BOUNDS ON CONVEX BODIES IN PAIRWISE INTERSECTING MINKOWSKI ARRANGEMENT OF ORDER μ

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ABSTRACT. The μ-kernel of an o-symmetric convex body is obtained by shrinking the body about its center by a factor of μ. As a generalization of pairwise intersecting Minkowski arrangement of o-symmetric convex bodies, we can define the pairwise intersecting Minkowski arrangement of order μ. Here, the homothetic copies of an o-symmetric convex body are so that none of their interiors intersect the μ-kernel of any other. We give general upper and lower bounds on the cardinality of such arrangements, and study two special cases: For d-dimensional translates in classical pairwise intersecting Minkowski arrangement we prove that the sharp upper bound is 3^d. The case $\mu = 1$ is the Bezdek–Pach Conjecture, which asserts that the maximum number of pairwise touching positive homothetic copies of a convex body in $\mathbb{R}^d$ is 2^d. We verify the conjecture on the plane, that is, when $d = 2$. Indeed, we show that the number in question is four for any planar convex body.

1. INTRODUCTION

A *positive homothetic copy* of a convex body (i.e. a compact convex set with non-empty interior) K in Euclidean d-space $\mathbb{R}^d$ is a set of the form $\lambda K + t$ where $\lambda > 0$ and $t \in \mathbb{R}^d$. Two sets in $\mathbb{R}^d$ are said to *touch* each other if they intersect but their interiors are disjoint.

The following notion was introduced by Fejes Tóth [19]: Pairwise intersecting homothets of a centrally symmetric convex body in the d-dimensional Euclidean space form a *Minkowski arrangement* if none of them contains the center of any other in its interior. In this paper, we only consider Minkowski arrangements that are pairwise intersecting.

Polyanskii [16] recently proved that such a family of convex bodies has at most 3^{d+1} members. This result was improved by Naszódi and Swanepoel [12] showing an upper bound of $2 \cdot 3^d$. It is natural to conjecture that the maximum number of elements is 3^d.

We prove the following upper bound on the cardinality of a family containing translates of a centrally symmetric convex body in pairwise intersecting Minkowski arrangement in Section 2:

Theorem 1. *In $\mathbb{R}^d$, a pairwise intersecting Minkowski arrangement consisting of translates of a centrally symmetric convex body K contains at most 3^d elements. This bound is sharp, equality holds if and only if K is a d-dimensional parallelotope.*

We show a construction for arbitrary centrally symmetric convex body that gives a linear lower bound on the cardinality of maximal pairwise intersecting Minkowski arrangements of translates. Although from Theorem 3 of [11] we can deduce the existence of an exponential lower bound, we now give a simple and deterministic construction.

Proposition 1. *For a centrally symmetric convex body K in $\mathbb{R}^d$ ($d \geq 2$), a maximum cardinality set consisting of translates of K in pairwise intersecting Minkowski arrangement has at least $2d + 3$ elements.*

We introduce some generalizations of the problem based on an idea of Böröczky and Szabó [6]: For $0 \leq \mu \leq 1$ they defined the *μ-kernel* of an o-symmetric convex body K as μK.

Using this notion, for homothets of an o-symmetric convex body we can consider a *pairwise intersecting Minkowski arrangement of order μ*, where the homothets are pairwise intersecting but none of their interiors intersect the center of any other.

We prove an upper bound on the cardinality of such an arrangement, then, for centrally symmetric convex bodies we verify the existence of an exponential lower bound.

Theorem 2. *In $\mathbb{R}^d$, a pairwise intersecting Minkowski arrangement of order μ consisting of translates of a centrally symmetric convex body K contains at most $\left(1 + \frac{2}{1+\mu}\right)^d$ elements.*

Proposition 2. *Let $M_\mu(K)$ denote the maximum number of translates of a d-dimensional, o-symmetric convex body K in pairwise intersecting μ-Minkowski arrangement. For $\mu < \sqrt{2} - 1$, there exists a lower bound $M_\mu(K) \geq e^{cd}$ for some universal constant c.*

In 1962, Danzer and Grünbaum [7] proved that the maximum cardinality of a family of pairwise touching translates of a convex body K in $\mathbb{R}^d$ is 2^d, which bound is attained if and only if K is an affine image of a cube. Petty [15] showed that every convex body in the plane (or in 3-space) has three (four) pairwise touching translates. As an extension of this problem, Bezdek and Pach [4] conjectured in 1988 that the maximum number of pairwise touching positive homothetic copies of a convex body in $\mathbb{R}^d$ is 2^d. They showed that any such family of homothetic copies has at most 3^d elements, and if K is a d-dimensional Euclidean ball, then the maximum is equal to $d+2$. Naszódi [13] improved the first estimate by proving the upper bound 2^{d+1}. In [10], Lángi and Naszódi proved (using a result [3] of Bezdek and Brass about one-sided Hadwiger numbers) the upper bound $3 \cdot 2^{d-1}$ in the case when K is centrally symmetric.

