CLOSING REPORT OF THE PROJECT OTKA K 68195

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This project entitled "Algebraic methods in models of quantum field theory" began in 2007 and ended, after a one year prolongation, in 2012. We have started with 3 participant researchers, Imre Bálint, Gabriella Böhm and Péter Vecsernyés, but soon (in 2008) our young colleague Imre Bálint left the project, and also our Institute, for a job in financial mathematics. During the five years the three of us have completed 16 research papers, most of which have already been published in leading mathematics and mathematical physics journals, and presented our results at about 25 conference talks. Most of these papers and talks are due to our most productive researcher, G. Böhm, who meanwhile has attained to get the DSc degree "MTA doktora" in mathematics from the Hungarian Academy of Sciences. Also, we are indebted to our collaborators Dragos Stefan (Univ. of Bucharest), José Gómez-Torrecillas (Univ. de Granada), Gábor Hofer-Szabó (King Sigismund College, Budapest), Steve Lack and Ross Street (Macquarie Univ., Sydney) without whom our goals could not have been achieved.

Our research area lies at the borders of quantum field theory, noncommutative geometry and category theory and is strongly determined by the results we have obtained in the past on quantum groupoids. The term "quantum groupoid" here is used to comprise such notions as the weak bialgebras and weak Hopf algebras of [Böhm-Nill-Szlachányi, 1998] and the bialgebroids of [Takeuchi, 1977] and [Lu,1996] and the Hopf algebroids of [Böhm-Szlachányi, 2004] and [Böhm, 2004]. Below we summarize our main results in somewhat arbitrary order.

Abstract field algebras [2]

The reconstruction problem in Algebraic Quantum Field Theory (AQFT) solved by the abstract duality theorem of S. Doplicher and J. E. Roberts contains, as an important ingredient, the construction of field algebras from observables. Although this construction relates fiber functors to field algebras, therefore relies only on Tannaka duality, it is all the more interesting in the quantum group(oid) world which lies beyond the reach of the Doplicher-Roberts Theorem.

In [2] a pure algebraic version was presented in which we used a ring B for the observable algebra and a monoidal category $\mathcal{C}$ of ring endomorphisms of B for the superselection sectors and determined a ring homomorphism (the field algebra extension) $\rho : B \to A$ for each Abelian group valued monoidal functor $F : \mathcal{C} \to \mathsf{Ab}$, i.e., for each 'fiber functor' in the weakest sense.

We have shown that the field algebra construction $F \mapsto \rho$ is the left adjoint of another familiar construction in AQFT, in which a fiber functor is constructed from the 'family of Hilbert spaces' within the field algebra. Moreover, we found that the field algebra A can be obtained as simply as the tensor product $\Upsilon \underset{\mathcal{C}}{\otimes} F$ of the fiber functor with a fixed contravariant monoidal functor $\Upsilon : \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$. The presheaf Υ is determined solely by the data $\langle B, \mathcal{C}\rangle$ and its existence characterizes the categories of interest.

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This paper also discusses two special, finitary cases of the construction. In the first, the extension $\rho : B \to A$ is right adjoint in the 2-category Ring. We have proved that this is the same thing as a Kleisli construction in Ring. In the second, the category $\mathcal{C}$ is generated by direct summand diagrams from a single right adjoint endomorphism. This latter case generalizes the concept of fusion category in the non-semisimple direction although the category is restricted to be a category of endomorphisms of a ring. In both cases the field algebra is a right H-Galois extension of the observable algebra B where the symmetry object $H = F^* \underset{\mathcal{C}}{\otimes} F$ is a right R-bialgebroid which is finite projective as left R-module.

Tannaka duality for bialgebroids [4]

Let R be a ring, let H be a right R-bialgebroid which is flat as a left R-module. Restricting the forgetful functor $\mathsf{Comod}(H) \to \mathsf{Bimod}(R)$ of right H-comodules to the full subcategory $\mathcal{C} \subset \mathsf{Comod}(H)$ consisting of comodules that are finite projective as left R-modules we obtain a functor $F : \mathcal{C} \to \mathsf{Bimod}(R)$ which is the archetype of *fiber functors* in noncommutative Tannaka duality. The basic question to answer is the following: Which properties of a monoidal category $\mathcal{C}$ and strong monoidal functor $F : \mathcal{C} \to \mathsf{Bimod}(R)$ can guarantee the existence of an R-bialgebroid H for which F becomes isomorphic to the archetypical fiber functor and $\mathcal{C}$ equivalent to the corresponding category of finite type H-comodules?

