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OSCILLATORY PROPERTIES OF EVEN-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. The aim of this paper is to investigate oscillatory properties of even-order advanced differential equations. The key idea of our approach is to conduct a comparison with first order equations and use the Riccati transformation technique. Some new oscillation criteria are shown. Two examples are presented in order to clarify the main results.

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1. INTRODUCTION

Advanced differential equations can find application in several real world problems where the evolution rate depends not only on the present, but also on the future. Hence, an advance could be introduced into the equation in order to take into account the influence of potential future actions. For instance, it is worth pointing out that there are a lot of applications to dynamical systems, mathematics of networks, optimization, and their application in the mathematical modelling of engineering problems, such as concerning electrical power systems, materials, energy (see [13]).

In this paper, we establish new oscillation criteria for all solutions of non-linear even-order differential equations with variable coefficients and advanced term of the form

$$\left(a(t)\left(w^{(n-1)}(t)\right)^{\beta}\right)' + \sum_{i=1}^{j} q_i(t)\, g\left(w\left(\eta_i(t)\right)\right) = 0, \quad t \geq t_0, \tag{1.1}$$

where $j \geq 1$ and β is a quotient of odd positive integers.

Throughout this paper, we suppose that:

- $a \in C^1([t_0,\infty),\mathbb{R})$,
- $a(t) > 0,\ a'(t) \geq 0,\ q_i, \eta_i \in C([t_0,\infty),\mathbb{R})$,
- $q_i(t) \geq 0$,
- $\eta_i(t) \geq t,\ \lim_{t\to\infty} \eta_i(t) = \infty,\ i = 1,2,..,j,$

- $g \in C(\mathbb{R}, \mathbb{R})$ such that $\frac{g(x)}{x^{\beta}} \geq k > 0$, for $x \neq 0$
- $\int_{t_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \mathrm{d}s = \infty$.

By a *solution* of (1.1) we mean a function $w \in C^{n-1}[t_w, \infty)$, $t_w \geq t_0$, which has the property $a(t)\left(w^{(n-1)}(t)\right)^{\beta} \in C^1[t_w, \infty)$, and satisfies (1.1) on $[t_w, \infty)$. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[t_w, \infty)$; otherwise, it is called *non-oscillatory*. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

2. The aim of this paper

In the last decade, the interest in studying of oscillation properties for differential equations increased (see for instance [3–8, 10–12, 14–19, 23–25]). On the other hand, the study of qualitative properties of solutions to differential equations were extensively studied also in [20–22].

The aim of this paper is to complement and enrich the results contained in [1,2,9]. For this reason, it is interesting to discuss briefly the aforementioned results.

By using the comparison technique, the equation

$$\left(\left(w^{(n-1)}(t)\right)^{\beta}\right)' + q(t)\, w^{\beta}(\eta(t)) = 0, \tag{2.1}$$

have been studied by Agarwal and Grace [1] and they proved that (2.1) is oscillatory if

$$\liminf_{t\to\infty} \int_t^{\eta(t)} (\eta(s) - s)^{n-2} \left(\int_s^{\infty} q(t)\, dt\right)^{1/\beta} ds > \frac{(n-2)!}{e}. \tag{2.2}$$

Agarwal et al. in [2] extended the Riccati transformation obtaining as new oscillation criterion for (2.1), the following condition:

$$\limsup_{t\to\infty} t^{\beta(n-1)} \int_t^{\infty} q(s)\, ds > ((n-1)!)^{\beta}. \tag{2.3}$$

In [9] the authors studied the oscillatory behaviour of (2.1) for $\beta = 1$. Using the Riccati transformation, they show that (2.1) is oscillatory if there exists a function $\mu \in C^1([t_0, \infty), (0, \infty))$, such that

$$\int_{t_0}^{\infty} \left(\mu(s)\, q(s) - \frac{(n-2)!\,(\mu'(s))^2}{2^{3-2n} s^{n-2} \mu(s)}\right) ds = \infty. \tag{2.4}$$

We apply the previous results to the equation

$$w^{(4)}(t) + \frac{q_0}{t^4} w(3t) = 0.\ t \geq 1, \tag{2.5}$$

(1) Applying the condition (2.2) contained in [1], we get

$$q_0 > 13.6.$$

(2) Applying condition (2.3) contained in [2], we get

$$q_0 > 18.$$

(3) Applying condition (2.4) contained in [9], we get

$$q_0 > 576.$$

From the above results, we can infer that the results in [2] improve the results contained in [9]. Moreover, the results in [1] improve the results [2, 9].

