NEIMARK-SACKER BIFURCATION AND STABILITY OF A PREY-PREDATOR SYSTEM

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Abstract. In this paper, we investigate the stability and bifurcation of a discrete-time prey-predator system which is subject to an efficiency of prey conversion into predators and minimum threshold prey consumption required before predators begin to reproduce. It is concluded that the system undergoes Neimark-Sacker bifurcations in a small neighborhood of the unique positive equilibrium which depends on the number of prey-predator. Moreover, the numerical simulations are done to demonstrate the theoretical results.

1. INTRODUCTION

Mathematical models which provide a way to design of changes in population size help to generate testable predictions. The changes in population size result from interactions between individuals of the same or different species, interactions with the environment, disease, food supply, etc. Predation, cooperative, mutualistic, commensural are different types of interactions of between individuals.

The discrete-time and continuous-time models are used for predicting the size of population. Especially, the discrete-time models are suitable to get more accurate numerical simulations for non-overlapping generations [20]. The prey-predator models have an importance at studies of mathematical biology [1, 7, 10, 25]. In recent years, studies on bifurcation and stability of systems involving population interaction have remarkable attention [2–6, 11, 13, 14, 16–19, 22, 27–29].

Bifurcation theory is a mathematical study of changes in the qualitative or topological structure of a given family. This theory is widely used in the mathematical study of dynamical systems. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour [21].

In [14], the Neimark-Sacker bifurcation of a dimensional discrete-time prey-predator model is presented as follows:

$$x_{n+1} = ax_n(1-x_n) - x_ny_n \tag{1.1}$$

$$y_{t+1} = \frac{1}{\beta}x_ny_n.$$

where x_n and y_n denote the numbers of prey and predator, respectively. Moreover the parameters a, β and the initial conditions x_0, y_0 are positive real numbers.

A predator needs availability of prey as well as resources, to be able to sustain life and mature of it. Biological facts include the possibility of time spent in hunting and consumption of prey. This case associated with reproductive rates of predator is important for competitive interaction models.

In this paper, we will focus on the stability and bifurcation of a discrete-time prey-predator system which is subject to an efficiency of prey conversion into predators and minimum threshold of prey *consumption required before predators begin to reproduce* in [1, 10]. The general a prey-predator system can be given as follows:

$$x_{n+1} = ax_n(1-x_n) - x_ny_n, \tag{1.2}$$

$$y_{t+1} = \frac{1}{b}x_ny_n - ey_n,$$

where x_n and y_n denote the numbers of prey and predator population in the n th generation, respectively. In this system, all the parameters a, b and e are positive real numbers. Moreover the parameter a denotes reproductive rates of prey population and the parameter b represents reproductive rates of predation depending on the growth rate of the prey. Also, the parameter e is the probability of the efficiency of prey conversion into predators and minimum threshold prey *consumption required before predators begin to reproduce* with $e < 1$.

The purpose of this work is to give the existence of the equilibrium point of the system (1.2); and to investigate the stability conditions of these equilibrium point. Also, the existence of Neimark-Sacker bifurcation of the system (1.2) is studied by choosing b as a bifurcation parameter. Moreover, we show the dynamical properties of the system (1.2), by means of trajectories, bifurcation diagrams and phase portraits.

This study consists of four section. In Section 2, we investigated the existence and local asymptotic stability of equilibrium points of the system (1.2) in $\mathbb{R}_+^2$. In Section 3, the existence of the system (1.2) undergoes a Neimark-Sacker bifurcation is discussed. Section 4 includes numerical simulations to support theoretical results. In the last section, the results are briefly presented.

2. THE EXISTENCE AND STABILITY OF EQUILIBRIUM POINTS

In this section, we consider the prey-predator system (1.2). Firstly, we discuss the existence of equilibrium points for the system (1.2), and then study the stability

of the equilibrium points by using the characteristic polynomial or the eigenvalues of the Jacobian matrix evaluated at the equilibrium points. Now, let us give some necessary information.

