



A NEW CLASS OF SPECIAL FUNCTIONS IN HIGHER DIMENSIONS

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Abstract. We introduce a new class of functions \mathcal{G}_f^ν that are defined via integrals over σ -finite measure spaces and their kernels are defined in terms of Bessel functions and measurable functions. We prove that for certain values of ν , the function \mathcal{G}_f^ν is the Fourier transform of the distribution function d_f .

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1. INTRODUCTION

Bessel function of first kind, J_ν , is defined to be the solution of the Bessel differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

see [6], a series formula of Bessel function can be given by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k}$$

In this paper we study a higher dimensional analog of the function

$$g_\nu(x) = \Gamma(\nu + 1) \left(\frac{2}{x}\right)^\nu J_\nu(x),$$

where $x \in \mathbb{R}$, $\nu > \frac{-1}{2}$ and J_ν stands for the Bessel function of the first kind, see [1]. Inequalities involving the function g_ν have been studied by many authors. For instance, Askey [2] wrote Grünbaum's inequality [4] in terms of the function g_ν . we refer the readers to [1, 5] for more background information.

For $n \geq 2$, let \mathbb{R}^n be the n -dimensional Euclidean space and \mathbb{S}^{n-1} denote the unit sphere in \mathbb{R}^n equipped with the surface measure $d\sigma$. Let (X, μ) be a σ -finite measure space, that is, there exists a sequence of measurable subsets $(A_n)_{n \in \mathbb{N}}$ such that $X =$

$\bigcup_{n \in \mathbb{N}} A_n$ and $\mu(A_n) < \infty$ for all n . Let $f : X \rightarrow \mathbb{C}$ be a measurable function. Now, if $f \in L^{\nu+1}(X, \mu)$ consider the function

$$\mathcal{G}_f^\nu(\xi) = \frac{1}{|\xi|^\nu} \int_X |f(x)| J_\nu(2\pi|f(x)||\xi|) \, d\mu(x),$$

where $\xi \in \mathbb{R}^n$, $\nu \geq \frac{-1}{2}$ and

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{its} (1-s^2)^\nu \frac{ds}{\sqrt{1-s^2}}.$$

The function $\mathcal{G}_f^\nu(\xi)$ is well defined. To see this, we need to recall the following properties of Bessel functions.

$$|J_\nu(t)| \leq C(\nu)t^\nu, \tag{1.1}$$

$$|J_\nu(t)| \leq C(\nu)t^{-1/2}, \tag{1.2}$$

where $t > 0$ and the constant $C(\nu)$ depends only on ν . Thus, by using (1.1) we have

$$\begin{aligned} |\mathcal{G}_f^\nu(\xi)| &= \left| \frac{1}{|\xi|^\nu} \int_X |f(x)| J_\nu(2\pi|f(x)||\xi|) \, d\mu(x) \right| \\ &\leq \frac{1}{|\xi|^\nu} \int_X |f(x)| |J_\nu(2\pi|f(x)||\xi|)| \, d\mu(x) \\ &\leq (2\pi)^\nu \|f\|_{\nu+1}^{\nu+1} < \infty, \end{aligned}$$

from which we have that \mathcal{G}_f^ν is bounded above. Notice also that if $f \equiv \frac{1}{2\pi}$ and X is a finite measure space with $\mu(X) = 2^{\nu+1}\pi\Gamma(\nu + 1)$, then $\mathcal{G}_f^\nu(x) = g_\nu(x)$ for $x > 0$.

In the remaining part of this paper we clarify how this class of functions came into being and how it is related to some other well-known functions.

2. MAIN RESULT

Functions are central objects in most fields of mathematics and science. One of the most important tools that we use to investigate the size of these objects, not its point-wise behavior is a function itself which is known as the distribution function. In the following, we review the main tools that we shall use to conclude our main result, see [3, 7–9].

We start by recalling the definition of the distribution function.

Definition 1. Let (X, μ) be a σ -finite measure space and $f : X \rightarrow \mathbb{C}$ be a measurable function. The distribution function d_f is given by

$$d_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\},$$

where $\alpha \geq 0$.

