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Colorful Helly-type theorems for the volume of intersections of convex bodies

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A B S T R A C T

We prove the following Helly-type result. Let $\mathcal{C}_1, \dots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \le k \le 2d$ with $1 \le i_1 < \dots < i_{2d} \le 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ is of volume at least 1. Then there is an $1 \le i \le 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ is of volume at least $d^{-O(d^2)}$.

1. Introduction

According to Helly's Theorem, *if the intersection of any $d+1$ members of a finite family of convex sets in $\mathbb{R}^d$ is non-empty, then the intersection of all members of the family is non-empty.*

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A generalization of Helly's Theorem, known as the **Colorful Helly Theorem**, was given by Lovász, and later by Bárány [2]: *If* $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ *are finite families (color classes) of convex sets in* $\mathbb{R}^d$*, such that for any colorful selection* $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$*, the intersection* $\bigcap_{i=1}^{d+1} C_i$ *is non-empty, then for some* j*, the intersection* $\bigcap_{C \in \mathcal{C}_j} C$ *is also non-empty.*

Another variant of Helly's Theorem was introduced by Bárány, Katchalski and Pach [3], whose **Quantitative Volume Theorem** states the following. *Assume that the intersection of any* $2d$ *members of a finite family of convex sets in* $\mathbb{R}^d$ *is of volume at least 1. Then the volume of the intersection of all members of the family is of volume at least* c_d*, a constant depending on* d *only.*

They proved that one can take $c_d = d^{-2d^2}$ and conjectured that it should hold with $c_d = d^{-cd}$ for an absolute constant $c > 0$. It was confirmed with $c_d \approx d^{-2d}$ in [12], whose argument was then refined by Brazitikos [5], who showed that one may take $c_d \approx d^{-3d/2}$. For more on quantitative Helly-type results, see the surveys [10,7].

In the present paper, we combine the two directions: colorful and quantitative.

1.1. Ellipsoids and volume

A well known consequence of John's Theorem (Corollary 2.2), is that any compact convex set K with non-empty interior contains a unique ellipsoid $\mathcal{E}$ of maximal volume. Moreover, $\mathcal{E}$ enlarged around its center by a factor d contains K (cf. [1]). It follows that the volume of the largest ellipsoid contained in K is of volume at least $d^{-d}\,\mathrm{Vol}(K)$. More precise bounds for this volume ratio are known (cf. [1]), but we will not need them.

As shown in [12, Section 3], in the Quantitative Volume Theorem, the d^{-cd} factor is sharp up to the absolute constant c. In particular, for every sufficiently large positive integer d, there is a family of convex sets satisfying the assumptions of the theorem whose intersection is of volume roughly $d^{-d/2}$.

John's Theorem and the fact above yield that bounding the volume of intersections and bounding the volume of ellipsoids contained in the intersections are essentially equivalent problems: the only difference is a multiplicative factor d^d which is of no consequence, unless one wants to find the best constants in the exponent. Thus, from this point on, we phrase our results in terms of the volume of ellipsoids contained in intersections. Its benefit is that this is how in the proofs we actually "find volume": we find ellipsoids of large volume.

1.2. Main result: few color classes

Our main result is the following.

Theorem 1.1 *(Colorful Quantitative Volume Theorem with ellipsoids – few color classes). Let* $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ *be finite families of convex bodies in* $\mathbb{R}^d$*. Assume that for any colorful* *selection of* $2d$ *sets,* $C_{i_k} \in \mathcal{C}_{i_k}$ *for each* $1 \le k \le 2d$ *with* $1 \le i_1 < \cdots < i_{2d} \le 3d$*, the intersection* $\bigcap_{k=1}^{2d} C_{i_k}$ *contains an ellipsoid of volume at least 1. Then, there exists an* $1 \le i \le 3d$ *such that* $\bigcap_{C \in \mathcal{C}_i} C$ *contains an ellipsoid of volume at least* $c^{d^2} d^{-5d^2/2}$ *with an absolute constant* $c \ge 0$*.*

We rephrase this theorem in terms of the volume of intersections, as this form may be more easily applicable.