Let us consider the two-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_n, y_n), \tag{2.1}$$
$$y_{n+1} = g(x_n, y_n), \quad n = 0, 1, 2, ...,$$

where $f : I \times J \to I$ and $g : I \times J \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (2.1) is uniquely determined by initial conditions $(x_0, y_0) \in I \times J$. An equilibrium point of (2.1) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f(\bar{x}, \bar{y}), \tag{2.2}$$
$$\bar{y} = g(\bar{x}, \bar{y}).$$

Let $(\bar{x}, \bar{y})$ be an equilibria of the map $F(x, y) = (f(x, y), g(x, y))$, where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2.1) about the equilibria $(\bar{x}, \bar{y})$ is given by $X_{n+1} = F(X_n) = F_J X_n$, where $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and F_J is a Jacobian matrix of system (2.1) about the equilibria $(\bar{x}, \bar{y})$.

Theorem 1 ([24]). *Assume that $X_{n+1} = F(X_n), n = 0, 1, \ldots$ is a system of difference equations and $\bar{x}$ is an equilibrium point of F. If all eigenvalues of the Jacobian matrix F_J about the equilibria $\bar{x}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable. If one of them has absolute value greater than one, then $\bar{x}$ is unstable.*

Theorem 2 ([8]). *Consider the second-degree polynomial equation*

$$\lambda^2 - p\lambda - q = 0 \tag{2.3}$$

where p and q are real numbers.

 (i) *If both roots of Equation (2.3) lie in the open unit disk $|\lambda| < 1$, then the equilibria $(\bar{x}, \bar{y})$ is locally asymptotically stable.*
 (ii) *If at least one of the roots of Equation (2.3) has absolute value greater than one, then the equilibria $(\bar{x}, \bar{y})$ is unstable.*
 (iii) *A necessary and sufficient condition for both roots of Equation (2.3) to lie inside the open disk $|\lambda| < 1$ is*

$$|p| < 1 - q < 2.$$

 In this case the locally asymptotically stable equilibria $(\bar{x}, \bar{y})$ is also called a sink.
 (iv) *A necessary and sufficient condition for both roots of Equation (2.3) to have absolute value greater than one is*

$$|q| > 1, |p| < |1 - q|.$$

In this case $(\bar{x}, \bar{y})$ is a repeller (source).

(v) *A necessary and sufficient condition for one root of Equation (2.3) to have absolute value greater than one and for the other to have absolute value less than one is $p^2 + 4q > 0$, $|p| > |1 - q|$. In this case the unstable equilibria $(\bar{x}, \bar{y})$ is called a saddle point.*

(vi) *A necessary and sufficient condition for a root of Equation (2.3) to have absolute value equal to one is*

$$|p| = |1 - q|.$$

In this case, the steady state $(\bar{x}, \bar{y})$ is called a non-hyperbolic point.

Theorem 3 ([18]). *Let $F(x) = x^2 + Bx + C$. Suppose that $F(1) > 0$, x_1 and x_2 are two roots of $F(x) = 0$. Then*

  (i) *$|x_1| < 1$ and $|x_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;*
  (ii) *$|x_1| < 1$ and $|x_2| > 1$ (or $|x_1| > 1$ and $|x_2| < 1$) if and only if $F(-1) < 0$;*
 (iii) *$|x_1| > 1$ and $|x_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;*
  (iv) *$x_1 = -1$ and $|x_2| \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;*
  (v) *x_1 and x_2 are complex and $|x_1| = |x_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.*

Theorem 4 ([2]). *The characteristic polynomial*

$$F(x) = x^2 + Bx + C$$

has all its roots inside the unit open disk ($|x| < 1$) if and only if

  (i) *$F(1) > 0$ and $F(-1) > 0$*
  (ii) *$D_1^+ = 1 + C > 0$ and $D_1^- = 1 - C > 0$.*

Now, we will investigate the equilibrium points of system (1.2) and analyze the stability of these equilibrium points.

We easily obtain the equilibrium points of system (1.2) by using (2.2) such that

$$\bar{x} = a\bar{x}(1 - \bar{x}) - \overline{xy}, \tag{2.4}$$

$$\bar{y} = \frac{1}{b}\overline{xy} - e\bar{y}.$$

Then, we have the following Lemma.

