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THE STRUCTURE OF THE UNIT GROUP OF SOME GROUP ALGEBRAS

ZS. BALOGH

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Abstract. Let FM be the group algebra of the modular 2-group M over a finite field F of characteristic two. In the present note we establish the structure of the unit group of the group algebra FM and verify the question of Johnson.

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1. Introduction and Results

Let F be a field of characteristic p and let G be a group such that G contains an element of order p. Let $U(FG)$ be the group of units of the group algebra FG. It is easy to see that $U(FG) = U(F) \times V(FG)$, in which

$$V(FG) := \left\{ x = \sum_{g \in G} \alpha_g g \in U(FG) \mid \chi(x) = \sum_{g \in G} \alpha_g = 1 \right\},$$

where $\chi(x)$ is the augmentation of $x \in FG$ (see [8, Chapters 2-3, p. 194-196]).

The structure of the group of units and its subgroup $V(FG)$ has been investigated by several authors, but the complete description is known only for certain group algebras (for example, see [1, 2, 10–13, 15, 16, 18–21]). For an overview in this topic we recommend the survey paper [8].

Let $\zeta(G)$ be the center and let G' be the commutator subgroup of G, respectively. It is well known [9, Theorem 2], if FG is a modular group algebra, then $G \cap \zeta(V(FG)) = \zeta(G)$ and $G \cap V(FG)' = G'$. The question whether $G \cap V(FG)^p = G^p$ is due to Johnson [15]. The Johnson's question was affirmatively confirmed for nonabelian groups in the following cases: (i) the group of exponent p and order p^3 [15, Theorem 7]; (ii) G is a finite p-group (p is an odd prime) with Frattini subgroup of order p [4]; and (iii) G is the modular 2-group of order 16 and F is the field of 2 elements [14, Theorem 2]. The structure of elements of order two in $V(FG)$, where

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G is a 2-group of maximal class and F is the field of elements two, was described in [5].

Let

$$M_n = \langle\, a,b \mid a^{2^{n-1}} = b^2 = 1, (a,b) = a^{2^{n-2}} \,\rangle = \langle\, a \,\rangle \rtimes \langle\, b \,\rangle, \qquad (n \geq 4) \qquad (1.1)$$

be the modular 2-group. The group M_n appears very frequently in the investigation of the group of units [3,6,7,14,17].

In the present note the structure of $V(FM_n)$ is established and affirmative answer for the Johnson's question is provided.

Theorem 1. *Let M_n be the modular group given in (1.1). If F is a field with $|F| = 2^m \geq 2$, then $V(FM_n)$ is a central extension of $C_2^{3m2^{n-3}}$ by*

$$C_{2^{n-2}}^{m} \times C_2^{7m2^{n-4}} \times \prod_{i=0}^{n-5} C_{2^{n-i-3}}^{2^i m}.$$

Corollary 1. *Let M_n be the modular group of order 2^n. If F is a field with $|F| = p^m \geq 2$, then*

$$M_n \cap V(FM_n)^2 = M_n^2.$$

2. Proof

Let H be a normal subgroup of a finite p-group G. The ideal of FG generated by the set $\{h-1 \mid h \in H\}$ is denoted by $I(H)$. Let $G[p^i]$ denote the subgroup of G generated by the elements of order p^i. We use the notation G^{p^i} for the subgroup $\langle\, g^{p^i} \mid g \in G \,\rangle$. Set $x^g := g^{-1}xg$, where $g \in G$ and $x \in FG$. Let $\widehat{S} = \sum_{s\in S} s \in FG$, where $S \subseteq G$ is a finite subset and let $|S|$ denote the cardinality of S. Furthermore, the order of $g \in G$ will be denoted by $|g|$.

If G is an abelian p-group, then the number of subgroups of order p^i in the decomposition of G into a direct product of cyclic groups will be denoted by $f_i(G)$.

Lemma 1. *Let F be a field with $|F| = p^m \geq p$. If G is a finite abelian p-group, then*

(i) $V(FG)^p = V(FG^p)$;
(ii) $V(FG)[p] = 1 + I(G[p])$; *and*
(iii) $f_i\big(V(FG)\big) = m\big(|G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}|\big)$.

Proof. (i) If $u = \sum_{g\in G} \alpha_g g \in V(FG)$, then $u^p = \sum_{g\in G} \alpha_g^p g^p \in V(FG^p)$, so $V(FG)^p \subseteq V(FG^p)$.

