$0<m\leq f(x)\leq M<\infty$$
$$0<n\leq g(x)\leq N<\infty$$

for $x\in[a,b]$. Then we have

$$|T(f,g;a,b)|\leq\frac{1}{4}\frac{(M-m)(N-n)}{\sqrt{mnMN}}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense it can not be replaced by a smaller constant.

Remark 1 (see [7]). Assume that the inequality in (1.2) holds with a constant $c>0$, i.e.,

$$|T(f,g;a,b)|\leq c\frac{(M-m)(N-n)}{\sqrt{mnMN}}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)$$

We choose the functions as $f=g$ with

$$f(x)=\begin{cases} cm, & x\in\left[a,\frac{a+b}{2}\right] \\ \\ M, & x\in[\frac{a+b}{2},b] \end{cases}$$

where $0 < m < M < \infty$, then

$$mM \leq c(M-m)^2 \tag{1.3}$$

for any $0 < m < M < \infty$. If in (1.3) we consider $m = 1-\varepsilon$, $M = 1+\varepsilon$, $\varepsilon \in (0,1)$, then we get $1-(\varepsilon)^2 \leq 4c$ for any $\varepsilon \in (0,1)$, which show that $c \geq \frac{1}{4}$.

Although fractional analysis origin dates back to the beginning of classical analysis, it has developed quite rapidly in recent years. Many mathematicians who researched in this field contributed to this development and made efforts to strengthen the relationship between fractional analysis and other fields. With the introduction of new fractional derivative and integral operators, the application opportunity for many real-world problems has been revealed. The majority of the new operators came to the fore with different features such as singularity, location and generalization, and gained functionality thanks to their effective use in application areas (see the papers [1, 4–6, 8–10, 12, 13, 16, 17, 19, 20]). Due to the intensive work on it, the Riemann-Liouville integral operator is a prominent operator and is defined as follows.

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx, \quad t > a$$

and

$$J_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (x-t)^{\alpha-1} f(x)dx, \quad t < b$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t) = \int_0^{\infty} e^{-t} t^{x-1} dx$. It is to be noted that $J_{a+}^{0} f(t) = J_{b-}^{0} f(t) = f(t)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

We will continue with the Generalized Proportional Fractional integral operator, which has been described recently and has been the main source of motivation for many studies in the literature with its use in many areas, especially the inequality theory. In [11], Jarad et al. identified the proportional generalized fractional integrals that satisfy many important features as follows:

Definition 2. The left and right generalized proportional fracitonal integral operators are respectively defined by

$${}_{a+}\mathfrak{J}^{\alpha,\lambda} f(t) = \frac{1}{\lambda^{\alpha}\Gamma(\alpha)} \int_a^t e^{\left[\frac{\lambda-1}{\lambda}(t-x)\right]} (t-x)^{\alpha-1} f(x)dx, \quad t > a$$

and

$${}_{b-}\mathfrak{J}^{\alpha,\lambda} f(t) = \frac{1}{\lambda^{\alpha}\Gamma(\alpha)} \int_t^b e^{\left[\frac{\lambda-1}{\lambda}(x-t)\right]} (x-t)^{\alpha-1} f(x)dx, \quad t < b$$

where $\lambda \in (0,1]$ and $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$.

The main aim of this study is to obtain new Pólya-Szegö type inequalities by using Generalized Proportional Fractional integral operators. Taylor series expansion of exponential function is used in addition to some classical inequalities to obtain main results. The study is enriched by giving special cases of our results.

2. Main results

In this section, we prove certain Pólya-Szegö type integral inequalities for positive integral functions involving Generalized Proportional Fractional integral operator.