Lemma 1. *For the system (1.2), the following cases hold:*

  (i) *The system (1.2) has a unique trivial (extinction) equilibria $E_0 = (0,0)$ for all parameters.*
  (ii) *If $a > 1$, then the system (1.2) has two equilibria. These are an extinction $E_0 = (0,0)$ and an exclusion $E_1 = (\frac{a-1}{a}, 0)$.*
 (iii) *If $a > 1$ and $b < \frac{a-1}{a(e+1)}$, then the system (1.2) has three equilibria. These are an extinction equilibria $E_0 = (0,0)$, an exclusion equilibria $E_1 = (\frac{a-1}{a}, 0)$ and a coexistence equilibria $E_2 = (b[e+1], a - ab[e+1] - 1)$.*

Now, let us study local dynamics of the equilibria obtaining the Jacobian matrix evaluated at E_0, E_1 and E_2. Firstly, by considering (1.2), we can get the Jacobian matrix as

$$J_{E_0} = \begin{pmatrix} a & 0 \\ 0 & -e \end{pmatrix} \tag{2.5}$$

evaluated at E_0, and the eigenvalues of J_{E_0} are

$$\lambda_1 = a, \quad \lambda_2 = -e.$$

By using Theorem 2-3, we can express the topological classification of the equilibria E_0 of the system (1.2) as follows:

Lemma 2. *For the extinction equilibrium points E_0, the following cases hold:*
  (i) *If $0 < a < 1$ and $0 < e < 1$, then the equilibria E_0 is a sink point.*
  (ii) *If $0 < e < 1 < a$, then the equilibria E_0 is a saddle point.*
  (iii) *If $a = 1$, E_0 is non-hyperbolic point.*

Note that if $a > 1$, then the equilibria E_0 can not be a source point with $e < 1$. Secondly, we can get the Jacobian matrix as

$$J_{E_1} = \begin{pmatrix} 2-a & -1+\frac{1}{\alpha} \\ 0 & \frac{-1+a-abe}{\alpha b} \end{pmatrix} \tag{2.6}$$

evaluated at E_1 and the eigenvalues of J_{E_1} are

$$\lambda_1 = 2-a, \quad \lambda_2 = \frac{-1+a-abe}{\alpha b}.$$

Similarly, by using Theorem 2-3, we can get the topological classification of the exclusion equilibria E_1 of the system (1.2) as follows:

Lemma 3. *For the extinction equilibrium points E_1, the following cases hold:*
  (i) *If $1 < a < 3$ and $\frac{a-1}{\alpha(e+1)} < b$, then the equilibria E_1 is a sink point.*
  (ii) *If $(a < 1$ or $a > 3)$ and $\frac{a-1}{\alpha(e+1)} > b$, then the equilibria E_1 is a source point.*
  (iii) *If $(a < 1$ or $a > 3)$ and $\frac{a-1}{\alpha(e+1)} < b$ or $1 < a < 3$ and $\frac{a-1}{\alpha(e+1)} > b$, then the equilibria E_1 is a saddle point.*
  (iv) *If $a = 3$, or $b = \frac{a-1}{\alpha(e+1)}$, E_1 is non-hyperbolic point.*

Finally, we can find the Jacobian matrix as

$$J_{E_2} = \begin{pmatrix} 1-ab(e+1) & -b(e+1) \\ \frac{-1+a-ab(e+1)}{b} & 1 \end{pmatrix} \tag{2.7}$$

evaluated at E_2 and the characteristic polynomial is

$$F(\lambda) = \lambda^2 + [-2+ab(1+e)]\lambda + [-e - a(1+e)(-1+b[2+e])].$$

So, we can analyze the dynamics of a unique positive coexistence equilibrium points of the system (1.2). By using Theorem 2-3, we can get the topological classification of the coexistence equilibria E_2 of system (1.2) as follows:

Lemma 4. *Suppose that $F(1) > 0$. In this case, $0 < b < \frac{a-1}{\alpha(e+1)}$ is provided for $a > 1$, $0 < e < 1$. For the coexistence equilibrium points E_2, the following cases hold:*

   (i)   *If $\frac{a-1}{2a+ae} < b < \frac{3+a-e+ae}{3a+4ae+ae^2}$, for $a > 3$ or $\frac{a-1}{2a+ae} < b < \frac{a-1}{\alpha(e+1)}$ for $1 < a < 3$, then the equilibria E_2 is a sink point.*

   (ii) *If $0 < b < \min\left\{\frac{a-1}{2a+ae}, \frac{3+a-e+ae}{3a+4ae+ae^2}\right\}$, then the equilibria E_2 is a source point.*

   (iii) *If $b > \frac{3+a-e+ae}{3a+4ae+ae^2}$, then the equilibria E_2 is a saddle point.*

