

APPLICATIONS OF AN INTERSECTION FORMULA TO DUAL CONES

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ABSTRACT. We give a succinct proof of a duality theorem obtained by Révész in 1991 [6] which concerns extremal quantities related to trigonometric polynomials. The key tool of our new proof is an intersection formula on dual cones in real Banach spaces. We show another application of this intersection formula which is related to the integral estimates of non-negative positive definite functions.

1. INTRODUCTION

Let X be a real Banach space and let X' denote its topological dual space endowed with the weak- $*$ topology. For any set $D \subseteq X$, the *dual cone* of D is denoted by D^+ and is defined as

$$D^+ = \{\varphi \in X' \mid \varphi(x) \geq 0 \quad \forall x \in D\},$$

see, e.g., [2, Section 2]. The *polar cone* (denoted by D^-) is defined as $D^- := -D^+$. Note that both D^+ and D^- are weak- $*$ closed convex cones in X' , no matter what the set D is. Moreover, by [2, Lemma 2.1.], if \mathcal{C} and \mathcal{P} are convex sets in X such that $0 \in \mathcal{C} \cap \mathcal{P}$ and $\mathcal{C} \cap \text{int} \mathcal{P} \neq \emptyset$, then

$$(1) \quad (\mathcal{C} \cap \mathcal{P})^+ = \mathcal{C}^+ + \mathcal{P}^+.$$

Consequently, in this case we have $(\mathcal{C} \cap \mathcal{P})^- = \mathcal{C}^- + \mathcal{P}^-$.

In this short note we show two applications of the formula (1) that describes the structure of the dual cone of the intersection of cones. Both applications are of a Fourier-analytic nature, hence we collect some basic facts and notation of this topic below.

- (i) For a locally compact abelian group G , the symbol $M(G)$ denotes the set of all complex-valued regular Borel measures on G with finite total variation. $M(G)$ is a commutative, unital Banach algebra, where the norm is defined as $\|\mu\| = |\mu|(G)$, and the multiplication is defined by the convolution [7, 1.3.2. Corollary].
- (ii) The symbol $L^1(G)$ stands for the set of all integrable functions on G (with respect to the Haar measure, which is denoted by λ .) We may consider $L^1(G)$ as a subset of $M(G)$ by the embedding

$$\mu_{(\cdot)} : L^1(G) \rightarrow M(G), f \mapsto \mu_f; \mu_f(E) = \int_E f d\lambda \text{ for any Borel set } E \subseteq G.$$

- In fact $L^1(G)$ is a Banach subalgebra of $M(G)$ [7, 1.3.5. Theorem]. Moreover, $L^1(G)$ is unital if and only if $L^1(G) = M(G)$ if and only if G is discrete [7, 1.7.3. Theorem].
- (iii) $L^\infty(G)$ stands for the set of all essentially bounded measurable functions on G .

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- (iv) A continuous group homomorphism from the locally compact abelian group G into the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is called a *character* of G . The set of all characters of G forms a group (with pointwise multiplication) which is called the *dual group* of G , and it is denoted by \hat{G} .
- (v) For any $\mu \in M(G)$ (or $f \in L^1(G)$), the symbol $\hat{\mu}$ (or \hat{f}) denotes the Fourier transform of μ (or f), that is,

$$\hat{\mu} \in \mathbb{C}^{\hat{G}}; \quad \hat{\mu}(\gamma) = \int_G \bar{\gamma} d\mu \quad (\gamma \in \hat{G})$$

and

$$\hat{f} \in \mathbb{C}^{\hat{G}}; \quad \hat{f}(\gamma) = \int_G f \bar{\gamma} d\lambda \quad (\gamma \in \hat{G}).$$

The Fourier transform is a continuous linear transformation from $L^1(G)$ into $C_0(\hat{G})$, where $C_0(\hat{G})$ denotes the set of all functions on \hat{G} vanishing at infinity (the topology on \hat{G} is the weak topology induced by the set of all functions \hat{f} obtained as Fourier-transforms of L^1 functions on G). Moreover, it is a contraction as $\|\hat{f}\|_\infty \leq \|f\|_1$. (For details, see [7, 1.2.4. Theorem].)

The following useful formula is an easy consequence of Fubini's theorem. If $\mu \in M(G)$, $\nu \in M(\hat{G})$ and $\phi(x) = \int_{\hat{G}} \gamma(x) d\nu(\gamma)$ ($x \in G$), then

$$(2) \quad \int_G \bar{\phi} d\mu = \int_{\hat{G}} \hat{\mu} d\bar{\nu}.$$

Now we turn to the detailed descriptions of the applications of the intersection formula (1). Section 2 is devoted to describe the first one, and Section 3 contains the second one.