Corollary 1.2 *(Colorful Quantitative Volume Theorem – few color classes). Let* $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ *be finite families of convex bodies in* $\mathbb{R}^d$*. Assume that for any colorful selection of* $2d$ *sets,* $C_{i_k} \in \mathcal{C}_{i_k}$ *for each* $1 \le k \le 2d$ *with* $1 \le i_1 < \cdots < i_{2d} \le 3d$*, the intersection* $\bigcap_{k=1}^{2d} C_{i_k}$ *is of volume at least 1.*

Then, there exists an $1 \le i \le 3d$ *such that* $\mathrm{Vol}\left(\bigcap_{C \in \mathcal{C}_i} C\right) \ge c^{d^2} d^{-7d^2/2}$ *with an absolute constant* $c \ge 0$*.*

Observe that the smaller the number of color classes in a colorful Helly-type theorem, the stronger the theorem is. For example, the Colorful Helly Theorem (see the top of the section) is stated with $d+1$ color classes, but it is easy to see that it implies the same result with $\ell \ge d+2$ color classes, as the last $\ell - (d+1)$ color classes make the assumption of the theorem stronger and the conclusion weaker. We note also that the Colorful Helly Theorem does not hold with less than $d+1$ color classes, as the number $d+1$ cannot be replaced by any smaller number in Helly's Theorem.

The novelty of the proof of Theorem 1.1 is the following. As we will see later, similar looking statements can be obtained by taking the Quantitative Volume Theorem as a "basic" Helly-type theorem, and combining it with John's Theorem and a combinatorial argument. This approach yields results with $d(d+3)/2$ color classes, but does not seem to yield results with fewer color classes. In order to achieve that, first, we introduce an ordering on the set of ellipsoids, and second, we give a finer geometric examination of the situation by comparing the maximum volume ellipsoid of a convex body K to other ellipsoids contained in K.

We find it an intriguing question whether one can decrease the number of color classes to $2d$ (possibly with an even weaker bound on the volume of the ellipsoid obtained), and whether an order d^{-cd} lower bound on the volume of the ellipsoid can be shown.

1.3. Earlier results and simple observations

In 1937, Behrend [4] (see also Section 6.17 of the survey [6] by Danzer, Grünbaum and Klee) proved a planar quantitative Helly-type result: *If the intersection of any 5 members of a finite family of convex sets in* $\mathbb{R}^2$ *contains an ellipse of area 1, then the* *intersection of all members of the family contains an ellipse of area 1.* We note that, since every convex set in $\mathbb{R}^2$ is the intersection of the half-planes containing it, the result is equivalent to the formally weaker statement where the family consists of half-planes only. This is the form in which it is stated in [6].

In [6, Section 6.17], it is mentioned that John's Theorem (Theorem 2.1) should be applicable to extend Behrend's result to higher dimensions. We spell out this argument, and present a straightforward proof of the following.

Proposition 1.3 *(Helly-type theorem with ellipsoids). Let $\mathcal{C}$ be a finite family of at least $d(d+3)/2$ convex sets in $\mathbb{R}^d$, and assume that for any selection $C_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1. Then $\bigcap_{C \in \mathcal{C}} C$ also contains an ellipsoid of volume 1.*

The number $d(d+3)/2$ is best possible. Indeed, for every dimension d, there exists a family of $d(d+3)/2$ half-spaces such that the unit ball $\mathbf{B}^d$ is the maximum volume ellipsoid contained in their intersection, but $\mathbf{B}^d$ is not the maximum volume ellipsoid contained in the intersection of any proper subfamily of them. That is, the intersection of any subfamily of $d(d+3)/2 - 1$ members contains an ellipsoid of larger volume than the volume of $\mathbf{B}^d$ (which we denote by $\omega_d = \mathrm{Vol}(\mathbf{B}^d)$), and yet, the intersection of all members of the family does not contain an ellipsoid of larger volume than ω_d. This follows from the much stronger result, Theorem 4 in [9] by Gruber.