   (iv) *If $0 < b < \frac{-2}{\alpha} + 2\sqrt{\frac{a+e}{a^2(e+1)}}$ and $b = b_1$ such that $b_1 = \frac{(a-1)}{(2a+ae)}$, then the eigenvalues of J_{E_2} are a pair of conjugate complex numbers whose modules are one.*

3. NEIMARK-SACKER BIFURCATIONS

In this section, the direction and the existence of Neimark–Sacker bifurcation are obtained for the system (1.2) ([16,27]). Also, if the system (1.2) provides eigenvalue assignment, transversality and non resonance conditions, then Neimark–Sacker bifurcation occurs at a bifurcation point ([9, 23]). In order to work Neimark–Sacker bifurcation in the system (1.2), we define the parameters providing non-hyperbolic conditions by

$$NSB_{E_2} = \left\{a, b, e \in \mathbb{R}^+ : 0 < b < \frac{-2}{\alpha} + 2\sqrt{\frac{a+e}{a^2(e+1)}} \text{ and } b = b_1\right\}. \quad (3.1)$$

By considering Lemma 8, the eigenvalues of J_{E_2} evaluated at a non-hyperbolic point E_2 has a pair of conjugate complex numbers whose modules are one. These eigenvalues are

$$\lambda, \bar{\lambda}\,|_{b=b_1} = \frac{1}{2}\left[2 - ab(1+e) \pm i\sqrt{(1+e)[(4a-4) - a^2b^2 - 4ab(1+e)]}\right]\Big|_{b=b_1} \quad (3.2)$$

with

$$|\lambda| = \left|\bar{\lambda}\right| = 1.$$

For $b \in NSB_{E_2}$, we get

$$\frac{\partial|\lambda_i(b)|}{\partial b}\Big|_{b=b_1} \neq 0 , \ i = 1,2. \quad (3.3)$$

Also, if

$$trJ_{E_2}\,|_{b=b_1} \neq 0, -1, \quad (3.4)$$

then, we reach

$$\lambda^k(b_1) \neq 1 , \ k = 1,2,3,4. \quad (3.5)$$

Let $q, p \in \mathbb{C}^2$ be two eigenvectors which corresponding to the eigenvalues λ of the matrix $J(NSB_{E_2})$ and the eigenvalues $\bar{\lambda}$ of the matrix $J(NSB_{E_2})^T$, respectively. If these eigenvectors are calculated with Maple program, then we get

$$q \sim \left(\frac{T - 1 - i\sqrt{2T - 1 - T^2 - 4Ka}}{2a}, 1 \right), \tag{3.6}$$

and

$$p \sim \left(\frac{T - 1 + i\sqrt{2T - 1 - T^2 - 4Ka}}{2K}, 1 \right), \tag{3.7}$$

such that $T = \frac{3 + 2e - a(e+1)}{2+e}$, $K = -\frac{(-1+a)(1+e)}{a(2+e)}$.

By using the scalar product in $\mathbb{C}^2 : < p, q >= \overline{p_1} q_1 + \overline{p_2} q_2$, we define the following vector in order to normalize p according to q

$$p \sim \left(\frac{(T-1) + i\sqrt{2T - 4aK - 1 - T^2}}{2K\overline{R}}, \frac{1}{\overline{R}} \right). \tag{3.8}$$

where $\overline{R} = (1 + \frac{(-1 + T - i\sqrt{-1 - 4aK + 2T - T^2})^2}{4aK})$ and $< p, q >= 1$.

In order to transform the equilibrium point E_2 of the system (1.2) into the origin $(0,0)$, we take

$$u_n = x_n - b[e+1], \ v_n = y_n - [a - ab(e+1) - 1)]. \tag{3.9}$$

Then, we get the following map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to J_{E_2} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix}, \tag{3.10}$$

where

$$F_1(u,v) = -au^2 - uv + O(\|U\|^3)$$

$$F_2(u,v) = \frac{1}{b} uv + O(\|U\|^3)$$

such that $U_t = (u,v)^T$. Also, the system (1.2) can be written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} \to J_{E_2} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \frac{1}{2} B(u_n, v_n) + \frac{1}{6} C(u_n, v_n, w_n) + O(\|U_n\|^4), \tag{3.11}$$

with the multilinear vector functions of $u; v; w \in \mathbb{R}^2$:

$$B(u,v) = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix}$$

and

$$C(u,v) = \begin{pmatrix} B_1(u,v,w) \\ B_2(u,v,w) \end{pmatrix}.$$

These vectors are expressed by

$$B_1(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_1}{\partial \xi_j \partial \xi_k} |_{\xi=0} u_{.j} v_k = -2au_1 v_1 - (u_2 v_1 + u_1 v_2)$$

$$B_2(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2}{\partial \xi_j \partial \xi_k} |_{\xi=0} u_{.j} v_k = \frac{1}{b}(u_2 v_1 + u_1 v_2)$$

$$C_1(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_1}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0} u_{.j} v_k w_l = 0$$

$$C_2(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_2}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0} u_{.j} v_k w_l = 0.$$

$\forall\, U \in \mathbb{R}^2$ can be uniqely offered as

$$U = zq + \overline{z}\overline{q} \tag{3.12}$$

for some $z \in \mathbb{C}$. Here, $\overline{z}$ is the conjugate of that complex number z, and $z = <p, U>$. For all sufficiently small $|b|$, we can transform the system (1.2) into the form

$$z \rightarrow \lambda(b)z + g(z,\overline{z},b), \tag{3.13}$$

where $\lambda(r) = (1 + \omega(b))e^{i\arctan(b)}$ with $\omega(b_1) = 0$ and $g(z,\overline{z},b)$ is a complex valued smooth function of z and $\overline{z}$. Taylor expression of g with respect to $g(z,\overline{z})$ is as follows:

$$g(z,\overline{z},b) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(b) z^k \overline{z}^l, \tag{3.14}$$

with

$$g_{20}(b_1) = <p, B(q,q)>$$
$$g_{11}(b_1) = <p, B(q,\overline{q})>$$
$$g_{02}(b_1) = <p, B(\overline{q},\overline{q})>$$
$$g_{21}(b_1) = <p, C(q,q,\overline{q})>.$$

In order to come out Neimark-Sacker bifurcation for the system (3.10), we need that the coefficient $\varphi(b_1)$ must not be zero. This coefficient is

$$\varphi(b_1) = Re\left(\frac{e^{-i\arctan(b_1)}}{2} g_{21}\right) - Re\left(\frac{(1 - 2e^{i\arctan(b_1)})e^{-2i\arctan(b_1)}}{2(1 - e^{i\arctan(b_1)})} g_{20}g_{11}\right)$$
$$- \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \tag{3.15}$$

where $e^{i\arctan(b_1)} = \lambda(b_1)$. Consequently, we have the following theorem on Neimark-Sacker bifurcation:

Theorem 5. *If (3.4) holds, $\varphi(b_1) \neq 0$ and the parameter b changes its value in small vicinity of NSB_{E_2}, then the system (1.2) passes through a Neimark-Sacker bifurcation at the only equilibrium point E_2. Moreover if $\varphi(b_1) < 0$ ($\varphi(b_1) > 0$), then there exists a unique attracting (repelling) invariant closed curve which bifurcates from E_2.*

4. NUMERICAL SIMULATIONS

In this section, theoretical results are supported with graphics by using Mathematica [15, 26] and SageMath [12] programming. Some numerical simulations is given to demonstrate existence of Neimark-Sacker bifurcation for the system (1.2). Here, trajectories, bifurcation diagrams and phase portraits are illustrated by taking b as bifurcation parameter.

Example 1. Let us consider the following system for the parameter values $e = 0.5$, $a = 2.5$ and $b = 0.24$,

$$x_{n+1} = 2.5x_n(1 - x_n) - x_n y_n \tag{4.1}$$

$$y_{n+1} = \frac{1}{0.24}x_n y_n - 0.5y_n.$$

The coexistence positive equilibrium point is $(\bar{x}, \bar{y}) = (0.36, 0.6)$, and the Jacobian matrix evaluated $(\bar{x}, \bar{y})$ is obtained as follows:

$$J_{(\bar{x}, \bar{y})} = \begin{pmatrix} 0.1 & -0.36 \\ 2.5 & 1 \end{pmatrix}.$$

If the eigenvalues are calculated, then we obtain

$$\lambda_{1,2} = 0.55 \mp 0.835165i$$

such that $|\lambda_{1,2}| = 1$. Let

$$q \sim (-0.3340658618 + 0.1800000000i, -i)^T,$$

and

$$p \sim (-0.3340658618 - 0.1800000000i, i)^T,$$

be complex eigenvectors corresponding to $\lambda_{1,2}$ respectively.