Lemma 1. *Assume that f and g are two positive integrable function on $[0,\infty)$. If v_1, v_2, w_1 and w_2 are positive functions such that*

$$0 < v_1(\tau) \leq f(\tau) \leq v_2(\tau) \tag{2.1}$$

$$0 < w_1(\tau) \leq g(\tau) \leq w_2(\tau)$$

for $\tau \in [0,x]$, $x > 0$, then we have the following inequality;

$$\frac{{}_{0}^{GPF}I^{\alpha,p_1}[w_1 w_2 f^2](x)\,{}_{0}^{GPF}I^{\alpha,p_1}[v_1 v_2 g^2](x)}{\left({}_{0}^{GPF}I^{\alpha,p_1}\left[(v_1 w_1 + v_2 w_2) f g\right](x)\right)^2} \leq \frac{1}{4}. \tag{2.2}$$

where $\alpha \in (n, n+1]$ and $n = 1,2,3,\ldots$.

Proof. From (2.1) for $\tau \in [0,x], x > 0$, we can write

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geq 0 \tag{2.3}$$

and

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \geq 0. \tag{2.4}$$

If we multiply (2.3) and (2.4) side by side, we have

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \geq 0.$$

This implies the following inequality,

$$\Big(v_1(\tau)w_1(\tau) + v_2(\tau)w_2(\tau)\Big) f(\tau)g(\tau) \geq w_1(\tau)w_2(\tau)f^2(\tau) + v_1(\tau)v_2(\tau)g^2(\tau). \tag{2.5}$$

Since all the functions are positive, $p_1 \in (0,1]$, $x \geq \tau$ and $x > 0$, by multiplying both sides of (2.5) by $\frac{1}{p_1^{\alpha}\Gamma(\alpha)} e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}$ and then integrating the resulting inequality with respect to τ over $(0,x)$, we get

$$\frac{1}{p_1^{\alpha}\Gamma(x)} \int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}\Big(v_1(\tau)w_1(\tau) + v_2(\tau)w_2(\tau)\Big) f(\tau)g(\tau)d\tau$$

$$\geq \frac{1}{p_1^{\alpha}\Gamma(x)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}\Big(w_1(\tau)w_2(\tau)f^2(\tau)\Big)d\tau$$
$$+\frac{1}{p_1^{\alpha}\Gamma(x)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}v_1(\tau)v_2(\tau)g^2(\tau)d\tau.$$

Namely,

$$^{GPF}_{0}I^{\alpha,p_1}\Big[(v_1w_1+v_2w_2)fg\Big](x) \geq {}^{GPF}_{0}I^{\alpha,p_1}[w_1w_2f^2](x)+{}^{GPF}_{0}I^{\alpha,p_1}[v_1v_2g^2](x). \tag{2.6}$$

Applying the A.M-G.M inequality i.e $(a+b\geq 2\sqrt{ab}, a,b\in\Re^+)$, we have

$$^{GPF}_{0}I^{\alpha,p_1}\Big[(v_1w_1+v_2w_2)fg\Big](x) \geq 2\sqrt{{}^{GPF}_{0}I^{\alpha,p_1}\Big[w_1w_2f^2\Big](x)\,{}^{GPF}_{0}I^{\alpha,p_1}\Big[v_1v_2g^2\Big](x)}.$$

This can be written as

$$^{GPF}_{0}I^{\alpha,p_1}\Big[w_1w_2f^2\Big](x)\,{}^{GPF}_{0}I^{\alpha,p_1}\Big[v_1v_2g^2\Big](x) \leq \frac{1}{4}\Big({}^{GPF}_{0}I^{\alpha,p_1}\Big[(v_1w_1+v_2w_2)fg\Big](x)\Big)^2.$$

□

Corollary 1. *If we take into account $v_1=m, v_2=m, w_1=n$ and $w_2=N$ in (2.2), then we have the following new inequality;*

$$\frac{\Big({}^{GPF}_{0}I^{\alpha,p_1}f^2\Big)(x)\Big({}^{GPF}_{0}I^{\alpha,p_1}g^2\Big)(x)}{\Big(\Big({}^{GPF}_{0}I^{\alpha,p_1}fg\Big)(x)\Big)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