We prove a colorful version of Proposition 1.3.

Proposition 1.4 *(Colorful Quantitative Volume Theorem with ellipsoids – many color classes). Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d(d+3)/2}$ be finite families of convex bodies in $\mathbb{R}^d$, and assume that for any colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1. Then for some j, the intersection $\bigcap_{C \in \mathcal{C}_j} C$ contains an ellipsoid of volume 1.*

The proof of Proposition 1.4 consists of two parts. First, as our contribution, in the *geometric part*, we introduce an ordering on the set of ellipsoids contained in a convex set, and study properties of this ordering, see Section 2.3. Second, a *combinatorial part* shows that this ordering yields the statement. This second part is essentially identical to the argument given by Lovász and Bárány [2] in their proof of the Colorful Helly Theorem, and it was presented in an abstract setting in [8, Theorem 5.3] by De Loera et al.

Sarkar, Xue and Soberón [13, Corollary 1.0.5], using matroids, recently obtained a result involving $d(d+3)/2$ color classes, but with the number of selected sets being $2d$.

Proposition 1.5 *(Sarkar, Xue and Soberón [13]). Let* $\mathcal{C}_1, \ldots, \mathcal{C}_{d(d+3)/2}$ *be finite families of convex bodies in* $\mathbb{R}^d$*. Assume that for any colorful selection of* $2d$ *sets,* $C_{i_k} \in \mathcal{C}_{i_k}$ *for each* $1 \le k \le 2d$ *with* $1 \le i_1 < \cdots < i_{2d} \le d(d+3)/2$*, the intersection* $\bigcap_{k=1}^{2d} C_{i_k}$ *contains an ellipsoid of volume at least 1. Then, there exists an* $1 \le i \le d(d+3)/2$ *such that* $\bigcap_{C \in \mathcal{C}_i} C$ *has volume at least* $d^{-O(d)}$*.*

For completeness, in Section 3.3, we sketch a brief argument showing that Proposition 1.5 immediately follows from our Proposition 1.4 and the Quantitative Volume Theorem.

The structure of the paper is the following. In Section 2, we introduce some preliminary facts and definitions, notably, an ordering on the family of ellipsoids of volume at least 1 that are contained in a convex body. Section 3 contains the proofs of our results.

2. Preliminaries

2.1. John's ellipsoid

Theorem 2.1 *(John [11]). Let* $K \subset \mathbb{R}^d$ *be a convex body. Then* K *contains a unique ellipsoid of maximal volume. This ellipsoid is* $\mathbf{B}^d$ *if and only if* $\mathbf{B}^d \subset K$ *and there are contact points* $u_1, \ldots, u_m \in bd(K) \cap bd(\mathbf{B}^d)$ *and positive numbers* $\lambda_1, \ldots, \lambda_m$ *with* $d + 1 \le m \le \frac{d(d+3)}{2}$ *such that*

$$\sum_{i=1}^{m} \lambda_i u_i = 0, \text{ and } I_d = \sum_{i=1}^{m} \lambda_i u_i u_i^T,$$

where I_d *denotes the* $d \times d$ *identity matrix and the* u_i *are column vectors.*

The following is a well known corollary, see [1, Lecture 3].

Corollary 2.2. *Assume that* $\mathbf{B}^d$ *is the unique maximal volume ellipsoid contained in a convex body* K *in* $\mathbb{R}^d$*. Then* $d\mathbf{B}^d \supseteq K$*.*

2.2. Colorful Helly theorem

We recall the Colorful Helly Theorem, as one of its straightforward corollaries will be used.