2. A NEW PROOF OF A DUALITY THEOREM

In 1991, Révész proved a duality theorem on certain extremal quantities related to multivariable trigonometric polynomials [6]. That theorem is general enough to cover the duality statements appearing in [4], [5] and [8]. The setting of the theorem is as follows.

Let d be a positive integer. Let us use the notation $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ and

$$\mathbb{Z}_+^d = \left\{ \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d \mid \exists j \in \{1, 2, \dots, d\} \text{ such that } n_k = 0 \text{ for any } k < j \text{ and } n_j > 0 \right\}.$$

Let $M \subseteq \mathbb{Z}_+^d$, $L \subseteq \mathbb{Z}_+^d$, and let $M^c := \mathbb{Z}_+^d \setminus M$ and $L^c := \mathbb{Z}_+^d \setminus L$.

Consider the real Banach space $L_{\mathbb{R},s}^1(\mathbb{Z}^d)$ of all symmetric real-valued absolutely summable functions on \mathbb{Z}^d with its topological dual space $L_{\mathbb{R},s}^\infty(\mathbb{Z}^d)$.

Set

$$\mathcal{C} := \left\{ f \in L_{\mathbb{R},s}^1(\mathbb{Z}^d) \mid f \text{ has finite support,} \right.$$

$$\left. \text{supp}(f_+) \subseteq \{\mathbf{0}\} \cup M \cup -M \text{ and } \text{supp}(f_-) \subseteq \{\mathbf{0}\} \cup L \cup -L \right\}.$$

The set $\mathcal{C} \subseteq L_{\mathbb{R},s}^1(\mathbb{Z}^d)$ is a convex set. It is easy to see that the dual cone of \mathcal{C} is

$$\mathcal{C}^+ = \left\{ t \in L_{\mathbb{R},s}^\infty(\mathbb{Z}^d) \mid \text{supp}(t_+) \subseteq L^c \cup -L^c \text{ and } \text{supp}(t_-) \subseteq M^c \cup -M^c \right\}.$$

Therefore, the polar cone of \mathcal{C} is

$$\mathcal{C}^- = \left\{ t \in L_{\mathbb{R},s}^\infty(\mathbb{Z}^d) \mid \text{supp}(t_+) \subseteq M^c \cup -M^c \text{ and } \text{supp}(t_-) \subseteq L^c \cup -L^c \right\}.$$

Set

$$\mathcal{P} := \left\{ f \in L^1_{\mathbb{R},s}(\mathbb{Z}^d) \left| \hat{f}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) e^{-i\mathbf{n} \cdot \mathbf{x}} = f(\mathbf{0}) + 2 \sum_{\mathbf{n} \in \mathbb{Z}^d_+} f(\mathbf{n}) \cos(\mathbf{n} \cdot \mathbf{x}) \geq 0 \forall \mathbf{x} \in \mathbb{T}^d \right. \right\}.$$

Clearly, \mathcal{P} is a convex set. The following Lemma is devoted to describe its dual cone.

Lemma 1.

$$\mathcal{P}^+ = \left\{ h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d) \mid h \gg 0, \text{ that is, } h \text{ is positive definite} \right\}.$$

Proof of Lemma 1. Recall that a function $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ is said to be positive definite if $\sum_{i,j=1}^m z_i \bar{z}_j g(\mathbf{n}_i - \mathbf{n}_j) \geq 0$ holds for any $\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^d$ and $z_1, \dots, z_m \in \mathbb{C}$. However, for symmetric real functions, positive definiteness is equivalent to the a priori weaker condition $\sum_{i,j=1}^m c_i c_j g(\mathbf{n}_i - \mathbf{n}_j) \geq 0$ ($\mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{Z}^d$, $c_1, \dots, c_m \in \mathbb{R}$).