Thus, the motivation of this paper is complement and enrich the results in [1, 2, 9].

The key idea of our approach is to conduct a comparison with first order equations whose oscillatory behaviours are already known, using Riccati transformation technique.

The rest of the paper is organized as follows. Section 3 is devoted to the main results of the paper. We present our investigations for equation (1.1). Meanwhile, a relevant result on the existence of oscillatory oscillatory behaviour of solutions for first order differential equations is stated. The proofs rely on some mathematical inequalities and lemmas which are given for the sake of completeness. In Section 4, we provide two examples with specific parameters to illustrate the applicability of our theorems. The paper ends with some concluding remarks.

The following lemmas will be very useful in the sequel:

Lemma 1 ([16]). *If the function w satisfies $w^{(i)}(t) > 0$, $i = 0,1,\ldots,n$, and $w^{(n+1)}(t) < 0$, then*

$$\frac{w(t)}{t^n/n!} \geq \frac{w'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 2 ([25]). *Suppose that $w \in C^n([t_0,\infty),(0,\infty))$, $w^{(n)}$ is of a fixed sign on $[t_0,\infty)$, $w^{(n)}$ not identically zero and there exists a $t_1 \geq t_0$ such that*

$$w^{(n-1)}(t)\, w^{(n)}(t) \leq 0,$$

for all $t \geq t_1$. If we have $\lim_{t\to\infty} w(t) \neq 0$, then there exists $t_\theta \geq t_1$ such that

$$w(t) \geq \frac{\theta}{(n-1)!} t^{n-1} \left| w^{(n-1)}(t) \right|,$$

for every $\theta \in (0,1)$ and $t \geq t_\theta$.

Lemma 3 ([3]). *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta},\ V > 0.$$

Lemma 4 ([15]). *Assume that w is an eventually positive solution of (1.1). Then, there exist two possible cases:*

$(\mathbf{S}_1)$ $w(t)>0,\ w'(t)>0,\ w''(t)>0,\ \ w^{(n-1)}(t)>0,\ \ w^{(n)}(t)<0,$
$(\mathbf{S}_2)$ $w(t)>0,\ w^{(r)}(t)>0,\ w^{(r+1)}(t)<0$ *for all odd integer* $r\in\{1,3,...,n-3\},\ w^{(n-1)}(t)>0,\ w^{(n)}(t)<0,$

for $t\geq t_1$, where $t_1\geq t_0$ is sufficiently large.

3. Oscillation criteria

In the following theorem, we compare the oscillatory behaviour of (1.1) with suitable first-order differential equations.

Theorem 1. *Assume that (1) holds. If the differential equations*

$$x'(t)+k\sum_{i=1}^{j}q_i(t)\left(\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\right)^{\beta}x(\eta(t))=0 \tag{3.1}$$

and

$$z'(t)+z(t)\frac{t}{(n-4)!}\int_t^{\infty}(\varsigma-t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma=0 \tag{3.2}$$

are oscillatory, then every solution of (1.1) is oscillatory.