This problem goes beyond the classical Tannaka duality not only in that the fiber functor has values in a bimodule category instead of vector spaces but also in that the fiber functor cannot be required to be exact and $\mathcal{C}$ cannot be required to be abelian. So, in fact, the real question is what are the fiber functors in this generality.

In a very long, technical paper [4], which is not intended to be published in this form, we collected the necessary tools to answer this question and did give an answer: In Definition 6.5 we defined a notion of fiber functor which then lead to the Representation Theorem 6.6. Furthemore we noticed an interesting relation with Grothendieck topologies. The bialgebroids arising from fiber functors were such that their comodule categories $\mathsf{Comod}(H)$ could also be described as categories $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ of abelian group valued sheaves on $\mathcal{C}$ with respect to a monoidal Grothendieck topology $\mathcal{T}$. The Grothendieck topology could be constructed directly from the flat fiber functor.

Skew-monoidal categories [16]

Needless to emphasize how important monoidal (or tensor) categories are in either mathematics or physics: tensors are everywhere. Therefore it appears to be a rather heretic ambition to generalize the very notion of monoidal category, unless there is an interesting example which calls for such a generalization.

In [16] we introduced categories with a "skew-monoidal product" $A \star B$ and "skew-monoidal unit" R such that the associativity and left and right unit constraints

$$A \star (B \star C) \to (A \star B) \star C, \qquad A \to R \star A, \qquad A \star R \to A$$

were just natural transformations and not natural isomorphisms. They were, however, required to obey the usual 5 axioms of a monoidal category. Such skew-monoidal categories can then be made the objects of a 2-category with 1-cells the skew-monoidal functors and with 2-cells the skew-monoidal natural transformations.

The surprising result was that the *closed* skew-monoidal structures on the category $\mathsf{Mod}(R)$ of one-sided R-modules (which does not possess any monoidal structure in general), with skew-monoidal unit being the regular R-module R, are in bijection with R-bialgebroids. The advantage of this description of bialgebroids H is that it allows one to treat H-modules and H-comodules on equal footing: As categories $\mathsf{Mod}(H)$ is the Eilenberg-Moore category of the canonical monad $R \star \_$ and $\mathsf{Comod}(H)$ is the Eilenberg-Moore category of the canonical comonad $\_ \star R$ on the skew-monoidal category. The automatic disadvantage is that the (ordinary) monoidal structures of $\mathsf{Mod}(H)$ and $\mathsf{Comod}(H)$ cannot be seen in terms of the skew-monoidal structure.

A few months after its appearence in the arXiv these results motivated two even more interesting papers. In the first Steve Lack and Ross Street proved that quantum categories are skew-monoidales and in the second Ross Street introduced *skew-closed categories* which fitted perfectly as being the base for a skew enriched category theory, thereby surpassing the *closed categories* of Eilenberg and Kelly, a cornerstone of category theory for 47 years. Street also proved that the monoidal skew-closed categories are precisely the closed skew-monoidal categories of [16].

Noncommuting common causes [7, 8, 9, 10]

States in algebraic quantum field theory (AQFT) typically establish correlations, $\phi(AB) \neq \phi(A)\phi(B)$, between local observable projections ('quantum events') $A \in \mathcal{A}(\mathcal{O}_A)$ and $B \in \mathcal{A}(\mathcal{O}_B)$ supported in spacelike separated double cones $\mathcal{O}_A$ and $\mathcal{O}_B$, respectively. Reichenbach's common cause principle, generalized to the quantum field theoretical setting, offers an apt tool to causally account for these superluminal correlations. Namely, in case of type III local von Neumann algebras, which occur in Poincaré covariant AQFTs in Minkowski spacetimes, the existence of a 'weak commuting common cause' $\{C_1, C_2 := \mathbf{1} - C_1\}$ has been previously established: for any locally normal and faithful state ϕ on $\mathcal{A}$ there exists a local observable projection C_1 in the union of the causal pasts of $\mathcal{O}_A$ and $\mathcal{O}_B$ commuting with A and B, such that the screening-off conditions $\phi(ABC_i)\phi(C_i) = \phi(AC_i)\phi(BC_i); i = 1, 2$ are fulfilled.