Theorem 2.3 *(Colorful Helly Theorem, Lovász, Bárány [2]). Let* $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ *be finite families of convex bodies in* $\mathbb{R}^d$*, and assume that for any colorful selection* $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$*, the intersection* $\bigcap_{i=1}^{d+1} C_i$ *is non-empty. Then for some* j*, the intersection* $\bigcap_{C \in \mathcal{C}_j} C$ *is also non-empty.*

Corollary 2.4. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be finite families of convex bodies, and L a convex body in $\mathbb{R}^d$. Assume that for any colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ contains a translate of L. Then for some j, the intersection $\bigcap_{C \in \mathcal{C}_j} C$ contains a translate of L.*

Proof of Corollary 2.4. We use the following operation, the Minkowski difference of two convex sets A and B:

$$A \sim B := \bigcap_{b \in B} (A - b).$$

It is easy to see that $A \sim B$ is the set of those vectors t such that $B + t \subseteq A$.

By the assumption, for any colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, we have $\bigcap_{i=1}^{d+1} (C_i \sim L) \neq \emptyset$. By Theorem 2.3, for some j, we have $\bigcap_{C \in \mathcal{C}_j} (C \sim L) \neq \emptyset$, and thus, $\bigcap_{C \in \mathcal{C}_j} C$ contains a translate of L. $\square$

2.3. Lowest ellipsoid

We will follow Lovász' idea of the proof of the Colorful Helly Theorem. The first step is to fix an ordering of the objects of study. This time, we are looking for an ellipsoid and not a point in the intersection, therefore we need an *ordering on the ellipsoids.*

For an ellipsoid $\mathcal{E}$, we define its *height* as the largest value of the orthogonal projection of $\mathcal{E}$ on the last coordinate axis, that is, $\max\{x^T e_d \mid x \in \mathcal{E}\}$, where $e_d = (0, 0, \ldots, 0, 1)^T$.

Lemma 2.5. *Let C be a convex body that contains an ellipsoid of volume $\omega_d := \mathrm{Vol}(\mathbf{B}^d)$. Then there is a unique ellipsoid of volume ω_d such that every other ellipsoid of volume ω_d in C has larger height. Furthermore, if $\tau \in \mathbb{R}$ denotes the height of this ellipsoid, then the largest volume ellipsoid of the convex body $H_\tau \cap C$ is this ellipsoid, where H_τ denotes the closed half-space $H_\tau = \{x \in \mathbb{R}^d \mid x^T e_d \leq \tau\}$.*

We call this ellipsoid the *lowest ellipsoid* in C.

Proof of Lemma 2.5. It is not difficult to see that $H_\tau \cap C$ does not contain any ellipsoid of volume larger than ω_d. Indeed, otherwise for a sufficiently small $\epsilon > 0$, the set $H_{\tau-\epsilon} \cap C$ would contain an ellipsoid of volume equal to ω_d, where $H_{\tau-\epsilon}$ denotes the closed half-space $H_{\tau-\epsilon} = \{x \in \mathbb{R}^d \mid x^T e_d \leq \tau - \epsilon\}$.

Thus, by Theorem 2.1, $\mathbf{B}^d$ is the unique largest volume ellipsoid of $H_\tau \cap C$. It follows that $\mathbf{B}^d$ is the unique lowest ellipsoid of C. $\square$

2.4. Quantitative volume theorem with ellipsoids

We will rely on the following quantitative Helly theorem.

Theorem 2.6 *(Quantitative Volume Theorem). Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$. Assume that the intersection of any $2d$ of them is of volume at least 1. Then* $\mathrm{Vol}\left(\bigcap_{i=1}^{n} C_i\right) \geq c^d d^{-3d/2}$ *with an absolute constant $c > 0$.*

As noted in Section 1, it is shown in [12] that the $d^{-3d/2}$ term cannot be improved further than $d^{-d/2}$.