Proof. On the contrary, let us assume that w is a positive solution of (1.1). Then, we can suppose that $w(t)$ and $w(\eta_i(t))$ are positive for all $t\geq t_1$ sufficiently large. From Lemma 4, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.
Let us assume that $(\mathbf{S}_1)$ holds. From Lemma 2, we get

$$w(t)\geq\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\left(a^{1/\beta}(t)\,w^{(n-1)}(t)\right),$$

for every $\theta\in(0,1)$ and for all large t. Thus, if we set

$$x(t)=a(t)\left(w^{(n-1)}(t)\right)^{\beta}>0,$$

then we see that δ is a positive solution of the inequality

$$x'(t)+k\sum_{i=1}^{j}q_i(t)\left(\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\right)^{\beta}x(\eta(t))\leq 0. \tag{3.3}$$

From [19, Theorem 1], we conclude that the corresponding equation (3.1) also has a positive solution, which is a contradiction.
Let us assume that $(\mathbf{S}_2)$ holds. Integrating (1.1) from t to m and using $w'(t)>0$, we obtain

$$a(m)\left(w^{(n-1)}(m)\right)^{\beta}-a(t)\left(w^{(n-1)}(t)\right)^{\beta}=-\int_t^{m}\sum_{i=1}^{j}q_i(s)\,g(w(\eta_i(s)))\,ds.$$

By virtue of $w'(t) > 0$ and $\eta_i(t) \geq t$, we get

$$a(m)\left(w^{(n-1)}(m)\right)^{\beta} - a(t)\left(w^{(n-1)}(t)\right)^{\beta} \leq -kw^{\beta}(t)\int_t^m \sum_{i=1}^{j} q_i(s)\,ds.$$

Letting $m \to \infty$, we see that

$$a(t)\left(w^{(n-1)}(t)\right)^{\beta} \geq kw^{\beta}(t)\int_t^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s$$

and so

$$w^{(n-1)}(t) \geq w(t)\left(\frac{k}{a(t)}\int_t^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}.$$

Integrating again from t to ∞ for a total of $n-4$ times, we get

$$w''(t) + \frac{w(t)}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma \leq 0. \tag{3.4}$$

Using Lemma 1, we get

$$w(t) \geq tw'(t), \tag{3.5}$$

From (3.4) and (3.5), we get

$$w''(t) + w'(t)\frac{t}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma \leq 0.$$

Thus, if we set

$$z(t) = w'(t),$$

then we see that δ is a positive solution of the inequality

$$z'(t) + z(t)\frac{t}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma \leq 0. \tag{3.6}$$

It is well known (see [19, Theorem 1]) that the corresponding equation (3.2) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 1. *Assume that* (1) *holds. If*

$$\liminf_{t\to\infty} \int_{\eta_i(t)}^{t} \sum_{i=1}^{j} q_i(s)\left(\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\right)^{\beta} \mathrm{d}s > \frac{((n-1)!)^{\beta}}{\mathrm{e}} \tag{3.7}$$

and

$$\liminf_{t\to\infty} \int_{\eta_i(t)}^{t} \frac{s}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{3.8}$$

are oscillatory, then every solution of (1.1) *is oscillatory.*

Lemma 5. *Assume that w be an eventually positive solution of (1.1) and $(\mathbf{S}_1)$ holds. If*

$$\phi(t) := \mu(t)\left(\frac{a(t)\left(w^{(n-1)}(t)\right)^{\beta}}{w^{\beta}(t)}\right), \tag{3.9}$$

where $\mu \in C^1([t_0,\infty),(0,\infty))$, then

$$\phi'(t) \le \frac{\mu'(t)}{\mu(t)}\phi(t) - k\mu(t)\sum_{i=1}^{j} q_i(t) - \frac{\beta\theta t^{n-2}}{(n-2)!\left(\mu(t)a(t)\right)^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}, \tag{3.10}$$

for all $t > t_1$, where t_1 large enough.

Proof. Let w is an eventually positive solution of (1.1) and let us assume that $(\mathbf{S}_1)$ holds. Thus, from Lemma 2, we get

$$w'(t) \ge \frac{\theta}{2}t^{n-2}w^{(n-1)}(t), \tag{3.11}$$

for every $\theta \in (0,1)$ and for all large t. From (3.9), we see that $\phi(t) > 0$ for $t \ge t_1$, and

$$\begin{aligned}\phi'(t) = \mu'(t)\frac{a(t)\left(w^{(n-1)}(t)\right)^{\beta}}{w^{\beta}(t)} + \mu(t)\frac{\left(a\left(w^{(n-1)}\right)^{\beta}\right)'(t)}{w^{\beta}(t)}\\ - \beta\mu(t)\frac{w^{\beta-1}(t)w'(t)a(t)\left(w^{(n-1)}(t)\right)^{\beta}}{w^{2\beta}(t)}.\end{aligned}$$