If the normalized vector is taken as

$$q \sim (-0.3340658618 + 0.1800000000i, -i)^T$$

to obtain the normalization $< p, q >= 1$, the vector is found

$$p \sim (0,331611 + 0.238793i, 0.139463 - 1.0678i)^T.$$

By transformation of variables

$$u_n = x_n - 0.36, \ v_n = y_n - 0.6$$

the system (4.1) can be written as follows:

$$u_{n+1} = -2.5u_n^2 + 0.1u_n - 0.36v_n - u_nv_n$$
$$y_{n+1} = 2.5u_n + v_n + 4.16667u_nv_n.$$

When the coefficient of the form (3.14) are calculated, we get

$$g_{20}(b_1) = -3.03009 + 2.14832i$$
$$g_{11}(b_1) = -0.328575 - 1.51574i$$
$$g_{02}(b_1) = 2.94708 + 1.41614i$$
$$g_{21}(b_1) = 0.$$

From (3.15), we get as $\varphi(b_{NS}) = -2.76917 < 0$ such that $\theta = 0.988432$. Consequently, the Neimark-Sacker bifurcation emerges at $b_{NS} = 0.24$.

To confirm the theoretical result, the trajectories, the bifurcation diagrams and phase portraits of the prey-predator system (1.2) are shown in Figure 1, Figure 2 and Figure 3.

5. CONCLUSIONS

This paper contains the complex dynamic behavior of the prey predator system (1.2) with an efficiency of prey conversion into predators and minimum threshold of prey *consumption required before predators begin to reproduce*. We investigate stability conditions of the equilibrium points of the system (1.2), and show that the system (1.2) displays a Neimark-Sacker bifurcations.