Corollary 2.7 *(Quantitative Volume Theorem with ellipsoids). Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$. Assume that the intersection of any $2d$ of them contains an ellipsoid of volume at least 1. Then $\bigcap_{i=1}^{n} C_i$ contains an ellipsoid of volume at least $c^d d^{-5d/2}$ with an absolute constant $c > 0$.*

Theorem 2.6 was proved by Bárány, Katchalski and Pach [3] with the weaker volume bound d^{-2d^2}. In [12], the volume bound $c^d d^{-2d}$ was shown, and this argument was later refined by Brazitikos [5] to obtain the bound presented above. An inspection of the argument in [12] shows that Corollary 2.7 holds with the slightly stronger bound $c^d d^{-3d/2}$ as well. However, as this constant in the exponent is of no consequence, we instead deduce Corollary 2.7 in the form presented above from Theorem 2.6.

We note that results like Theorem 2.6 are sometimes stated only for halfspaces and not convex sets in general, as is the case for example in [5]. However, this yields no loss of generality, as any convex set can be approximated (in any meaningful metric) by the intersection of finitely many halfspaces.

Proof of Corollary 2.7. Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$ satisfying the assumptions of Corollary 2.7. In particular, they satisfy the assumptions of Theorem 2.6, and hence, $\mathrm{Vol}\left(\bigcap_{i=1}^{n} C_i\right) \geq c^d d^{-3d/2}$. Finally, Corollary 2.2 yields that $\bigcap_{i=1}^{n} C_i$ contains an ellipsoid of volume at least $c^d d^{-5d/2}$ completing the proof of Corollary 2.7. □

3. Proofs

3.1. Proof of Proposition 1.3

We will prove the following statement, which is clearly equivalent to Proposition 1.3.

Assume that the largest volume ellipsoid contained in $\bigcap_{C \in \mathcal{C}} C$ is of volume $\omega_d := \mathrm{Vol}(\mathbf{B}^d)$. Then there are $d(d+3)/2$ sets in $\mathcal{C}$ such that the largest volume ellipsoid in their intersection is of volume ω_d.

The problem is clearly affine invariant, and thus, we may assume that the largest volume ellipsoid in $\bigcap_{C\in\mathcal{C}} C$ is the unit ball $\mathbf{B}^d$.

By one direction of Theorem 2.1, there are contact points $u_1, \ldots, u_m \in bd(\bigcap_{C\in\mathcal{C}} C) \cap bd(\mathbf{B}^d)$ and positive numbers $\lambda_1, \ldots, \lambda_m$ with $d+1 \le m \le \frac{d(d+3)}{2}$ satisfying the equations in Theorem 2.1. We can choose $C_1, \ldots C_m \in \mathcal{C}$ such that $u_i \in bd(C_i)$ for $i = 1, \ldots, m$.

By the other direction of Theorem 2.1, $\mathbf{B}^d$ is the largest volume ellipsoid of $\bigcap_{i=1}^{m} C_i$, completing the proof of Proposition 1.3.

3.2. Proof of Proposition 1.4

Lemma 3.1. *Let $C_1, \ldots, C_{d(d+3)/2}$ be convex bodies in $\mathbb{R}^d$. Assume that $K := \bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume $\omega_d := \mathrm{Vol}(\mathbf{B}^d)$. Set $K_j := \bigcap_{i=1, i\neq j}^{d(d+3)/2} C_i$, and let $\mathcal{E}$ denote the lowest ellipsoid in K. Then there exists a j such that $\mathcal{E}$ is also the lowest ellipsoid of K_j.*

Proof of Lemma 3.1. Let τ denote the height of $\mathcal{E}$. By Lemma 2.5, $\mathcal{E}$ is the largest volume ellipsoid of $K \cap H_\tau$, where H_τ is the half-space defined in Lemma 2.5.