We have shown, that this property, called 'weak commuting common cause principle' in the literature, is not valid in AQFTs with locally finite degrees of freedom in general. However, the noncommutative generalization (dropping the unreasonable commutativity requirement between the common cause $\{C_1, C_2\}$ and the events A, B) of the principle has been proved for such AQFTs.

Applying to commuting subsystems of quantum theories Bell inequalities are fulfilled for separable, i.e. unentangled states. However, Bell inequalities can also be understood as constraints between classical conditional probabilities, and can be derived from a set of assumptions representing a joint common causal explanation of classical correlations. A similar derivation of the 'non-classical' Bell inequalities was expected in the AQFT setting from the existence of a joint common cause. We have shown that although the classical notion of joint common causal explanation can readily be generalized for the non-classical case, the Bell inequalities used in quantum theories can be derived only from the non-classical *commuting* common causes. In the *noncommuting* case just the opposite is true: a joint noncommuting common causal explanation can be given for a set of correlations even if it violates the Bell inequalities, i.e. if the corresponding state is entangled. Namely, one can reproduce the EPR-Bohm scenario in the AQFT setting with locally finite degrees of freedom and a joint noncommuting common cause can be constructed for the set of spacelike separated correlating events (projections) violating maximally the Clauser–Horne inequality. The local noncommuting common cause is supported in the common past of the correlating events.

Phases of Hopf spin chains [17, 18]

Phases in AQFTs can be defined as inequivalent extensions of the observable algebra $\mathcal{A}$ provided by the so-called dual algebra $\mathcal{A}^d_\pi$, which is the inductive limit C^*-algebra of the net of local extensions $\mathcal{A}^d_\pi(\mathcal{O}) := \pi(\mathcal{A}(\mathcal{O}'))' \cap \mathcal{B}(\mathcal{H})$ within a double cone $\mathcal{O}$ in an irreducible representation $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and which measures the violation of Haag duality in the representation π. In case of Hopf spin chains $\mathcal{A} \equiv \mathcal{A}(H)$ based on finite dimensional Hopf C^*-algebras H, which generalize the Ising and Potts quantum chains based on the group algebras $H = \mathbf{C}Z_N$ of cyclic groups Z_N of order N, the extensions $\mathcal{A} \subseteq \mathcal{A}^d_\pi$ are shown to be completely characterized by the cohomology classes of intermediate right Δ-cocycles of the Drinfeld double $\mathcal{D}(H)$ of the underlying Hopf algebra H. This means that the possible local extensions $\mathcal{A}^d_\pi(\mathcal{O})$ can be given in terms of the one-sided cocycle-deformed left coideal subalgebras of the dual $\hat{\mathcal{D}}(H)$ of $\mathcal{D}(H)$.

Having introduced the notion of asymptotic commutants of α-invariant subalgebras in norm asymptotically abelian (NAA) C^*-algebras $\mathcal{A}$ with respect to the automorphism $\alpha \in \operatorname{Aut}\mathcal{A}$ it was shown that the asymptotic commutant $\mathcal{A}^{ac}_\omega$ of $\pi_\omega(\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_\omega)$ arising from a pure α-invariant state ω on $\mathcal{A}$ is a strongly asymptotically abelian C^*-extension of $\mathcal{A}$. Using that the pair $(\mathcal{A}, \alpha)$ of an observable algebra and a spacelike translation automorphism $\alpha \in \operatorname{Aut}\mathcal{A}$ is a NAA algebra this result implies that $\mathcal{A}^{ac}_\omega \supseteq \mathcal{A}^d_{\pi_\omega}$ cannot contain nonlocal, i.e. non-bosonic field algebra elements. In case of a Hopf spin chain the result implies that the dual net $\mathcal{A}^d_{\pi_\omega}$ is local, that is it implies the absence of non-local translation invariant phases. The possible translation invariant phases are characterized by the bosonic intermediate right Δ-cocycles of the quasitrangular Drinfeld double $\mathcal{D}(H)$.