In Section 3, we show that the conjecture holds on the plane, moreover, every planar convex body has four pairwise touching homothets.

Theorem 3. *For any convex body K in $\mathbb{R}^2$, the maximum number of pairwise touching positive homothetic copies of K is four.*

The generalized notion of Minkowski arrangement provides a connection between the original problem of pairwise intersecting Minkowski arrangements and the Bezdek–Pach Conjecture [4]. In both problems we consider pairwise intersecting Minkowski arrangements of order μ, in the first case $\mu = 0$, while in the latter case $\mu = 1$.

For two points a, b in $\mathbb{R}^d$, we denote the closed and the open line segment connecting them by $[a, b]$ and (a, b), respectively. We use the standard notations conv, bd and int to denote the convex hull, the boundary and the interior of a set in $\mathbb{R}^d$, respectively.

In Section 2, we prove Theorems 1, 2, and Propositions 1 and 2. Sections 3 and 4 together give the proof of Theorem 3. Finally, in Section 5, we verify Proposition 8, a topological note that yields to an alternative version of the proof of Theorem 3.

2. Bounds on pairwise intersecting Minkowski arrangements

It is natural to conjecture that in $\mathbb{R}^d$, a pairwise intersecting Minkowski arrangement consisting of homothets of a centrally symmetric convex body contains at most 3^d elements. Here we prove this upper bound – and a generalization – for the case when all the homothets in the arrangement are translates of the given body.

2.1. **Proof of Theorem 1 and 2.**

First, we verify Theorem 2, then Theorem 1 will follow as a corollary.

Any o-symmetric convex body K can be considered as the unit ball of a normed space $(\mathbb{R}^d, \|.\|_K)$, where for any $x \in \mathbb{R}^d$, $\|x\|_K = \inf\{\lambda \in \mathbb{R}^+ | x \in \lambda K\}$. It is easy to see that having a pairwise intersecting Minkowski arrangement of order μ is equivalent to the following two conditions on the distances between centers: none of them can be farther than 2, nor closer than $1 + \mu$ to any other. After applying a homothety, this is equivalent to the problem when the distances are between 1 and $\frac{2}{1+\mu}$.

Lemma 3. *Consider a centrally symmetric convex body K in $\mathbb{R}^d$ and $v_1, v_2, ..., v_n \in \mathbb{R}^d$, so that $1 \leq \|v_i - v_j\|_K \leq \lambda$ for any $i \neq j$. Then $n \leq (\lambda + 1)^d$.*

Proof. By the assumption, for different indices the bodies $v_i + \frac{1}{2}K$ are pairwise non-overlapping. Let $Q = \text{conv}\left[\bigcup\limits_{i=1}^{n} \left(v_i + \frac{1}{2}K\right)\right]$. Since $\text{diam}_K(Q) \leq \lambda + 1$, using the isodiametric inequality for Minkowski spaces [8] we get that

$$
(1) \qquad \frac{n}{2^d}\text{Vol}\,(K) \leq \text{Vol}\,(Q) \leq \text{Vol}\left(\frac{\lambda + 1}{2}K\right).
$$

From this, $n \leq (\lambda + 1)^d$ follows. $\qquad\qquad\square$

Applying this lemma for $\lambda = \frac{2}{1+\mu}$, we get that the number of points with pairwise distances between 1 and $\frac{2}{1+\mu}$ is at most $\left(1 + \frac{2}{1+\mu}\right)^d$, which is equivalent to the statement of Theorem 2. $\qquad\square$

Theorem 1 is the special case of Theorem 2 with $\mu = 0$, so the upper bound 3^d follows easily.

To reach this, (1) has to hold with two equalities. From the following lemma of Groemer [9] we can see that this happens if and only if K is a d-dimensional parallelotope.

Lemma 4. *Suppose that K is a convex body in $\mathbb{R}^d$ such that for some $1 < t \in \mathbb{R}$ the body tK can be decomposed into translates of K. Then K is a d-dimensional parallelotope and t is an integer. The partition is unique.*

$\qquad\qquad\square$

Remark 5. The bound in Theorem 2 gives the known result 2^d for the pairwise touching case, when $\mu = 1$.