Let us recall Bochner's theorem [7, 1.4.3. Theorem] which says that a function $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ is positive definite if and only if there is a non-negative symmetric measure $\nu \in M_{\mathbb{R},s}(\mathbb{T}^d)$ such that

$$h(\mathbf{n}) = \int_{\mathbb{T}^d} e^{i\mathbf{n} \cdot \mathbf{x}} d\nu(\mathbf{x}) \quad (\mathbf{n} \in \mathbb{Z}^d).$$

Therefore, the positive definite functions are in \mathcal{P}^+ as any positive definite $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ can be written in the form $h(\mathbf{n}) = \int_{\mathbb{T}^d} e^{i\mathbf{n} \cdot \mathbf{x}} d\nu(\mathbf{x})$ ($\mathbf{n} \in \mathbb{Z}^d$) for some $0 \leq \nu \in M_{\mathbb{R},s}(\mathbb{T}^d)$, and hence, by equation (2), the inequality

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) h(\mathbf{n}) = \int_{\mathbb{T}^d} \hat{f}(\mathbf{x}) d\nu(\mathbf{x}) \geq 0$$

for any $f \in \mathcal{P} \subset L^1_{\mathbb{R},s}(\mathbb{Z}^d)$.

Conversely, if $g \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ and g is not positive definite, that is,

$$\sum_{i,j=1}^m c_i c_j g(\mathbf{n}_i - \mathbf{n}_j) < 0$$

for some $\{\mathbf{n}_1, \dots, \mathbf{n}_m\} \subset \mathbb{Z}^d$ and $\{c_1, \dots, c_m\} \subset \mathbb{R}$, then

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} (x * \tilde{x})(\mathbf{n}) g(\mathbf{n}) < 0,$$

where $x = \sum_{i=1}^m c_i \chi_{\{\mathbf{n}_i\}}$ and \tilde{x} is defined by $\tilde{x}(\mathbf{n}) = x(-\mathbf{n})$ ($\mathbf{n} \in \mathbb{Z}^d$). Clearly, $x * \tilde{x} \in \mathcal{P} \subset L^1_{\mathbb{R},s}(\mathbb{Z}^d)$, hence this means that $g \notin \mathcal{P}^+$. \square

Now, let $r \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ with $r(\mathbf{0}) = 1$ be fixed and let us define the affine subspace

$$\mathcal{H} := \left\{ f \in L^1_{\mathbb{R},s}(\mathbb{Z}^d) \mid f(\mathbf{0}) = 1 \right\}.$$

According to [6, eq. (5) and eq. (12)], let us define the extremal quantities

$$\alpha := \inf \left\{ \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) r(\mathbf{n}) \mid f \in \mathcal{C} \cap \mathcal{P} \cap \mathcal{H} \right\}$$

and

$$\begin{aligned} \omega &:= \sup \{ \delta \in \mathbb{R} \mid \exists t \in \mathcal{C}^- \text{ such that } r + t - \delta \chi_{\{\mathbf{0}\}} \in \mathcal{P}^+ \} \\ &= \sup \{ \delta \in \mathbb{R} \mid \exists t \in \mathcal{C}^- \text{ such that } \delta \chi_{\{\mathbf{0}\}} - r - t \in \mathcal{P}^- \} \\ &= \sup \{ \delta \in \mathbb{R} \mid \delta \chi_{\{\mathbf{0}\}} - r \in \mathcal{C}^- + \mathcal{P}^- \} \end{aligned}$$

(It is clear that the definition on α coincides with the definition given in [6, eq. (5)]. It is less obvious that the definition of ω is the same as the one given in [6, eq. (12)]. However, the fact that the nonnegative symmetric measures on \mathbb{T}^d are in one-to-one correspondence with the real positive definite functions on \mathbb{Z}^d by the Fourier transform may convince the reader that the definition of ω is also correct.)

We mentioned before that \mathcal{C} and \mathcal{P} are convex sets in the real Banach space $L^1_{\mathbb{R},s}(\mathbb{Z}^d)$. It is clear that $0 \in \mathcal{C} \cap \mathcal{P}$ and $\mathcal{C} \cap \text{int}\mathcal{P} \neq \emptyset$, as $\chi_{\{0\}} \in \mathcal{C} \cap \text{int}\mathcal{P}$. (The fact that $\chi_{\{0\}} \in \text{int}\mathcal{P}$ can be easily seen by the following. The Fourier transform is a contraction from $L^1_{\mathbb{R},s}(\mathbb{Z}^d)$ into $C_{\mathbb{R},s}(\mathbb{T}^d)$, and $\widehat{\chi_{\{0\}}} = 1$.) Therefore, by [2, Lemma 2.1.], the intersection formula

$$(\mathcal{C} \cap \mathcal{P})^+ = \mathcal{C}^+ + \mathcal{P}^+$$

holds. Consequently, we have $(\mathcal{C} \cap \mathcal{P})^- = \mathcal{C}^- + \mathcal{P}^-$. So, by this intersection formula, ω can be rewritten as

$$\omega = \sup \{ \delta \in \mathbb{R} \mid \delta \chi_{\{0\}} - r \in (\mathcal{C} \cap \mathcal{P})^- \}.$$

