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Rigid realizations of graphs with few locations in the plane

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ABSTRACT

Adiprasito and Nevo (2018) proved that there exists a set of 76 points in $\mathbb{R}^3$ such that every triangulated planar graph has an infinitesimally rigid realization in which each vertex is mapped to a point in this set.

In this paper we show that there exists a set of 26 points in the plane such that every planar graph which is generically rigid in $\mathbb{R}^2$ has an infinitesimally rigid realization in which each vertex is mapped to a point in this set.

It is known that a similar result, with a set of constant size, does not hold for the family of all generically rigid graphs in $\mathbb{R}^d$, $d \geq 2$. We show that there exists a constant c such that for every positive integer n there is a set of $c(\sqrt{n})$ points in the plane such that every generically rigid graph in $\mathbb{R}^2$ on n vertices has an infinitesimally rigid realization on this set. This bound is tight up to a constant factor.

1. Introduction

Adiprasito and Nevo [1] asked the following question: "How generic does the realization of a generically rigid graph need to be to guarantee that it is infinitesimally rigid?" In fact, Adiprasito and Nevo considered a more exact question. Which graph classes have infinitesimally rigid realizations for each of its members on a given subset of $\mathbb{R}^d$ of constant cardinality? They showed that triangulated planar graphs have such realizations on 76 points in $\mathbb{R}^3$. They also gave similar results

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for 3-dimensional realizations of triangulations of closed surfaces. The problem whether a similar statement is true for planar rigid graphs in $\mathbb{R}^2$ was left open in [1].

The first result of this paper is that there exists a set A of 26 points in the plane such that every planar graph which is generically rigid in $\mathbb{R}^2$ has an infinitesimally rigid realization on A. Furthermore, a similar result follows when we change the class of rigid planar graphs to the class of rigid graphs whose members can be embedded in a given closed surface. This implies our second result that states that for every positive integer n there exists a set A_n of $O(\sqrt{n})$ points of the plane such that every graph on n vertices which is rigid in the plane has an infinitesimally rigid realization on A_n. We note that the above question of Adiprasito and Nevo was also considered before by Fekete and Jordán [2] who proved that instead of using generic points one can always find an infinitesimally rigid *injective* realization on a grid of size $(\sqrt{n}+O(1))\times(\sqrt{n}+O(1))$.

Before introducing the above problems formally, we summarize some basics of rigidity theory. We refer to [4] for more details. A ***d*-dimensional framework** is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We will also refer to (G, p) (or less precisely to p) as a **realization** of G and to $p(v)$ as the **location** of v for a vertex $v \in V$.

We assign to (G, p) a matrix, called the **rigidity matrix** $R(G, p) \in \mathbb{R}^{|E|\times d|V|}$, which is defined as follows. We assign a row of $R(G, p)$ to each edge $uv \in E$ and d columns to each $v \in V$. The row of $R(G, p)$ assigned to $uv \in E$ contains the $d+d$ coordinates of $p(u)-p(v)$ and $p(v)-p(u)$ in the d columns assigned to u and in the d columns assigned to v, respectively, while the other entries are zeros.

An **infinitesimal motion** of a framework (G, p) is an assignment $m : V \to \mathbb{R}^d$ of infinitesimal velocities to the vertices, such that

$$\langle p(u)-p(v), m(u)-m(v)\rangle = 0 \text{ for all edges } uv \in E, \tag{1}$$

that is, $R(G, p)\,m = 0$. An infinitesimal motion m is **trivial** if $m(v) = S\,p(v) + t$ holds for all $v \in V$, for a $d \times d$ skew-symmetric matrix S and a vector $t \in \mathbb{R}^d$, that is, if m is in the kernel of $R(K_V, p)$ where K_V is the complete graph on V. (G, p) is **infinitesimally rigid** in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. We also note that the dimension of the vector space of the trivial infinitesimal motions of a d-dimensional framework is $\binom{d+1}{2}$ when the underlying graph has at least d vertices. Thus, assuming that $|V| \geq d$, (G, p) is infinitesimally rigid if and only if $\operatorname{rank}(R(G, p)) = d|V| - \binom{d+1}{2}$.

A set of points $A \subseteq \mathbb{R}^d$ is said to be **generic** if the (multi)set of the coordinates of the points in A is algebraically independent over $\mathbb{Q}$. A realization p of G is said to be **generic** if p is injective and its image is a generic set. It follows by the definition of a generic realization that if the determinant of a square submatrix of $R(G, p_0)$ is 0 for a generic realization p_0, then the determinant of the same submatrix of $R(G, p)$ is also 0 for every other realization p. Thus $\operatorname{rank}(R(G, p_0)) = \max\{\operatorname{rank}(R(G, p)) : p : V \to \mathbb{R}^d\}$. Therefore, the infinitesimal rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the infinitesimal rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic (see [10]). We say that the graph G is **rigid** in $\mathbb{R}^d$ if all (or equivalently, if some) generic realizations of G in $\mathbb{R}^d$ are infinitesimally rigid. $G = (V, E)$ is said to be **minimally rigid** in $\mathbb{R}^d$ if G is rigid but $G - e$ is not rigid in $\mathbb{R}^d$ for each $e \in E$. It is easy to see that if $G = (V, E)$ is minimally rigid and p is an infinitesimally rigid realization of G, then the rows of $R(G, p)$ are linearly independent. Let $\boldsymbol{E(X)}$ denote the set of edges in a graph $G = (V, E)$ induced by a set $X \subseteq V$, let $\boldsymbol{i_G(X)}$:= $|E(X)|$, and let $\boldsymbol{d_G(v)}$ or $\boldsymbol{d_E(v)}$ denote the degree of a vertex $v \in V$. We have the following necessary conditions for minimal rigidity.