If $f \in L^p(X, \mu)$ for $0 < p < \infty$, then we have the following integral formula which correlates between the L^p -norm of f and its distribution function.

$$\|f\|_{L^p}^p = p \int_0^\infty d_f(\alpha) \alpha^{p-1} d\alpha. \tag{2.1}$$

We know that if η is a radial function, then

$$\int_{\mathbb{R}^n} \eta(x) dx = \sigma(\mathbb{S}^{n-1}) \int_0^\infty \eta(r) r^{n-1} dr, \tag{2.2}$$

where $\sigma(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Thus, in view of (2.1) and (2.2) we can think about d_f as an integrable radial function on \mathbb{R}^p . To be more precise, we have

$$\|f\|_{L^p}^p = \frac{p}{\sigma(\mathbb{S}^{p-1})} \int_{\mathbb{R}^p} d_f(|y|) dy. \tag{2.3}$$

Equation (2.3) says that $d_f \in L^1(\mathbb{R}^p)$ whenever $f \in L^p(X, \mu)$ and so its Fourier transform exists. The following result shows how the class \mathcal{G}_f^v did arise.

Theorem 1. *Let X be a σ -finite measure space and $f \in L^2(X, \mu)$. Then $\mathcal{G}_f^1(\xi) = \hat{d}_f(\xi)$, $\xi \in \mathbb{R}^2$.*

Proof. By using the fact that the Fourier transform of a radial function η is given by

$$\hat{\eta}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \eta(r) J_{\frac{n}{2}-1}(2\pi r|\xi|) r^{\frac{n}{2}} dr,$$

we have

$$\begin{aligned} \hat{d}_f(\xi) &= \int_{\mathbb{R}^2} e^{-2\pi i y \cdot \xi} d_f(|y|) dy \\ &= 2\pi \int_0^\infty d_f(r) J_0(2\pi r|\xi|) r dr. \end{aligned} \tag{2.4}$$

The definition of distribution function and (2.4) give

$$\begin{aligned} \hat{d}_f(\xi) &= 2\pi \int_0^\infty m\{x \in X : |f(x)| > r\} J_0(2\pi r|\xi|) r dr \\ &= 2\pi \int_0^\infty \left(\int_{\{z \in X : |f(z)| > r\}} d\mu \right) J_0(2\pi r|\xi|) r dr. \end{aligned} \tag{2.5}$$

Noting that $\mathbb{1}_{\{z \in X : |f(z)| > r\}}(x) \equiv \mathbb{1}_{(0, |f(x)|)}(r)$ and applying Fubini's theorem we obtain

$$\hat{d}_f(\xi) = 2\pi \int_0^\infty \left(\int_X \mathbb{1}_{\{z \in X : |f(z)| > r\}}(x) d\mu(x) \right) J_0(2\pi r|\xi|) r dr$$

$$\begin{aligned}
 &= 2\pi \int_X \left(\int_0^\infty \mathbb{1}_{(0,|f(x)|)}(r) J_0(2\pi r|\xi|) r \, dr \right) \, d\mu(x) \\
 &= 2\pi \int_X \left(\int_0^{|f(x)|} J_0(2\pi r|\xi|) r \, dr \right) \, d\mu(x). \tag{2.6}
 \end{aligned}$$

Now, by using the change of variables, $s = r/|f(x)|$, we have

$$\hat{d}_f(\xi) = 2\pi \int_X |f(x)|^2 \left(\int_0^1 J_0(2\pi s|f(x)||\xi|) s \, ds \right) \, d\mu(x). \tag{2.7}$$

Next, applying the integral identity

$$\int_0^1 J_\alpha(ts) s^{\alpha+1} (1-s^2)^\beta \, ds = \frac{\Gamma(\beta+1)2^\beta}{t^{\beta+1}} J_{\alpha+\beta+1}(t) \tag{2.8}$$

with $\alpha = \beta = 0$, we get

$$\begin{aligned}
 \hat{d}_f(\xi) &= 2\pi \int_X |f(x)|^2 \left(\frac{1}{2\pi|f(x)||\xi|} J_1(2\pi|f(x)||\xi|) \right) \, d\mu(x) \\
 &= \frac{1}{|\xi|} \int_X |f(x)| J_1(2\pi|f(x)||\xi|) \, d\mu(x) \\
 &= \mathcal{G}_f^1(\xi). \tag{2.9}
 \end{aligned}$$

□

Remark 1. If p is an integer greater than 2, then using the integral identity

$$\int t^\nu J_{\nu-1}(t) \, dt = t^\nu J_\nu(t) + C.$$

in place of (2.8) and applying the same technique that was used in the proof of Theorem 1 we have $\mathcal{G}_f^{p/2}(\xi) = \hat{d}_f(\xi)$, $\xi \in \mathbb{R}^p$.