Lemma 2. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exists four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds:*

$$^{GPF}_{0}I^{\alpha,p_1}[v_1v_2](x)\,{}^{GPF}_{0}I^{\beta,p_2}[w_1w_2](x)\times{}^{GPF}_{0}I^{\alpha,p_1}[f^2](x)\,{}^{GPF}_{0}I^{\beta,p_2}[g^2](x) \tag{2.7}$$
$$\leq \frac{1}{4}\Big({}^{GPF}_{0}I^{\alpha,p_1}[v_1f](x)\,{}^{GPF}_{0}I^{\beta,p_2}[w_1g](x)+{}^{GPF}_{0}I^{\alpha,p_1}[v_2f](x)\,{}^{GPF}_{0}I^{\beta,p_2}[w_2g](x)\Big)^2$$

where $\alpha\in(n,n+1]$ and $\beta\in(k,k+1]$, $n,k=0,1,2,3,\ldots$.

Proof. From (2.1), we get

$$\left(\frac{v_2(\tau)}{w_1(\xi)}-\frac{f(\tau)}{g(\xi)}\right)\geq 0$$

and

$$\left(\frac{f(\tau)}{g(\xi)}-\frac{v_1(\tau)}{w_2(\xi)}\right)\geq 0.$$

Which leads to

$$\left(\frac{v_1(\tau)}{w_2(\xi)}+\frac{v_2(\tau)}{w_1(\xi)}\right)\frac{f(\tau)}{g(\xi)}\geq\frac{f^2(\tau)}{g^2(\xi)}+\frac{v_1(\tau)v_2(\tau)}{w_1(\xi)w_2(\xi)}. \tag{2.8}$$

Multiplying both sides of (2.8) by $w_1(\xi)w_2(\xi)g^2(\xi)$, we have

$$v_1(\tau)f(\tau)w_1(\xi)g(\xi)+v_2(\tau)f(\tau)w_2(\xi)g(\xi)\geq w_1(\xi)w_2(\xi)f^2(\tau)+v_1(\tau)v_2(\tau)g^2(\xi). \tag{2.9}$$

Multiplying both sides (2.9) by

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}$$

and integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$\begin{aligned}
&\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}v_1(\tau)f(\tau)w_1(\xi)g(\xi)d\tau d\xi\\
&+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}v_2(\tau)f(\tau)w_2(\xi)g(\xi)d\tau d\xi\\
&\geq\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}w_1(\xi)w_2(\xi)f^2(\tau)d\tau d\xi\\
&+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}v_1(\tau)v_2(\tau)g^(\xi)d\tau d\xi.
\end{aligned}$$

If we re-write the above inequality with the help of the definition of operator, we get

$$\begin{aligned}
&{}_0^{GPF}I^{\alpha,p_1}[v_1f](x)\,{}_0^{GPF}I^{\beta,p_2}[w_1g](x)+{}_0^{GPF}I^{\alpha,p_1}[v_2f](x)\,{}_0^{GPF}I^{\beta,p_2}[w_2g](x)\\
&\geq{}_0^{GPF}I^{\alpha,p_1}[f^2](x)\,{}_0^{GPF}I^{\beta,p_2}[w_1w_2](x)+{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x)\,{}_0^{GPF}I^{\beta,p_2}[g^2](x).
\end{aligned}$$

Applying the A.M-G.M inequality, we have

$$\begin{aligned}
&{}_0^{GPF}I^{\alpha,p_1}[v_1f](x)\,{}_0^{GPF}I^{\beta,p_2}[w_1g](x)+{}_0^{GPF}I^{\alpha,p_1}[v_2f](x)\,{}_0^{GPF}I^{\beta,p_2}[w_2g](x)\\
&\geq 2\sqrt{{}_0^{GPF}I^{\alpha,p_1}[f^2](x)\,{}_0^{GPF}I^{\beta,p_2}[w_1w_2](x)\times{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x)\,{}_0^{GPF}I^{\beta,p_2}[g^2](x)}.
\end{aligned}$$