A new cyclic cohomology theory [1, 3, 12]

Non-commutative differential geometry is a relatively new branch of mathematics which combines operator algebraic techniques with ideas from differential geometry. As an alternative to renormalization, the study of quantum field theories on (topological) spaces obeying non-commutative geometry, was suggested already by Heisenberg. Seiberg and Witten pointed out that also some string theories lead to non-commutative geometries. On a long scale, the theory is hoped to lead to the insertion of relativity theory into the standard model.

Cyclic (co)homology theory is a tool to describe the (Hopf algebraic or more general) symmetry in non-commutative geometry. In our joint research with Dragos Stefan we proposed a general (categorical) approach to this theory. Our construction covers, on one hand, the known examples of para-(co)cyclic objects (e.g. those by Connes and Moscovici). But on the other hand, it could be smoothly applied to develop the cyclic cohomology theory of bialgebroids.

Bialgebroids generalize bialgebras from commutative to non-commutative base rings. This means that – while in the study of a model with bialgebra symmetry one deals with the symmetric monoidal category of modules over the commutative base ring – in the presence of a bialgebroid symmetry the appropriate framework is the monoidal category of bimodules over the non-commutative base ring. However, a bimodule category is no longer symmetric, not even braided. This implies that the methods that are available for bialgebras can no longer be applied.

The new feature of our method is to work, instead of algebras, with their induced monads. Taking the tensor product with it on the left and on the right, an algebra determines two monads which are related by a (trivial) distributive law. In contrast to the earlier approaches using the symmetry of the base category, we constructed the cyclic structure in terms of this distributive law. The advantage is the applicability not only for algebras but also for rings over non-commutative base (what is the case in bialgebroids); hence in the absence of a symmetry operation interchanging the factors in the tensor product.

Generalizing the tensor products with algebras, we constructed examples of cyclic models in terms of locally braided morphisms of monads.

As an application of our method, we computed the Hochschild and cyclic cohomology groups of a groupoid algebra.

Connes' cyclic duality relates a para-cyclic, and a para-cocyclic theory. We worked out a categorical description of this duality. For that, first we re-formulated our functor-based approach to para-(co)cyclic objects to an arbitrary 2-category K. We presented 2-functors from an appropriate 2-category of 2-functors to the 2-category of para-cyclic and para-cocyclic objects in K, respectively. We proved that the functor representing cyclic duality admits a lift to these 2-categories.

As an application of the general theory, we wrote down explicitly the duals of the para-(co)cyclic models associated to (co)module (co)algebras over a Hopf algebroid with a bijective antipode. This provides new examples of non-commutative geometries with quantum symmetries.

Weak bimonads and weak Hopf monads [6]

It is a classical result that an algebra admits a bialgebra structure if and only if its category of representations is monoidal (corresponding to the superposition of physical charges) and the forgetful functor respects this monoidal structure. The generalization of bialgebras to so-called *bimonads* is due to Moerdijk and is based on this property: a bimonad is a monad on a monoidal category whose Eilenberg-Moore category is monoidal and the forgetful functor respects this monoidal structure.

In low dimensional quantum field theories one observes not only bialgebra symmetries but also more general, weak bialgebra symmetries. The representation category of a weak bialgebra is also monoidal but this monoidal structure is no longer respected by the forgetful functor. It was proved by Szlachanyi that a weak bialgebra structure corresponds to a separable Frobenius structure of the forgetful functor.

Unifying both above described notions of symmetry, jointly with Stephen Lack and Ross Street we studied such monads – called *weak bimonads* – on (Cauchy complete) monoidal categories whose Eilenberg-Moore category is monoidal and the forgetful functor possesses a separable Frobenius structure. We formulated sufficient and necessary conditions for these properties to hold, which conditions generalize the axioms of a weak bialgebra. We proved an equivalence between weak bimonads and bimonads on bimodule categories over separable Frobenius monoids. We proved that weak bialgebras in braided monoidal categories induce examples of weak bimonads. We studied the antipode on a weak bimonad (which corresponds to the charge conjugation in the quantum field theory language).