Using (3.11) and (3.9), we obtain

$$\begin{aligned}\phi'(t) &\le \frac{\mu'_{+}(t)}{\mu(t)}\phi(t) + \mu(t)\frac{\left(a(t)\left(w^{(n-1)}(t)\right)^{\beta}\right)'}{w^{\beta}(t)}\\ &\quad - \beta\mu(t)\frac{\theta}{(n-2)!}t^{n-2}\frac{a(t)\left(w^{(n-1)}(t)\right)^{\beta+1}}{w^{\beta+1}(t)}\\ &\le \frac{\mu'(t)}{\mu(t)}\phi(t) + \mu(t)\frac{\left(a(t)\left(w^{(n-1)}(t)\right)^{\beta}\right)'}{w^{\beta}(t)}\\ &\quad - \frac{\beta\theta t^{n-2}}{(n-2)!\left(\mu(t)a(t)\right)^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}.\end{aligned} \tag{3.12}$$

From (1.1) and (3.12), we obtain

$$\phi'(t) \le \frac{\mu'(t)}{\mu(t)}\phi(t) - k\mu(t)\frac{\sum_{i=1}^{j} q_i(t)w^{\beta}(\eta_i(t))}{w^{\beta}(t)} - \frac{\beta\theta t^{n-2}}{(n-2)!\left(\mu(t)a(t)\right)^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}.$$

Note that $w'(t) > 0$ and $\eta_i(t) \geq t$, thus, we find

$$\phi'(t) \leq \frac{\mu'(t)}{\mu(t)}\phi(t) - k\mu(t)\sum_{i=1}^{j} q_i(t) - \frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)\,a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}.$$

The proof is complete. □

Lemma 6. *Assume that w be an eventually positive solution of (1.1) and $(\mathbf{S}_2)$ holds. If*

$$\delta(t) := \vartheta(t)\frac{w'(t)}{w(t)}. \tag{3.13}$$

where $\vartheta \in C^1([t_0,\infty),(0,\infty))$, then

$$\delta'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) - \Phi(t) - \frac{1}{\vartheta(t)}\delta(t)^2, \tag{3.14}$$

for all $t > t_1$, where t_1 large enough and

$$\Phi(t) = \frac{\vartheta(t)}{(n-4)!}\int_t^{\infty}(\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma.$$

Proof. Let w is an eventually positive solution of (1.1) and let us assume that $(\mathbf{S}_2)$ holds. From the definition of $\delta(t)$, we see that $\delta(t) > 0$ for $t \geq t_1$. By differentiating, we find

$$\delta'(t) = \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) + \vartheta(t)\frac{w''(t)}{w(t)} - \frac{1}{\vartheta(t)}\delta(t)^2. \tag{3.15}$$

From (3.4) and (3.15), we obtain

$$\delta'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) - \frac{\vartheta(t)}{(n-4)!}\int_t^{\infty}(\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma - \frac{1}{\vartheta(t)}\delta(t)^2.$$

Thus,

$$\delta'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) - \Phi(t) - \frac{1}{\vartheta(t)}\delta(t)^2.$$

The proof is complete. □

In this theorem, we will establish an oscillation criterion for equation (1.1).

Theorem 2. *Assume that there exist positive functions $\mu, \vartheta \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\int_{t_0}^{\infty}\left(k\mu(s)\sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta}\,a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\,(t^{n-2}\theta\mu(t))^{\beta}}\right)\mathrm{d}s = \infty, \tag{3.16}$$

for some $\theta \in (0,1)$, and either

$$\int_{t_0}^{\infty}\sum_{i=1}^{j} q_i(s)\,ds = \infty \tag{3.17}$$

or

$$\int_{t_0}^{\infty}\left(\Phi(s)-\frac{1}{4\vartheta(s)}\left(\vartheta'(s)\right)^2\right)\mathrm{d}s=\infty. \tag{3.18}$$