Which leads to the desired inequality in (2.7). The proof is completed. □

Corollary 2. *If we set $v_1 = M$, $v_2 = M$, $w_1 = n$ and $w_2 = N$ in (2.7), then we have the following inequality;*

$$
{}_{0}^{GPF}I^{\alpha,p_1}(x){}_{0}^{GPF}I^{\beta,p_2}(x)\frac{\left({}_{0}^{GPF}I^{\alpha,p_1}f^2\right)(x)\left({}_{0}^{GPF}I^{\beta,p_2}g^2\right)(x)}{\left(\left({}_{0}^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_{0}^{GPF}I^{\beta,p_2}g\right)(x)\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2
$$

Let $a = \frac{p_1-1}{p_1}$. The Taylor Series of $\exp(a(x-\tau))$ at the point x is given by

$$
\begin{aligned}
{}_{0}^{GPF}I^{\alpha,p_1}(x) &= \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}d\tau \\
&= \frac{1}{p_1\Gamma(\alpha)}\int_0^x \sum_{k=0}^{\infty}\frac{\left(a(x-\tau)\right)^{k_1}}{k_1!}(x-\tau)^{\alpha-1}d\tau \\
&= \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{a^{k_1}x^{\alpha+k_1}}{\alpha+k_1}
\end{aligned}
$$

$$
{}_{0}^{GPF}I^{\beta,p_2}(x) = \frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{b^{k_2}x^{\beta+k_2}}{k_2!(\beta+k_2)}
$$

$$
\begin{aligned}
&\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}x^{\alpha+k_1}}{(\alpha+k_1)k!}\sum_{k_2=0}^{\infty}\frac{b^{k_2}x^{\beta+k_2}}{(\beta+k_2)k_2!} \\
&\times\frac{\left({}_{0}^{GPF}I^{\alpha,p_1}f^2\right)(x)\left({}_{0}^{GPF}I^{\beta,p_2}g^2\right)(x)}{\left(\left({}_{0}^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_{0}^{GPF}I^{\beta,p_2}g\right)(x)\right)^2} \\
&\le \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.
\end{aligned}
$$

Lemma 3. *Let f and g be two positive integrable function on $[0,\infty)$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1) then the following inequality holds.*

$$
{}_{0}^{GPF}I^{\alpha,p_1}[f^2](x){}_{0}^{GPF}I^{\beta,p_2}[g^2](x) \le {}_{0}^{GPF}I^{\alpha,p_1}[\frac{v_2 fg}{w_1}](x){}_{0}^{GPF}I^{\beta,p_2}[\frac{w_2 fg}{v_1}](x). \tag{2.10}
$$

where $\alpha \in (n, n+1]$, $\beta \in (k, k+1]$, $n, k = 0, 1, 2, 3, \ldots$.

Proof. Using the condition (2.1), we get

$$
f^2(\tau) \le \frac{v_2(\tau)}{w_1(\tau)}f(\tau)g(\tau). \tag{2.11}
$$

Multiplying both sides of (2.11) by $\frac{1}{p_1^{\alpha}\Gamma(\alpha)}e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}$ and integrating the resulting inequality with respect to τ over $(0,x)$, we get

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}f^2(\tau)d\tau$$

$$\leq \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}\frac{v_2\tau}{w_1(\tau)}f(\tau)g(\tau)d\tau$$

$${}_0^{GPF}I^{\alpha,p_1}[f^2](x) \leq {}_0^{GPF}I^{\alpha,p}[\frac{v_2 fg}{w_1}](x). \tag{2.12}$$

Similarly, we can write

$$g^2(\xi) \leq \frac{w_2(\xi)}{v_1(\xi)}f(\xi)g(\xi).$$

By a similar argument, we have

$$\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\xi)^{\beta-1}g^2(\xi)d\xi$$

$$\leq \frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\xi)^{\beta-1}\frac{w_2(\xi)}{v_1(\xi)}f(\xi)g(\xi)d\xi.$$