2.2. **Proof of Proposition 1.**

First, we show a construction of seven bodies in $\mathbb{R}^2$, then the lower bound $2d+3$ for the higher dimensional cases will follow recursively. In $\mathbb{R}^2$, consider an affine-regular hexagon inscribed in K that is symmetric about the center of K (see for example [14, Lemma 4.3]). There exist seven translates of this hexagon in Minkowski arrangement, shown in Figure 1. Translate K in a way that the center points are the same as the centers of the above hexagons. Now a center of any translate is either not contained in another body, or lies on its boundary. Furthermore, these translates share a common point, so they are pairwise intersecting. This means, that the construction gives a Minkowski arrangement.
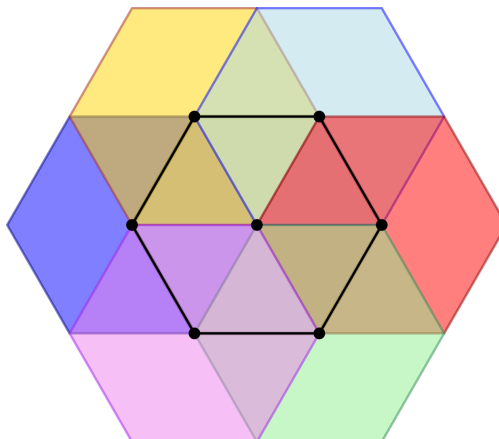

FIGURE 1. Seven translates of an affine-regular hexagon in Minkowski arrangement

For a centrally symmetric convex body K in $\mathbb{R}^d$, denote by $M(K)$ the maximal number of translates in a pairwise intersecting Minkowski arrangement. It is easy to see, that for any K in $\mathbb{R}^1$, $M(K) = 3$, and we showed that for K in $\mathbb{R}^2$, $M(K) \geq 7$.

Let $e_1, ..., e_d$ be an Auerbach basis [18, Chapter 3] of the space $(\mathbb{R}^d, \|.\|_K)$. In dimension $d \geq 3$, using the above planar construction, we can take 7 translates of K in a Minkowski arrangement such that their centers lie in the plane of the first two basis vectors e_1 and e_2. Along each direction $e_3, ..., e_d$ we can add two further translates of K to the arrangement so that they contain o on their boundary.   □

Now we verify Proposition 2. Note that Theorem 3. of [11] by Naszódi, Pach and Swanepoel implies the existence of an exponential lower bound for translates in pairwise intersecting Minkowski arrangement. Their idea was based on a result of Arias-de-Reyna, Ball, and Villa [1]. Here we give a similar argument for μ-Minkowski arrangements.

Proof of Proposition 2. The statement follows from a result of Bourgain [5]. He showed that on the unit sphere of any normed space, there is an exponentially large number of points so that the distance of any two is more than $\sqrt{2} - \varepsilon$. Consider the o-symmetric convex body K as the unit ball of the normed space $(\mathbb{R}^d, \|.\|_K)$. Choosing $\mu < \sqrt{2} - 1$, we get exponentially many points on the sphere so that their pairwise distances are between $1 + \mu$ and 2. Considering these points as centers, we verify the statement.   □

3. Proof of the upper bound in Theorem 3

Let $\mathcal{K} = \{K_1, K_2, \ldots, K_n\}$ be a family of pairwise touching positive homothetic copies of a planar convex body K.

If there is a point that belongs to four of the homothets, then we can enlarge (or shrink) each of the four bodies from that point as a center, to obtain four touching translates of K. By the result of Danzer and Grünbaum [7], this implies that K is a parallelogram. It is easy to see that in this case, the family does not have a fifth member. Thus, from this point on, we will assume that no point belongs to four of the homothets.

If there is a point that belongs to three of the homothets and $\mathcal{K}$ has at least four members, then we will show that this point also belongs to a fourth body.

Proposition 6. *Let K_1, K_2, K_3, K_4 be pairwise touching positive homothets of the convex body K in $\mathbb{R}^2$. If $K_1 \cap K_2 \cap K_3 \neq \emptyset$, then $K_1 \cap K_2 \cap K_3 \cap K_4 \neq \emptyset$.*

Proof. Let $p \in K_1 \cap K_2 \cap K_3$, and C_i be the smallest angular region with vertex p containing K_i.

We show that $\operatorname{int} C_i \cap \operatorname{int} C_j = \emptyset$ for any $i \neq j$, $i, j \in \{1, 2, 3\}$.