Theorem 2 (Révész, [6]).

$$\alpha = \omega.$$

A short proof. If $\delta \chi_{\{0\}} - r \in (\mathcal{P} \cap \mathcal{C})^-$, then

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) r(\mathbf{n}) \geq \delta \text{ for any } f \in \mathcal{C} \cap \mathcal{P} \cap \mathcal{H},$$

as in this case

$$0 \geq \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) (\delta \chi_{\{0\}}(\mathbf{n}) - r(\mathbf{n})) = \delta f(\mathbf{0}) - \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) r(\mathbf{n}) = \delta - \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) r(\mathbf{n}).$$

Therefore, $\omega \leq \alpha$.

On the contrary, if $\beta > \omega$, then $\beta \chi_{\{0\}} - r \notin (\mathcal{C} \cap \mathcal{P})^-$, that is, there exists some $f \in \mathcal{C} \cap \mathcal{P}$ such that

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) (\beta \chi_{\{0\}}(\mathbf{n}) - r(\mathbf{n})) > 0.$$

This f is necessarily a non-zero element of \mathcal{P} , hence $f(\mathbf{0}) > 0$. Therefore, without loss of generality, we can assume that $f(\mathbf{0}) = 1$, so there exists some $f \in \mathcal{C} \cap \mathcal{P} \cap \mathcal{H}$ such that

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) (\beta \chi_{\{0\}}(\mathbf{n}) - r(\mathbf{n})) > 0.$$

That is,

$$\beta > \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) r(\mathbf{n})$$

for some $f \in \mathcal{C} \cap \mathcal{P} \cap \mathcal{H}$, which means that $\beta > \alpha$. So, we deduced that $\beta > \omega$ implies $\beta > \alpha$, therefore, $\alpha \leq \omega$. The proof is done. \square

3. ANOTHER APPLICATION OF THE INTERSECTION FORMULA

The second application concerns integral estimates of non-negative positive definite functions. This problem is related to Wiener's problem [9, 11] and to the recent works [1, 10]. The arguments in this section are partially parallel to the arguments presented in the previous section.

Let $L_{\mathbb{R},s}^1(\mathbb{Z})$ denote the real Banach space of all real-valued, symmetric, summable functions on \mathbb{Z} and let us consider its topological dual space $L_{\mathbb{R},s}^\infty(\mathbb{Z})$ endowed with the weak- $*$ topology.

Let us define $\mathcal{C} = \{f \in L_{\mathbb{R},s}^1(\mathbb{Z}) \mid f \geq 0\}$ and $\mathcal{P} = \{f \in L_{\mathbb{R},s}^1(\mathbb{Z}) \mid \hat{f} \geq 0\}$. Clearly, \mathcal{C} and \mathcal{P} are convex cones in $L_{\mathbb{R},s}^1(\mathbb{Z})$. The closedness of \mathcal{C} is obvious, and \mathcal{P} is also closed as the Fourier transform is a continuous (moreover, norm-non-increasing) linear transformation from $L_{\mathbb{R},s}^1(\mathbb{Z})$ into $C_{\mathbb{R},s}(\mathbb{T})$, and the nonnegative functions form a closed set of $C_{\mathbb{R},s}(\mathbb{T})$ with respect to the maximum norm topology. (The symbol \mathbb{T} denotes the additive group of real numbers modulo 2π and $C_{\mathbb{R},s}(\mathbb{T})$ stands for the Banach space of all continuous, symmetric real functions on \mathbb{T} .)