We find that the system (1.2) has a trivial (extinction) equilibria E_0, an exclusion equilibria E_1 and a coexistence equilibria E_2. We give asymptotic stability conditions of these equilibria by using linearization method. It is clear that there is unique positive coexistence equilibria E_2 of the system (1.2) with $a > 1$ and $b < \frac{a-1}{a(e+1)}$. Moreover, it is proved that system (1.2) undergoes Neimark-Sacker bifurcation under the condition $b = \frac{a-1}{2a+ae}$ by using mathematical techniques of bifurcation theory. It is seen that Neimark–Saker bifurcation appears when the parameters vary on the neighborhood $NSB_{E_2} = \left\{ a, b, e \in \mathbb{R}^+ : 0 < b < \frac{-2}{\alpha} + 2\sqrt{\frac{a+e}{a^2(e+1)}} \text{ and } b = b_1 \right\}$. Some figures present dynamical properties of system (1.2) which has an attracting invariant curve for $b < 0.24$, $e = 0.5$, $a = 2.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$. As the parameter b exceed 0.24, the system becomes stable. If the parameter b exceeds 0.4, the predation population disappears. We conclude that the parameter b has a different effect on the dynamics on system (1.2) with the biological effect of e.
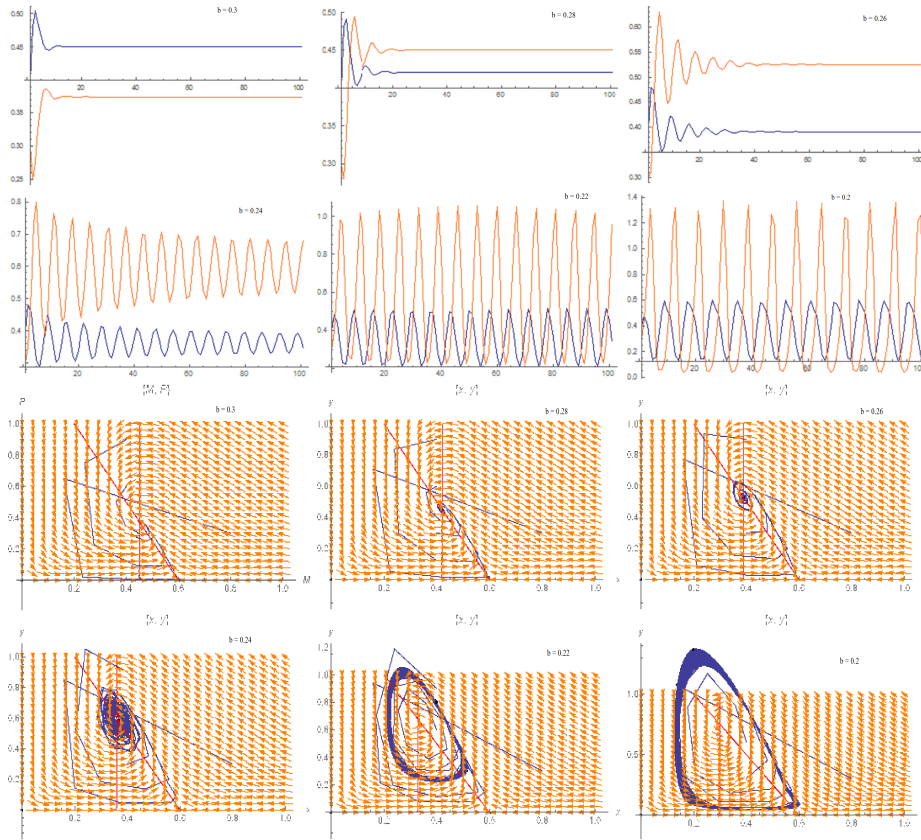
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FIGURE 1. Trajectories of the prey-predator system (1.2) for different values b with the parameter values $a = 2.5$, $e = 0.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$.
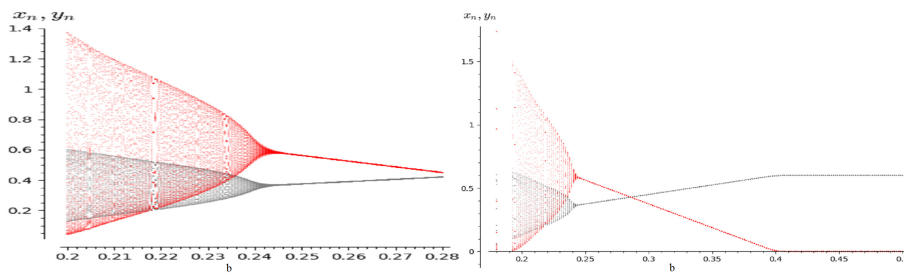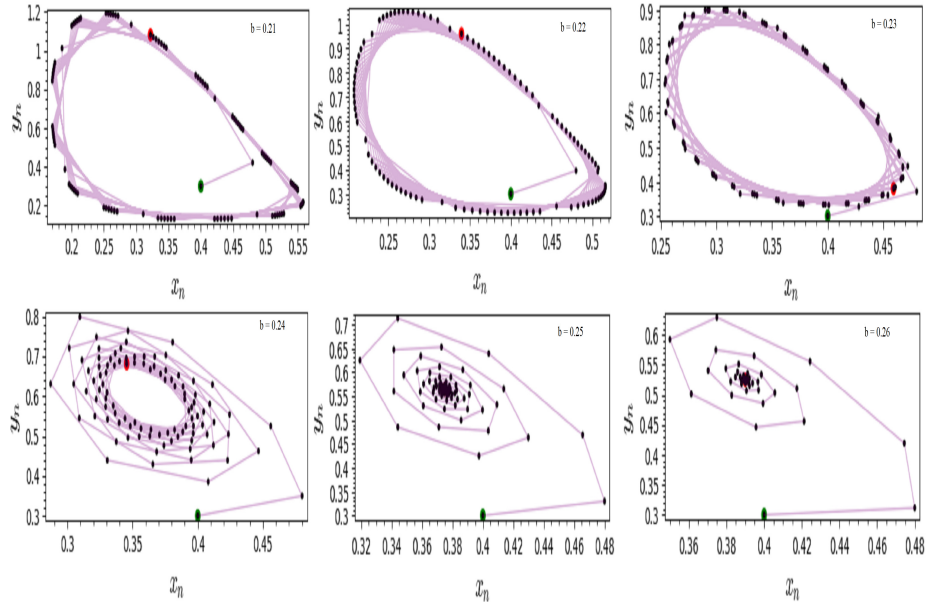

FIGURE 2. Trajectories of the prey-predator system (1.2) for different values b with the parameter values $a = 2.5$, $e = 0.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$.

FIGURE 3. The phase portraits of the prey-predator system (1.2) with the parameter values $a = 2.5$, $e = 0.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$.

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