Suppose that for a pair $i \neq j$ there exists $c \in (\operatorname{int} C_i \cap \operatorname{int} C_j)$. Then the line pc intersects the interior of both K_i and K_j because C_i and C_j are the smallest angular regions containing K_i and K_j respectively. Hence due to the convexity of the bodies, K_i overlaps K_j, which is a contradiction.

Suppose that $K_1 \cap K_2 \cap K_3 \cap K_4 = \emptyset$. Then $p \notin K_4$, thus there exists a supporting line ℓ of K_4 that does not go through p and separates K_4 from p. K_4 touches K_1, K_2 and K_3, hence each of these three bodies has a point in both of the closed half-planes bounded by ℓ. From this it follows that ℓ intersects the angular regions C_1, C_2 and C_3. For every $i \in \{1, 2, 3\}$, $\ell \cap C_i$ is a connected subset of ℓ, thus there is a middle one of them. Without loss of generality we can assume that this one is K_1. Let $v_1 = p - x_1$. The image of p by the homothety that maps K_1 to K_4 is the point $x_4 + \frac{\lambda_4}{\lambda_1} \cdot v_1 \in \ell$. The same homothety maps C_1 to the angular region $C_1' := C_1 + \left(x_4 - x_1 + v_1 \cdot \left(\frac{\lambda_4}{\lambda_1} - 1\right)\right)$. (Figure 2) As $K_1 \subset C_1$ and the bodies are positive homothets, $K_4 \subset C_1'$ follows. At least one pair of the bounding lines of C_1 and C_1' are different, thus due to the fact that $\operatorname{int} C_i \cap \operatorname{int} C_j = \emptyset$ for any $i \neq j$, $i, j \in \{1, 2, 3\}$ C_1' is disjoint to at least one of the angular regions C_2 and C_3. But in this case K_4 cannot touch the body lying in this angular region, which is a contradiction.

                                                                                □

Thus, it is enough to consider the case when no point belongs to three of the homothets.

Proposition 7. *Let K_1, K_2,...,K_n be pairwise touching convex bodies in $\mathbb{R}^2$, such that no three share a common point. Then $n \leq 4$.*

Proof. For each $i \in \{1, ..., n\}$, choose an interior point $p_i \in K_i$. The bodies are pairwise touching, so we can draw a curve between any two of the chosen points p_i, p_j so that it lies in $K_i \cup K_j$. Since no three of the bodies share a common point, these curves intersect only in the interior of the bodies. It is easy to see that we can eliminate these intersections with a perturbation. This way we draw the complete graph of n vertices on the plane, from which $n \leq 4$ follows immediately. $\square$
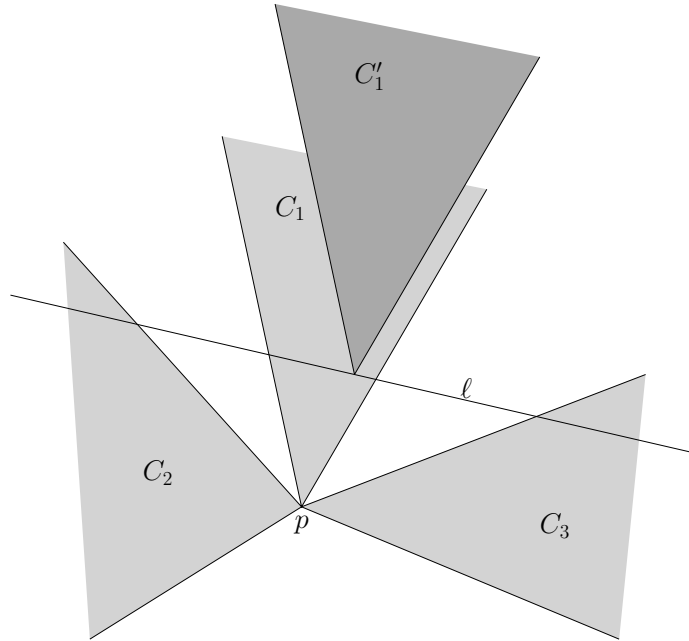

FIGURE 2. The smallest angular regions containing the bodies

4. PROOF OF THE LOWER BOUND IN THEOREM 3

In this section, we show that for any planar convex body K, there are four pairwise touching homothets of K.

Consider two distinct parallel support lines of K that each touch K at one point: x_1 and x_2. The existence of such pair of lines follows from Theorem 2.2.9. of [17], but may also be proved as an exercise.