Lemma 3.

$$\mathcal{C}^+ = \{g \in L_{\mathbb{R},s}^\infty(\mathbb{Z}) \mid g \geq 0\},$$

and

$$\mathcal{P}^+ = \{h \in L_{\mathbb{R},s}^\infty(\mathbb{Z}) \mid h \gg 0, \text{ that is, } h \text{ is positive definite}\}.$$

Proof of Lemma 3. The first statement of Lemma 3 is obvious. The proof of the second statement is very similar to the proof of Lemma 1. \square

Let L and N be positive integers. Let us define the extremal quantities

$$C(L, N) := \inf \left\{ C \in \mathbb{R} \mid \sum_{k=-LN}^{LN} f(k) \leq (C+1) \sum_{k=-N}^N f(k) \text{ for any } f \in \mathcal{C} \cap \mathcal{P} \right\}$$

and

$$K(L, N) := \inf \{h(0) \mid h \in L_{\mathbb{R},s}^\infty(\mathbb{Z}), h \gg 0 \text{ and } h(k) \leq -\chi_{\{-LN, \dots, LN\}}(k), \text{ if } |k| > N\}.$$

Let us introduce

$$\mathcal{S}_{L,N} := \{h \in L_{\mathbb{R},s}^\infty(\mathbb{Z}) \mid h(k) \leq -\chi_{\{-LN, \dots, LN\}}(k), \text{ if } |k| > N\},$$

and observe that $\mathcal{S}_{L,N}$ is closed in the weak- $*$ topology as it is the intersection of weak- $*$ closed sets.

Note that

$$C(L, N) = \inf \{C \in \mathbb{R} \mid (C+1)\chi_{\{-N, \dots, N\}} - \chi_{\{-LN, \dots, LN\}} \in (\mathcal{C} \cap \mathcal{P})^+\},$$

and by the result of Lemma 3,

$$K(L, N) = \inf \{h(0) \mid h \in \mathcal{P}^+ \cap \mathcal{S}_{L,N}\}.$$

Remark 4. Let us note that $K(L, N)$ is finite as the set $\mathcal{P}^+ \cap \mathcal{S}_{L,N}$ is not empty. Indeed, one can easily check that the function

$$w_{L,N}(k) := \begin{cases} 2(L-1)N & \text{if } k = 0 \\ -1 & \text{if } N < |k| \leq LN \\ 0 & \text{otherwise} \end{cases}$$

is positive definite, and therefore, it is an element of $\mathcal{P}^+ \cap \mathcal{S}_{L,N}$.

Theorem 5.

$$C(L, N) = K(L, N).$$

Proof. The key idea is the observation that \mathcal{C} and \mathcal{P} are convex sets in $L_{\mathbb{R},s}^1(\mathbb{Z})$ such that $0 \in \mathcal{C} \cap \mathcal{P}$ and $\mathcal{C} \cap \text{int}\mathcal{P} \neq \emptyset$ as $\chi_{\{0\}} \in \mathcal{C} \cap \text{int}\mathcal{P}$. Therefore, the intersection formula

$$(\mathcal{C} \cap \mathcal{P})^+ = \mathcal{C}^+ + \mathcal{P}^+$$

holds.

On the one hand, if $h \in \mathcal{P}^+ \cap \mathcal{S}_{L,N}$ then

$$h \leq (h(0) + 1) \chi_{\{-N, \dots, N\}} - \chi_{\{-LN, \dots, LN\}}$$

as $h(0) \geq h(n)$ ($n \in \mathbb{Z}$) holds for any positive definite function $h \in L_{\mathbb{R},s}^\infty(\mathbb{Z})$. Therefore, in this case

$$(h(0) + 1) \chi_{\{-N, \dots, N\}} - \chi_{\{-LN, \dots, LN\}} \in \mathcal{C}^+ + \mathcal{P}^+ = (\mathcal{C} \cap \mathcal{P})^+,$$

hence $C(L, N) \leq K(L, N)$.

On the other hand, by the intersection formula,

$$C(L, N) = \inf \{ C \in \mathbb{R} \mid (C + 1) \chi_{\{-N, \dots, N\}} - \chi_{\{-LN, \dots, LN\}} \in \mathcal{C}^+ + \mathcal{P}^+ \}$$

holds, hence the following argument shows the opposite inequality. If

$$(C + 1) \chi_{\{-N, \dots, N\}} - \chi_{\{-LN, \dots, LN\}} \in \mathcal{C}^+ + \mathcal{P}^+$$

then $(C + 1) \chi_{\{-N, \dots, N\}} - \chi_{\{-LN, \dots, LN\}} = g + h$ for some $g \in \mathcal{C}^+$ and $h \in \mathcal{P}^+$. Clearly, this h is an element of $\mathcal{P}^+ \cap \mathcal{S}_{L,N}$, and $h(0) \leq C$, hence $K(L, N) \leq C(L, N)$. \square

Remark 6. We have noted (see Remark 4) that $K(L, N)$ is finite. Therefore, the result of Theorem 5 directly implies the finiteness of $C(L, N)$.

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