Suppose that $\mathcal{E}$ is not the lowest ellipsoid in K_j for every $j \in \{1, \ldots, d(d+3)/2\}$. Since $\mathcal{E} \subset K \subset K_j$, this means that each K_j contains a lower ellipsoid than $\mathcal{E}$ of volume ω_d. Therefore we can choose a small $\epsilon > 0$ such that $K_j \cap H_{\tau-\epsilon}$ contains an ellipsoid of volume ω_d for each j, where $H_{\tau-\epsilon}$ denotes the closed half-space $H_{\tau-\epsilon} = \{x \in \mathbb{R}^d \mid x^T e_d \le \tau - \epsilon\}$.

Let us consider now the following $\frac{d(d+3)}{2} + 1$ sets: $K_1, K_2, \ldots, K_{d(d+3)/2}, H_{\tau-\epsilon}$. If we take the intersection of $\frac{d(d+3)}{2}$ of these sets, we obtain either K, or $K_j \cap H_{\tau-\epsilon}$ for some j. By our assumption, K contains an ellipsoid of volume ω_d. By the choice of ϵ, we have that $K_j \cap H_{\tau-\epsilon}$ also contains an ellipsoid of volume ω_d. Hence, we can apply Proposition 1.3, which yields that $C_1 \cap \cdots \cap C_{d(d+3)/2} \cap H_{\tau-\epsilon} = K \cap H_{\tau-\epsilon}$ also contains an ellipsoid of volume ω_d. This contradicts the fact that $\mathcal{E}$ is the lowest ellipsoid in K, and thus, Lemma 3.1 follows. □

We will prove the following statement, which is clearly equivalent to Proposition 1.4.

Assume that for every colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume ω_d. We will show that for some j, the intersection $\bigcap_{C\in\mathcal{C}_j} C$ contains an ellipsoid of volume ω_d.

By Lemma 2.5, we can choose the lowest ellipsoid in each of these intersections. Let us denote the set of these ellipsoids as $\mathcal{B}$. Since we have finitely many intersections, there is a highest one among these ellipsoids. Let us denote this ellipsoid by $\mathcal{E}_{max}$.

$\mathcal{E}_{max}$ is defined by some $C_1 \in \mathcal{C}_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$. Once again let $K_j = \bigcap_{i=1, i \neq j}^{d(d+3)/2} C_i$ and $K = \bigcap_{i=1}^{d(d+3)/2} C_i$. By Lemma 3.1, there is a j such that $\mathcal{E}_{max}$ is the lowest ellipsoid in K_j. We will show that $\mathcal{E}_{max}$ lies in every element of $\mathcal{C}_j$ for this j.

Fix a member C_0 of $\mathcal{C}_j$. Suppose that $\mathcal{E}_{max} \not\subset C_0$. Then $\mathcal{E}_{max} \not\subset C_0 \cap K_j$. By the assumption of Proposition 1.4, $C_0 \cap K_j$ contains an ellipsoid of volume ω_d, since it is the intersection of a colorful selection of sets. Since $C_0 \cap K_j \subset K_j$, the lowest ellipsoid of $C_0 \cap K_j$ is at least as high as the lowest ellipsoid of K_j. But the unique lowest ellipsoid of K_j is $\mathcal{E}_{max}$, and $\mathcal{E}_{max} \not\subset C_0 \cap K_j$. So the lowest ellipsoid of $C_0 \cap K_j$ lies higher than $\mathcal{E}_{max}$. This contradicts that $\mathcal{E}_{max}$ was chosen to be the highest among the ellipsoids in $\mathcal{B}$. So $\mathcal{E}_{max} \subset C_0$. Since $C_0 \in \mathcal{C}_j$ was chosen arbitrarily, we obtain that $\mathcal{E}_{max} \subset \bigcap_{C \in \mathcal{C}_j} C$, completing the proof of Proposition 1.4.