2-categories of monads and colimit completions [15]

Many constructions based on Hopf algebras and occurring in Hopf-Galois theory are known to fit the so-called formal theory of monads, i.e. the abstract treatment of monads in 2-categories due to Stephen Lack and Ross Street. For example, crossed products with bialgebras are examples of wreath products with monads. As another example, the so-called relative Hopf modules can be understood as coalgebras of a lifted comonad on the Eilenberg-Moore category of some monad.

In her paper [The weak theory of monads, (Adv. Math. 225 (2010), 1-32] G. Böhm investigated how to extend the formal theory of monads in such a way that the resulting new theory covers constructions in terms of weak Hopf algebras as well. She has shown that for this – in the case of an arbitrary 2-category K – Lack and Steet's 2-category of monads in K needs to be replaced by a larger 2-category denoted by $\mathsf{EMW}(K)$.

In a classical paper, Lack and Street proved that their 2-category of monads in K arises as a completion of K under some (2-categorical) colimits. It was a naturally arising question if $\mathsf{EMW}(K)$ can be obtained as an appropriate completion of K. In collaboration with Lack and Street we proved that $\mathsf{EMW}(K)$ arises as the completion of the Cauchy completion of K under some new, bicategorical colimits. In order to prove that, we worked out in detail a systematic theory of monads in the Cauchy completion of an arbitrary 2-category. This put my earlier results in a new light, some earlier proven statements became easy corollaries.

Bearing the weak bialgebra axioms on representation theory [5]

Florian Nill, in his work [Axioms for Weak Bialgebras, arxiv.org/abs/ math/9805104] derived the axioms of weak bialgebras from requirements on the category of representations. In the proofs he made essential use of the assumption that only finite dimensional weak bialgebras are treated, hence the linear dual is a weak bialgebra, too.

In our joint work with Stefaan Caenepeel and Kris Jansen we applied Nill's representation theoretical approach to weak bialgebras without making any assumption on the dimension. We systematically analyzed how the individual weak bialgebra axioms influence the properties of the representation category and of the forgetful functor. Evidently, this required new proofs since no arguments referring to the dual were applicable.

Weak product algebras [11, 13, 14]

There is an extended mathematical literature on algebras arising as tensor products of two algebras, providing a common extension of both factors. Such algebras describe e.g. the observable quantities in spin models whose spins take their values in a finite dimensional Hopf algebra or in a finite group. It is known that an algebra is isomorphic to a tensor product of two algebras if and only if it satisfies a so-called strong factorization property. Such algebras are in bijective correspondence with the so-called *distributive laws* between the constituent algebras.

In the spin models in which the possible spin-values lie in a weak Hopf algebra, the observable algebra of a union of two neighbouring space-time regions is not the tensor product of the observable algebras of the two regions. It can be obtained by a more general construction corresponding to a *weak distributive law*.

In our joint work with Jose Gomez-Torrecillas, we investigated the factorization problem behind such weak product algebras. We proved a biequivalence between the bicategory of weak distributive laws and the bicategory of algebras obeying the so-called bilinear factorization property. We listed a number of examples of such algebras, some of them of interest in physics.

The algebra of observables in a spin chain is an iterated product of the observable algebras associated to lattice points. By this motivation, G. Böhm investigated the compatibility conditions that the distributive laws describing the subsequent steps must obey in order for the product construction could be iterated. She proved that they must be objects of an appropriate 2-category. She interpreted the product construction as the object map of an appropriate 2-functor. Then the associativity of the iterated product was proven from a natural equivalence between the iterated 2-functors. These abstract results were applied to the observable algebras of spin chains based on weak Hopf algebras.

Bialgebras are, in fact, monoids in the monoidal category of coalgebras. Their so-called *double crossed products* can be described by distributive laws in this category. There are many examples of weak bialgebras (for a good reason called double crossed products, too) whose algebra structure is induced by a weak distributive law. (For example, one can think of the Drinfeld double, describing the superselection symmetry of a weak Hopf spin model.) However, weak bialgebras are not known to be monoids in any well-chosen monoidal category. So their double crossed products can not be described by (weak) distributive laws in such a category. The purpose of our latest joint project with Jose Gomez-Torrecillas was to find sufficient conditions on weak distributive laws between weak bialgebras, under which the resulting product algebra is a weak bialgebra too. About a large number of examples we showed that they obey our sufficient conditions.

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