Then every solution of (1.1) *is oscillatory.*

Proof. Assume that w is an eventually positive solution of (1.1). Then, we can suppose that $w(t)$ and $w(\eta_i(t))$ are positive for all $t\geq t_1$ sufficiently large. From Lemma 4, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.
Assume that $(\mathbf{S}_1)$ holds. From Lemma 5, we get that (3.10) holds. Using Lemma 3 with

$$U=\frac{\mu'(t)}{\mu(t)},\quad V=\frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)\,a(t))^{\frac{1}{\beta}}}\quad\text{and } x=\phi(t),$$

we get

$$\frac{\mu'(t)}{\mu(t)}\phi(t)-\frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)\,a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}\leq-\frac{((n-2)!)^{\beta}a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\left(t^{n-2}\theta\mu(t)\right)^{\beta}}. \tag{3.19}$$

From (3.10) and (3.19), we obtain

$$\phi'(t)\leq-k\mu(t)\sum_{i=1}^{j}q_i(t)+\frac{((n-2)!)^{\beta}a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\left(t^{n-2}\theta\mu(t)\right)^{\beta}}.$$

Integrating from t_1 to t, we get

$$\int_{t_1}^{t}\left(k\mu(s)\sum_{i=1}^{j}q_i(s)-\frac{((n-2)!)^{\beta}a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\left(t^{n-2}\theta\mu(t)\right)^{\beta}}\right)\mathrm{d}s\leq\phi(t_1),$$

for every $\theta\in(0,\ 1)$, which contradicts (3.16). Assume that $(\mathbf{S}_2)$ holds. Integrating (1.1) from m to t, we conclude that

$$-a(m)\left(w^{(n-1)}(m)\right)^{\beta}=-\int_m^t\sum_{i=1}^{j}q_i(s)\,g(w(\eta_i(s)))\,ds.$$

By virtue of $w'(t)>0$ and $\eta_i(t)\geq t$, we get

$$\int_m^t\sum_{i=1}^{j}q_i(s)\,ds\leq\frac{a(m)\left(w^{(n-1)}(m)\right)^{\beta}}{kw^{\beta}(m)},$$

which contradicts (3.17).
From Lemma 6, we get that (3.14) holds. Using Lemma 3 with

$$U=\vartheta'(t)/\vartheta(t),\ V=1/\vartheta(t),\beta=1\text{ and } x=\delta(t),$$

we get

$$\frac{\vartheta'(t)}{\vartheta(t)}\delta(t)-\frac{1}{\vartheta(t)}\delta^2(t)\leq-\frac{1}{4\vartheta(t)}\left(\vartheta'(t)\right)^2. \tag{3.20}$$

From (3.14) and (3.20), we obtain

$$\delta'(t) \leq -\Phi(t) + \frac{1}{4\vartheta(t)} \left(\vartheta'(t)\right)^2. \tag{3.21}$$

Integrating from t_1 to t, we get

$$\int_{t_1}^{t} \left(\Phi(s) - \frac{1}{4\vartheta(s)} \left(\vartheta'(s)\right)^2 \right) \mathrm{d}s \leq \delta(t_1),$$

which contradicts (3.18). The proof is complete. □

Putting $\mu(t) = t^{n-1}$ and $\vartheta(t) = t$ into Theorem 2, we get the following oscillation criterion:

Corollary 2. *Let* (1.1) *hold. Assume that*

$$\int_{t_0}^{\infty} \left(s^{n-1} \sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta} (n-1)^{\beta+1} s^{-n\beta+n+\beta-2} a(s)}{(\beta+1)^{\beta+1} \theta^{\beta}} \right) \mathrm{d}s = \infty, \tag{3.22}$$

or some $\theta \in (0,1)$. *If* (3.17) *holds and*

$$\int_{t_0}^{\infty} \left(\frac{s}{(n-4)!} \int_{t}^{\infty} (\varsigma - t)^{n-4} \left(\frac{k}{a(\varsigma)} \int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s \right)^{1/\beta} \mathrm{d}\varsigma - \frac{1}{4s} \right) \mathrm{d}s = \infty, \tag{3.23}$$

then every solution of (1.1) *is oscillatory.*

4. Examples and concluding remarks

In this section we show two numerical examples as application of the theoretical findings discussed in the previous section. Finally, we list some perspectives for future works.