3.3. Proof of Proposition 1.5

Consider an arbitrary colorful selection of $d(d+3)/2$ convex bodies. By Corollary 2.7, their intersection contains an ellipsoid of volume at least $c^d d^{-5d/2}$. It follows immediately from Proposition 1.4, that the intersection of one of the color classes contains an ellipsoid of volume at least $c^d d^{-5d/2}$, completing the proof of Proposition 1.5.

3.4. Proof of Theorem 1.1

We will prove the following statement, which is clearly equivalent to Theorem 1.1.

Assume that the intersection of all colorful selections of $2d$ sets contains an ellipsoid of volume at least $\omega_d := \mathrm{Vol}(\mathbf{B}^d)$. Then, there is an $1 \leq i \leq 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume at least $c^{d^2} d^{-5d^2/2} \omega_d$ with an absolute constant $c \geq 0$.

Lemma 3.2. *Assume that $\mathbf{B}^d$ is the largest volume ellipsoid contained in the convex set C in $\mathbb{R}^d$. Let $\mathcal{E}$ be another ellipsoid in C of volume at least $\delta\omega_d$ with $0 < \delta < 1$, where $\omega_d = \mathrm{Vol}(\mathbf{B}^d)$. Then there is a translate of $\frac{\delta}{d^{d-1}}\mathbf{B}^d$ which is contained in $\mathcal{E}$.*

Proof of Lemma 3.2. If the lengths of all d semi-axes $a_1, \ldots, a_d$ of $\mathcal{E}$ are at least λ for some $\lambda > 0$, then clearly, $\lambda\mathbf{B}^d + c \subset \mathcal{E}$, where c denotes the center of $\mathcal{E}$. We will show that all the semi-axes are long enough.

By Corollary 2.2, $\mathcal{E} \subset C \subset d\mathbf{B}^d$. Therefore, $a_i \leq d$ for every $i = 1, \ldots, d$. Since the volume of $\mathcal{E}$ is $a_1 \cdots a_d \omega_d \geq \delta\omega_d$, we have $a_i \geq \frac{\delta}{d^{d-1}}$ for every $i = 1, \ldots, d$, completing the proof of Lemma 3.2. □

Consider the lowest ellipsoid in the intersection of all colorful selections of $2d - 1$ sets. We may assume that the highest one of these ellipsoids is $\mathbf{B}^d$. By possibly changing the indices of the families, we may assume that the selection is $C_1 \in \mathcal{C}_1, \dots, C_{2d-1} \in \mathcal{C}_{2d-1}$. We call $\mathcal{C}_{2d}, \mathcal{C}_{2d+1}, \dots, \mathcal{C}_{3d}$ the *remaining families*.

Consider the half-space $H_1 = \{x \in \mathbb{R}^d \mid x^T e_d \leq 1\} \supset \mathbf{B}^d$. By Lemma 2.5, $\mathbf{B}^d$ is the largest volume ellipsoid contained in $M := C_1 \cap \dots \cap C_{2d-1} \cap H_1$.

Next, take an arbitrary colorful selection $C_{2d} \in \mathcal{C}_{2d}, C_{2d+1} \in \mathcal{C}_{2d+1}, \dots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d+1$ families. We claim that the intersection of any $2d$ sets of

$$C_1, \dots, C_{2d-1}, H_1, C_{2d}, \dots, C_{3d}$$

contains an ellipsoid of volume at least ω_d. Indeed, if H_1 is not among those $2d$ sets, then our assumption ensures this. If H_1 is among them, then by the choice of H_1, the claim holds.

Therefore, by Theorem 2.7, the intersection

$$\bigcap_{i=1}^{3d} C_i \cap H_1$$

contains an ellipsoid $\mathcal{E}$ of volume at least $\delta\omega_d$, where $\delta := c^d d^{-3d/2}$. Clearly, $\mathcal{E} \subset M$.

Since $\mathbf{B}^d$ is the maximum volume ellipsoid contained in M, by Lemma 3.2, we have that there is a translate of $\frac{\delta}{d^{d-1}}\mathbf{B}^d$ which is contained in $\mathcal{E}$ and thus in $\bigcap_{i=2d}^{3d} C_i$.