Which implies

$${}_0^{GPF}I^{\beta,p_2}[g^2](x) \leq {}_0^{GPF}I^{\beta,p_2}[\frac{w_2 fg}{v_1}](x). \tag{2.13}$$

Multiplying (2.12) and (2.13), we get then (2.10). Then the desired inequality is obtained such that

$${}_0^{GPF}I^{\alpha,p_1}[f^2](x){}_0^{GPF}I^{\beta,p_2}[g^2](x) \leq {}_0^{GPF}I^{\alpha,p_1}[\frac{v_2 fg}{w_1}](x){}_0^{GPF}I^{\beta,p_2}[\frac{w_2 fg}{v_1}](x).$$

□

Corollary 3. *If we choose $v_1 = m$, $v_2 = M, w_1 = n$ and $w_2 = N$ in (2.10), then we have the following inequality;*

$$\frac{\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g^2\right)(x)}{\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)} \leq \frac{MN}{mn}.$$

Theorem 2. *Let f and g be two positive integrable function on $[0,\infty)$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition*

(2.1) *then the following inequality holds:*

$$\begin{aligned}&\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}fg\right)(x)\\&+\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)\\&-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)-\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\\ \le&\left|A_1(f,v_1,v_2)(x)+A_2(f,v_1,v_2)(x)\right|^{\frac12}\times\left|A_1(g,w_1,w_2)(x)+A_2(g,w_1,w_2)(x)\right|^{\frac12},\end{aligned}$$

for $\alpha\in(n,n+1]$, $\beta\in(k,k+1]$, $n,k=0,1,2,3,...,$ *where*

$$\begin{aligned}A_1(u,v,w)(x)=&\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\\&\times\frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v+w)u](x)\right)^2}{4\,{}_0^{GPF}I^{\alpha,p_1}[vw](x)}-\left({}_0^{GPF}I^{\alpha,p_1}u\right)(x)\left({}_0^{GPF}I^{\beta,p_2}u\right)(x)\end{aligned}$$

and

$$\begin{aligned}A_2(u,v,w)(x)=&\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\\&\times\frac{\left({}_0^{GPF}I^{\beta,p_2}[(v+w)u](x)\right)^2}{4\,{}_0^{GPF}I^{\beta,p_2}[vw](x)}-\left({}_0^{GPF}I^{\alpha,p_1}u\right)(x)\left({}_0^{GPF}I^{\beta,p_2}u\right)(x).\end{aligned}$$

Proof. Let f and g be two positive integrable functions on $[0,\infty)$ for $\tau,\xi\in(0,x)$ with $x>0$, we define $H(\tau,\xi)$ as

$$H(\tau,\xi)=\Big(f(\tau)-f(\xi)\Big)\Big(g(\tau)-g(\xi)\Big).$$

Namely

$$H(\tau,\xi)=f(\tau)g(\tau)+f(\xi)g(\xi)-f(\tau)g(\xi)-f(\xi)g(\tau). \tag{2.14}$$

Multiplying both sides of (2.14) by

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}.$$

Then by integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}H(\tau,\xi)d\tau d\xi$$

$$
\begin{aligned}
&= \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\tau)g(\tau)d\tau d\xi\\
&+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\xi)g(\xi)d\tau d\xi\\
&-\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\tau)g(\xi)d\tau d\xi\\
&-\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\xi)g(\tau)d\tau d\xi\\
&\qquad\qquad= \left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)\\
&\qquad\qquad+\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}fg\right)(x)\\
&\quad-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)-\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x).
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
&\left|\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}H(\tau,\xi)d\tau d\xi\right| \\
&\leq\left[\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f^2(\tau)d\tau d\xi\right. \\
&\quad+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f^2(\xi)d\tau d\xi \\
&\left.-2\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\tau)f(\xi)d\tau d\xi\right]^{\frac{1}{2}} \\
&\times\left[\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}g^2(\tau)d\tau d\xi\right. \\
&\quad+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}g^2(\xi)d\tau d\xi \\
&\left.-2\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}g(\tau)g(\xi)d\tau d\xi\right]^{\frac{1}{2}}.
\end{aligned}$$