Let $K_1 = K$ and $K_2 = K + x_2 - x_1$. Let f be the line through the single point of contact, x_2, of K_1 and K_2 parallel to $x_2 - x_1$. On both sides of f, there is a translate of K that touches both K_1 and K_2. Indeed, if we push K around K_1 so that it always touches K_1 then, by continuity, such two positions will be found.

If on both sides we can find such translates of K that also contains x_2 then x_2 is a common point of four translates of K and we are done. Thus we assume that at least one of these translates does not contain x_2. We call this translate K_3.

Now, K_1, K_2, K_3 are pairwise touching translates of K that do not share a common point. It follows that they surround a bounded region R with non-empty interior. Consider the largest homothet K_4 of K contained in R. To finish the proof, we claim that K_4 touches K_1, K_2 and K_3. Indeed, assume K_4 touches only two of them, say K_1 and K_2. Consider a line that separates K_4 and K_1, and another line that separates K_4 and K_2. Let u_1 and u_2 be the unit normal vectors of these two lines respectively, pointing away from K_4. Clearly, if the origin is not in $\text{conv}(u_1, u_2)$ then K_4 can be moved a little inside R so that it does not touch either K_1, K_2 or K_3. Then, we may enlarge K_4 slightly within R contradicting the maximality of K_4. Thus $o \in \text{conv}(u_1, u_2)$, that is $u_1 = -u_2$. However, in this case, K_1 and K_2 are strictly separated, which is a contradiction, finishing the proof of the lower bound in Theorem 3.

5. A topological note

In this section we present Proposition 8, a topological observation, which may be used in place of Proposition 7 to prove the upper bound in Theorem 3.

An *arc* in the plane is the image of an injective continuous map of the $[0, 1]$ interval into the plane. A *Jordan curve* in the plane is the image of an injective continuous map of the circle into the plane. We will call the closed bounded region bounded by a Jordan curve a *Jordan region.*

Let K_1, K_2, K_3 be three pairwise touching Jordan regions whose pairwise intersections are non-empty arcs (which may be degenerate, that is a single point). Using the Jordan curve theorem, it is easy to show that the complement of $K_1 \cup K_2 \cup K_3$ in the plane has two connected components, one bounded and one unbounded. We call the closure of the bounded component the *internal region* surrounded by K_1, K_2, K_3, and the closure of the unbounded component the *external region.*

Proposition 8. *Let K_1, K_2, K_3, K_4 be four pairwise touching Jordan regions whose pairwise intersections are non-empty arcs (which may be degenerate, that is a single point). Suppose that $K_1 \cap K_2 \cap K_3 \cap K_4 = \emptyset$. Then one of them lies in the internal region surrounded by the other three.*

Proof. We will call the image of the non-negative reals under an injective mapping into the plane an *unbounded path* if it is an unbounded subset of the plane. The image of 0 is the *starting point* of the unbounded path.

Assume that K_1 is not in the internal region surrounded by K_2, K_3, K_4. Then there is a point p_1 on the boundary of K_1 that does not belong to either of the other three sets, and from which there is an unbounded path, γ_1 disjoint from the other three sets. Similarly, if K_2 is not in the internal region surrounded by the other three, then there is a point p_2 on the boundary of K_2 that does not belong to either of the other three sets, and from which there is an unbounded path, γ_2 disjoint from the other three sets. And the same holds for K_3 yielding p_3 and γ_3.

We may assume that γ_1, γ_2 and γ_3 are pairwise disjoint. Now, $\gamma_1 \cup \gamma_2 \cup \gamma_3$ partition the external region of K_1, K_2, K_3 into three parts. And K_4 is in one of these three parts. However, each part only intersects two of the sets K_1, K_2, K_3, which is a contradiction. $\square$

Remark 9. Although Proposition 8 has been observed in [2], their proof does not work for the special case when three of the bodies share a common point. Therefore we found it necessary to give a general proof.

Observe that the conclusion of Proposition 7 follows from Proposition 8. Indeed, we may assume that K_4 is in the internal region I surrounded by K_1, K_2, K_3. Suppose that $n \geq 5$. Since K_5 touches K_4, it must also lie in I. On the other hand, K_5 touches K_1, K_2, K_3 at points that do not belong to K_4. Now, $(\operatorname{bd} I) \setminus K_4$ is the union of three open arcs, and K_4 must have a point on at least two of these arcs to touch K_1, K_2, K_3. However, then the interior of K_5 intersects the interior of at least one set from $K_1, \ldots, K_4$, a contradiction.

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