Thus, we have shown that any colorful selection $C_{2d} \in \mathcal{C}_{2d}, C_{2d+1} \in \mathcal{C}_{2d+1}, \dots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d+1$ families, $\bigcap_{i=2d}^{3d} C_i$ contains a translate of the same convex body $c^d d^{-5d/2}\mathbf{B}^d$. It follows from Corollary 2.4 that there is an index $2d \leq i \leq 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains a translate of $c^d d^{-5d/2}\mathbf{B}^d$, which is an ellipsoid of volume $c^{d^2} d^{-5d^2/2}\omega_d$, finishing the proof of Theorem 1.1.

3.5. Proof of Corollary 1.2

By Corollary 2.2, the volume of the largest ellipsoid in a convex body is at least d^{-d} times the volume of the body. Corollary 1.2 now follows immediately from Theorem 1.1.

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References

[1] Keith Ball, An elementary introduction to modern convex geometry, in: Flavors of Geometry, in: Math. Sci. Res. Inst. Publ., vol. 31, Cambridge Univ. Press, Cambridge, 1997, pp. 1–58.
[2] Imre Bárány, A generalization of Carathéodory's theorem, Discrete Math. 40 (2–3) (1982) 141–152, https://doi.org/10.1016/0012-365X(82)90115-7.
[3] Imre Bárány, Meir Katchalski, János Pach, Quantitative Helly-type theorems, Proc. Am. Math. Soc. 86 (1) (1982) 109–114, https://doi.org/10.2307/2044407.
[4] Felix Behrend, Über einige Affininvarianten konvexer Bereiche, Math. Ann. 113 (1) (1937) 713–747, https://doi.org/10.1007/BF01571662.
[5] Silouanos Brazitikos, Brascamp-Lieb inequality and quantitative versions of Helly's theorem, Mathematika 63 (1) (2017) 272–291, https://doi.org/10.1112/S0025579316000255.
[6] Ludwig Danzer, Branko Grünbaum, Victor Klee, Helly's theorem and its relatives, in: Proc. Sympos. Pure Math., vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180.
[7] Jesús A. De Loera, Xavier Goaoc, Frédéric Meunier, Nabil H. Mustafa, The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg, Bull. Am. Math. Soc. (N.S.) 56 (3) (2019) 415–511, https://doi.org/10.1090/bull/1653.
[8] Jesús A. De Loera, Reuben N. La Haye, Deborah Oliveros, Edgardo Roldán-Pensado, Helly numbers of algebraic subsets of $\mathbb{R}^d$ and an extension of Doignon's theorem, Adv. Geom. 17 (4) (2017) 473–482, https://doi.org/10.1515/advgeom-2017-0028.
[9] Peter M. Gruber, Minimal ellipsoids and their duals, Rend. Circ. Mat. Palermo (2) 37 (1) (1988) 35–64, https://doi.org/10.1007/BF02844267.
[10] Andreas Holmsen, Rephael Wenger, Helly-type theorems and geometric transversals, in: Jacob E. Goodman, Joseph O'Rourke, Csaba D. Tóth (Eds.), Handbook of Discrete and Computational Geometry, third edition, CRC Press, Boca Raton, FL, 2018, pp. 91–123.
[11] Fritz John, Extremum problems with inequalities as subsidiary conditions, in: Studies and Essays Presented to R. Courant on His 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N.Y., 1948, pp. 187–204.
[12] Márton Naszódi, Proof of a conjecture of Bárány, Katchalski and Pach, Discrete Comput. Geom. 55 (1) (2016) 243–248, https://doi.org/10.1007/s00454-015-9753-3.
[13] Sherry Sarkar, Alexander Xue, Pablo Soberón, Quantitative combinatorial geometry for concave functions, arXiv e-prints, arXiv:1908.04438, Aug 2019.