Remark 2. Theorem 1 and Remark 1 show that $\mathcal{G}_f^{p/2}(\xi)$ enjoys the property that it is uniformly continuous on \mathbb{R}^p . Moreover, $\mathcal{G}_f^{p/2}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

Theorem 2. *If X is a finite measure space and $f \in L^2(X, \mu)$, then $\mathcal{G}_f^1 \in L^2(\mathbb{R}^2)$.*

Proof. Notice that

$$\begin{aligned}
 \left(\int_{\mathbb{R}^2} |\mathcal{G}_f^1(\xi)|^2 \, d\xi \right)^{1/2} &= \left(\int_{|\xi|<1} |\mathcal{G}_f^1(\xi)|^2 \, d\xi + \int_{|\xi|>1} |\mathcal{G}_f^1(\xi)|^2 \, d\xi \right)^{1/2} \\
 &= \left(\int_{|\xi|<1} |\mathcal{G}_f^1(\xi)|^2 \, d\xi \right)^{1/2} + \left(\int_{|\xi|>1} |\mathcal{G}_f^1(\xi)|^2 \, d\xi \right)^{1/2} \\
 &:= I + II. \tag{2.10}
 \end{aligned}$$

By using (1.1), we have

$$\begin{aligned}
 I &= \left(\int_{|\xi|<1} |\mathcal{G}_f^1(\xi)|^2 d\xi \right)^{1/2} \\
 &\leq \left(\int_{|\xi|<1} \left(\frac{1}{|\xi|} \int_X |f(x)| |J_1(2\pi|f(x)||\xi|)| d\mu(x) \right)^2 d\xi \right)^{1/2} \\
 &\leq 2\pi \left(\int_{|\xi|<1} \left(\frac{1}{|\xi|} \int_X |f(x)|^2 |\xi| d\mu(x) \right)^2 d\xi \right)^{1/2} \\
 &\leq 2\pi^{3/2} \|f\|_{L^2(X,\mu)}.
 \end{aligned}
 \tag{2.11}$$

Again, applying (1.2) we obtain

$$\begin{aligned}
 II &= \left(\int_{|\xi|>1} |\mathcal{G}_f^1(\xi)|^2 d\xi \right)^{1/2} \\
 &\leq \left(\int_{|\xi|>1} \left(\frac{1}{|\xi|} \int_X |f(x)| |J_1(2\pi|f(x)||\xi|)| d\mu(x) \right)^2 d\xi \right)^{1/2} \\
 &\leq \frac{1}{\sqrt{2\pi}} \left(\int_{|\xi|>1} \frac{1}{|\xi|^3} \left(\int_X |f(x)|^{1/2} d\mu(x) \right)^2 d\xi \right)^{1/2} \\
 &\leq \frac{1}{\sqrt{2\pi}} \left(\int_X |f(x)|^{1/2} d\mu(x) \right) \left(\int_{|\xi|>1} \frac{1}{|\xi|^3} d\xi \right)^{1/2} \\
 &= (\mu(X))^{3/4} \sqrt{\|f\|_{L^2(X,\mu)}}.
 \end{aligned}
 \tag{2.12}$$

Now, combining (2.10), (2.11) and (2.12) we get the desired result. □

3. CONCLUSION

We have introduced a new class of functions which are defined via integrals. We have proved that the function \mathcal{G}_f^v is uniformly continuous and can be an element of the Lebesgue spaces for some values of v . Fourier analysis techniques were used to show that Bessel functions are related to the statistical terminology "distribution function". The main findings of this paper should open the door for studying these functions extensively.

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