Example 1. Let us consider the differential equation

$$\left(t\left(w'''(t)\right)\right)' + \frac{d_0}{t^3} w(ct) = 0,\ t \geq 1, \tag{4.1}$$

where $c > 0$ and $d_0 > 1$ is a constant. Note that $\beta = 1,\ n = 4,\ a(t) = t,\ q(t) = d_0/t^3$. If we set $k = 1$, and $\mu(s) = s^3$,then

$$\begin{aligned}
&\int_{t_0}^{\infty} \left(k\mu(s) \sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta} a(s) \left(\mu'(s)\right)^{\beta+1}}{(\beta+1)^{\beta+1} \left(s^{n-2}\theta\mu(s)\right)^{\beta}} \right) \mathrm{d}s \\
&= \int_{t_0}^{\infty} \left(d_0 \frac{s^3}{s^3} - \frac{3^2 s^5}{2\theta s^5} \right) \mathrm{d}s = \int_{t_0}^{\infty} \left(d_0 - \frac{9}{2\theta} \right) ds \\
&= \infty \quad \text{if} \quad d_0 > \frac{9}{2\theta},
\end{aligned}$$

for some constant $\theta \in (0,\ 1)$. Hence, by Theorem 2, every solution of equation (4.1) is oscillatory if

$$d_0 > \frac{9}{2\theta}.$$

Remark 1. Applying the condition (3.22) to the equation (2.5), we find

$$q_0 > 4.5.$$

Therefore, our result improves the results contained in [1, 2, 9].

Example 2. Let us consider the differential equation

$$w^{(4)}(t) + \frac{q_0}{t^4} w(2t) = 0,\ t \geq 1, \tag{4.2}$$

where $q_0 > 0$ is a constant. Note that $\beta = 1,\ n = 4,\ a(t) = 1,\ q(t) = q_0/t^4$ and $\eta(t) = 2t$. If we set $k = 1$, then condition (3.16) becomes

$$\begin{aligned} &\int_{t_0}^{\infty} \left(s^{n-1} \sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta} (n-1)^{\beta+1} s^{-n\beta+n+\beta-2} a(s)}{(\beta+1)^{\beta+1} \theta^{\beta}} \right) \mathrm{d}s \\ &= \int_{t_0}^{\infty} \left(\frac{q_0}{s} - \frac{9}{2\theta s} \right) ds = \left(q_0 - \frac{9}{2\theta} \right) \int_{t_0}^{\infty} \frac{1}{s} ds \\ &= \infty \quad \text{if} \quad q_0 > 4.55 \ \ (\text{let } \theta = 99/100) \end{aligned}$$

and condition (3.18) becomes

$$\begin{aligned} &\left(\frac{s}{(n-4)!} \int_{t}^{\infty} (\varsigma - t)^{n-4} \left(\frac{k}{a(\varsigma)} \int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s) \, \mathrm{d}s \right)^{1/\beta} \mathrm{d}\varsigma - \frac{1}{4s} \right) \mathrm{d}s \\ &= \int_{t_0}^{\infty} \left(\frac{q_0}{6s} - \frac{1}{4s} \right) \mathrm{d}s \\ &= \infty, \quad \text{if} \quad q_0 > \frac{3}{2}. \end{aligned}$$

Therefore, from Corollary 2, every solution to the equation (4.2) is oscillatory if $q_0 > 4.55$.

Remark 2. Now we compare our result with the known related criteria for oscillation of the equation under consideration.

(1) Applying condition (2.2) in [1], we get

$$q_0 > 25.5.$$

(2) Applying condition (2.3) in [2], we obtain

$$q_0 > 18.$$

(3) Applying condition (2.4) in [9], we find

$$q_0 > 1728.$$

Therefore, our result improves the results contained in [1, 2, 9].

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