Let $u = \sum_{g\in G^p} \alpha_g g \in V(FG^p)$. Obviously, the mapping $\tau(\alpha) = \alpha^p$ is an automorphism of F. Therefore there exists a $\beta_g \in F$ and $h \in G$ for every $g \in G^p$ such that $\beta_g^p = \alpha_g$ and $h^p = g$. We have that $u = \sum_{g\in G^p} \alpha_g g = \sum_{h\in G} \beta_g^p h^p \in V(FG)^p$, which completes the proof.

(ii) If $u \in 1 + I(G[p])$, then $u^p = 1$ and $1 + I(G[p]) \subseteq V(FG)[p]$.

Let $u \in V(FG)[p]$. Clearly, $u-1$ can be written as $x_1h_1+x_2h_2+\cdots+x_sh_s$ for some s, where $x_i \in FG[p]$ and the set $\{h_i\}$ is a complete set of right coset representatives of $G[p]$ in G. We have that $x_1^ph_1^p+x_2^ph_2^p+\cdots+x_s^ph_s^p=(u-1)^p=u^p-1=0$. Suppose that $h_i^p=h_j^p$ for some i, j and $i\neq j$. Clearly, $h_ih_j^{-1}\in G[p]$ which is impossible. Without loss of generality we can assume that $h_1=1$ and $h_i^p\neq 1$ if $1<i\leq s$. Therefore $x_i^p=0$ if $1<i\leq s$ and $u-1\in I(G[p])$ which proves that $V(FG)[p]\subseteq 1+I(G[p])$.

(iii) It is true when $|F|=p$ [18, Theorem 2.4]. Now we extend it to any finite field.

If $V(FG)=\langle\, a_1\,\rangle\times\cdots\times\langle\, a_s\,\rangle$, then $V(FG)[p]=\langle\, a_1^{b_1-1}\,\rangle\times\langle\, a_2^{b_2-1}\,\rangle\times\cdots\times\langle\, a_s^{b_s-1}\,\rangle$ in which $b_j=|a_j|$. The number of elements in $1+I(G[p])$ equals $|I(G[p])|$. Evidently, $I(G[p])$ can be considered as a vector space over F with the basis $\{u(h-1)\mid u\in T(G/G[p]),\ h\in G[p]\}$ in which $T(G/G[p])$ is a complete set of right coset representatives of $G[p]$ in G. Thus

$$|I(G[p])|=p^{m\frac{|G|}{|G[p]|}(G[p]-1)}=p^{m(|G|-|G^p|)}.$$

According to part (ii), the p-rank of $V(FG)$ is $m(|G|-|G^p|)$.

The part (i) shows that the p-rank of $V(FG)^p$ is $m(|G^p|-|G^{p^2}|)$, so

$$f_1\big(V(FG)\big)=m(|G|-|G^p|)-m(|G^p|-|G^{p^2}|)=m\big(|G|-2|G^p|+|G^{p^2}|\big).$$

The proof can be easily completed using part (i) and induction on $V(FG^{p^i})$. □

Lemma 2. *If F is a field with $|F|=2^m\geq 2$, then*

$$\zeta\big(V\big)=V\big(F\zeta(M_n)\big)\times N,$$

where $N\cong C_2^{3m2^{n-3}}$ and $f_i\Big(V\big(F\zeta(M_n)\big)\Big)=\begin{cases} m & \text{if}\quad i=n-2;\\ m\cdot 2^{n-3-i} & \text{if}\quad i<n-2.\end{cases}$

Proof. Let C_g be the conjugacy class of $g\in M_n\setminus\zeta(M_n)$. Clearly, $|C_g|=2$, $M_n'=\{1,a^{2^{n-2}}\}$ and $\widehat{C_g}=g\widehat{M_n'}$. Let N be defined by

$$N=\langle\ 1+\beta_ia^{2i+1}\widehat{M_n'}\mid 0\leq i<2^{n-3},\beta_i\in F\ \rangle\times\langle\ 1+\gamma_ia^ib\widehat{M_n'}\mid 0\leq i<2^{n-2},\gamma_i\in F\ \rangle.$$

Since $(1+x\widehat{M_n'})(1+y\widehat{M_n'})=1+x\widehat{M_n'}+y\widehat{M_n'}$ and $(1+x\widehat{M_n'})^2=1$ for every $x,y\in FM_n$, the group $N\cong C_2^{3m2^{n-3}}$ is an elementary abelian 2-group and

$$\zeta\big(V\big)=V\big(F\zeta(M_n)\big)\times N.$$

Indeed, $V\big(F\zeta(M_n)\big)\times N\subseteq\zeta\big(V\big)$. Since $\zeta(M_n)=M_n^2$, each element $x\in\zeta\big(V\big)$ can be written as $x=x_1+x_2$ in which

$$x_1=\textstyle\sum_{i=0}^{2^{n-2}-1}\alpha_ia^{2i},\quad x_2=\sum_{i=0}^{2^{n-3}-1}\beta_ia^{2i+1}\widehat{M'}+\sum_{i=0}^{2^{n-2}-1}\gamma_ia^ib\widehat{M'},\qquad(\alpha_i,\beta_i,\gamma_i\in F).$$