As a consequence

$$\left|\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}H(\tau,\xi)d\tau d\xi\right|$$
$$\leq\left[\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)\right.$$
$$+\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}f^2\right)(x)$$
$$\left.-2\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\right]^{\frac{1}{2}}$$
$$\left[\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}g^2\right)(x)\right.$$
$$+\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}g^2\right)(x)$$
$$\left.-2\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\right]^{\frac{1}{2}}.$$

Applying Lemma 1 with $w_1(\tau)=w_2(\tau)=g(\tau)=1$, we get

$$\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)$$
$$\leq\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v_1+v_2)f](x)\right)^2}{4{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x)}.$$

This implies that

$$\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)$$
$$\leq\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v_1+v_2)f](x)\right)^2}{4{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x)}$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)=A_1(f,v_1,v_2) \tag{2.15}$$

and

$$\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}f^2\right)(x)-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)$$
$$\leq\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\frac{\left({}_0^{GPF}I^{\beta,p_2}[(v_1+v_2)f](x)\right)^2}{4{}_0^{GPF}I^{\beta,p_2}[v_1v_2](x)}$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)=A_2(f,v_1,v_2). \tag{2.16}$$

Similarly, applying Lemma 1 with $v_1(\tau)=v_2(\tau)=f(\tau)=1$, we have

$$\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right){}_0^{GPF}I^{\alpha,p_1}[g^2](x)$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\leq A_1(g,w_1,w_2) \tag{2.17}$$

and

$$\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right){}_0^{GPF}I^{\beta,p_2}[g^2](x)$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\leq A_2(g,w_1,w_2). \tag{2.18}$$

Using (2.15)-(2.18), we conclude the result. $\square$

Theorem 3. *Let f and g be two positive integrable function on $[0,\infty)$. Assume that there exist four positive integrable functions v_1,v_2,w_1 and w_2 satisfying condition (2.1) then the following inequality holds:*

$$\left|\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right){}_0^{GPF}I^{\alpha,p_1}[fg](x)-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)({}_0^{GPF}I^{\alpha,p_1}g)(x)\right| \tag{2.19}$$
$$\leq\left|A(f,v_1,v_2)(x)A(g,w_1,w_2)(x)\right|^{\frac{1}{2}}$$

for $\alpha\in(n,n+1]$, $\beta\in(k,k+1]$, $n,k=0,1,2,3,\ldots$, where

$$A(u,v,w)(x)=\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)$$

$$\times\frac{\left({}_{0}^{GPF}I^{\alpha,p_1}[(v+w)u](x)\right)^2}{4{}_{0}^{GPF}I^{\alpha,p_1}[vw](x)}-\left(\left({}_{0}^{GPF}I^{\alpha,p_1}u\right)(x)\right)^2.$$

Proof. Setting $\alpha=\theta$ in Theorem 2, we obtain (2.19). □

Corollary 4. *Assume that all the assumptions of Theorem 3 satify, then we have the following inequality;*

$$\left|\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right){}_{0}^{GPF}I^{\alpha,p_1}[fg](x)-\left({}_{0}^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_{0}^{GPF}I^{\alpha,p_1}g\right)(x)\right|$$
$$\leq\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\frac{(M-m)(N-n)}{4\sqrt{MmNn}}\times\left({}_{0}^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_{0}^{GPF}I^{\alpha,p_1}g\right)(x).$$

Proof. If we set $v_1=m$, $v_2=M$, $w_1=n$ and $w_2=N$ in (2.19), then the proof is completed. We omit the details. □

3. Conflict of interest

All authors declare no conflicts of interest in this paper.

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