It is clear, that the augmentation of x_2 equals 0 therefore x_1 is an invertible element with augmentation 1. Obviously, $1+a^{2i}\cdot x_2\in N$ therefore $x_1^{-1}x=x_1^{-1}(x_1+x_2)=$

$1+x_1^{-1}x_2 \in N$. Since $V\big(F\zeta(M_n)\big)\cap N=\{1\}$ we have proved that $\zeta(V)\subseteq V\big(F\zeta(M_n)\big)\times N$.

Since $\zeta(M_n)=M_n^2\cong C_{2^{n-2}}$, Lemma 1(iii) ensures that

$$f_i\Big(V\big(F\zeta(M_n)\big)\Big)=m(2^{n-1-i}-2\cdot 2^{n-2-i}+2^{n-3-i})=m2^{n-i-3}$$

for $i<n-2$ and $f_i\Big(V\big(F\zeta(M_n)\big)\Big)=m$ for $i=n-2$. □

Lemma 3. *Let F be a field with $|F|=2^m\geq 2$. Then $|\zeta(V)|=2^{5m2^{n-3}-m}$ and*

$$\zeta(V)\cong C_{2^{n-2}}^{m}\times C_2^{7m2^{n-4}}\times\prod_{i=0}^{n-5}C_{2^{n-i-3}}^{2^im}.$$

Proof. According to the previous lemma, $\zeta(V)\cong V\big(F\zeta(M_n)\big)\times N$. Since $|N|=2^{3m2^{n-3}}$ and $|V\big(F\zeta(M_n)\big)|=|F|^{|\zeta(M_n)|-1}=2^{m(2^{n-2}-1)}$, we can easily compute that

$$|\zeta(V)|=2^{3m2^{n-3}+m2^{n-2}-m}=2^{5m2^{n-3}-m}.$$

Finally, using Lemma 2, it is easy to check that

$$\zeta(V)\cong C_{2^{n-2}}^{m}\times C_{2^{n-3}}^{m}\times C_{2^{n-4}}^{2m}\times C_{2^{n-5}}^{2^2m}\times C_{2^{n-6}}^{2^3m}\times\cdots\times C_{2^2}^{m2^{n-5}}\times C_2^{7m2^{n-4}}.$$

□

Proof of Theorem. Each $x\in FM_n$ can be written as $x=x_1+x_2b$, where $x_1,x_2\in F\langle\, a\,\rangle$ (see (1.1)) and

$$x^2=x_1^2+x_2x_2^b+(x_1+x_1^b)x_2b.$$

Obviously, we can write $x_1=y_1+y_2a$ and $x_2=z_1+z_2a$, where $y_1,y_2,z_1,z_2\in F\langle\, a^2\,\rangle$, so

$$x_1+x_1^b=y_2a(1+a^{2^{n-2}})\in I(M_n')\quad\text{ and }$$

$$x_2x_2^b=z_1^2+z_2^2a^{2^{n-2}+2}+z_1z_2a(1+a^{2^{n-2}})\in\zeta(FM_n).$$

Consequently, $(x_1+x_1^b)x_2b\in I(M_n')\subseteq\zeta(FM_n)$ and $x^2\in\zeta\big(V(FM_n)\big)$ for every $x\in V(FM_n)$. Hence $V(FM_n)/\zeta(V)$ is an elementary abelian 2-group of order $2^{3m2^{n-3}}$ and

$$\zeta(V)\cong C_{2^{n-2}}^{m}\times C_{2^{n-3}}^{m}\times C_{2^{n-4}}^{2m}\times C_{2^{n-5}}^{2^2m}\times C_{2^{n-6}}^{2^3m}\times\cdots\times C_{2^2}^{m2^{n-5}}\times C_2^{7m2^{n-4}}$$

by Lemma 3 which is the desired conclusion. □

Proof of Corollary. Since $V(FM_n)^2\subseteq\zeta(V)$ and $\zeta(M_n)=M_n^2$,

$$M_n\cap V(FM_n)^2\subseteq M_n\cap\zeta(V)\subseteq\zeta(M_n)=M_n^2.$$

□

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Author’s address

Zs. Balogh

Department of Math. Sci., College of Science, United Arab Emirates University, Al Ain, United Arab Emirates,

E-mail address: baloghzsa@gmail.com