Theorem 1.1 ([10]). *Let $G = (V, E)$ be minimally rigid in $\mathbb{R}^d$ with $|V| \geq d$. Then*
(i) $|E| = d|V| - \binom{d+1}{2}$,
(ii) $i_G(X) \leq d|X| - \binom{d+1}{2}$ for every $X \subseteq V$ with $|X| \geq d$. □

Pollaczek-Geiringer [7] (and Laman [5]) showed that these necessary conditions are also sufficient for minimal rigidity when $d = 2$.

Theorem 1.2 ([5,7]). *A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if*
(L1) $|E| = 2|V| - 3$,
(L2) $i_G(X) \leq 2|X| - 3$ for every $X \subseteq V$ with $|X| \geq 2$. □

A graph $G = (V, E)$ for which (L2) holds is called **sparse**. A graph for which both (L1) and (L2) hold is called **tight** or **Laman**.

Formally, the problem posed by Adiprasito and Nevo [1] is the following.

Problem 1.3 (*[1]*)**.** Let $\mathcal{G}$ be a graph class and $c \in \mathbb{Z}_+$. $\mathcal{G}$ is called **rigid in** $\mathbb{R}^d$ **with** c **locations** if there exists a set $A \subset \mathbb{R}^d$ with $|A| = c$ such that, for each $G = (V, E) \in \mathcal{G}$, there exists an infinitesimally rigid realization $p : V \to A$ of G. Which graph classes are rigid in $\mathbb{R}^d$ with c locations for a constant c?

The main result of Adiprasito and Nevo [1] is the following.

Theorem 1.4 (*[1]*)**.** *Let $A \subset \mathbb{R}^3$ be a generic set with $|A| = 76$. Then, for every triangulated planar graph $G = (V, E)$, there exists an infinitesimally rigid realization $p : V \to A$ of G.* □

We note that Fekete and Jordán [2] observed that the class of graphs which are rigid on the line (that is, the class of connected graphs) is rigid on the line with 2 locations. A similar result with a constant c does not hold in $\mathbb{R}^2$ by the following result.

Proposition 1.5 (*[2]*)**.** *For every positive integer c there exists a graph G on $c + 1 + \binom{c+1}{2}$ vertices which is rigid in $\mathbb{R}^2$ and has no infinitesimally rigid realization in $\mathbb{R}^2$ with c locations.* □

Based on Theorem 1.4 and Proposition 1.5, Walter Whiteley asked the authors of [1] whether they can prove a result similar to Theorem 1.4 for planar Laman graphs. This problem was left open in [1]. Our main result gives an affirmative answer to this problem.

Theorem 1.6. *Let $A \subset \mathbb{R}^2$ be a generic set with $|A| = 26$. Then, for every planar graph $G = (V, E)$ which is rigid in $\mathbb{R}^2$, there exists an infinitesimally rigid realization $p : V \to A$ of G.*

We prove Theorem 1.6 in Section 3. In Section 4 we obtain the following result by using some observations on graph embeddings in closed surfaces.

Theorem 1.7. *There exists a constant $c > 0$ such that, for every graph $G = (V, E)$ which is rigid in $\mathbb{R}^2$ and every set A of generic points in $\mathbb{R}^2$ with $|A| = c\sqrt{|V|}$, there exists an infinitesimally rigid realization $p : V \to A$ of G.*

Note that Proposition 1.5 shows that the above bound on the cardinality of A is sharp up to a constant factor.

Finally, in Section 5, we show the following theorem by using another idea of Fekete and Jordán [2].

Theorem 1.8. *Let $A \subseteq \mathbb{R}^d$ and let $G = (V, E)$ be a graph. Assume that there exists a map $p : V \to A$ such that (G, p) is infinitesimally rigid. Then there exists a set of integral points $B_G \subseteq \{1, \ldots, |V|\}^d$ with $|B_G| \leq |A|$ and a map $p' : V \to B_G$ such that (G, p') is infinitesimally rigid.* □

This result implies that some slightly weaker statements remain true if we change 'generic' in Theorems 1.4, 1.6, or 1.7 to 'integral'. However, note that, in Theorem 1.8, the image set B_G of p' depends on the graph G.

2. Preliminaries

In this section, we list the main lemmas which we use to prove Theorem 1.6. In what follows, we will say that a set $X \subseteq V$ is **tight** in a sparse graph $G = (V, E)$ if the subgraph $G[X]$ induced by X is tight. The following two lemmas follow from the supermodularity of the function i_G, see [4].

Lemma 2.1. *Let $G = (V, E)$ be a sparse graph and let $X, Y \subset V$ be two tight sets in G with $|X \cap Y| \geq 1$. Then $i_G(X \cup Y) \geq 2|X \cup Y| - 4$.* □

Lemma 2.2. *Let $G = (V, E)$ be a sparse graph and let $X, Y, Z \subset V$ be three tight sets in G such that $X \cap Y - Z \neq \emptyset$, $X \cap Z - Y \neq \emptyset$, and $Y \cap Z - X \neq \emptyset$. Then $X \cup Y \cup Z$ is also tight in G.* $\square$

Our main tool in the proof of Theorem 1.6 is the following generalization of the key lemma of Adiprasito and Nevo [1] from $d = 3$ to general d. Its proof, which we include here for completeness, is also a straightforward generalization of that of [1, Proposition 4.5].

Lemma 2.3. *Assume that (G, p) is an infinitesimally rigid framework in $\mathbb{R}^d$ and v is a vertex of degree c. Let $A \subset \mathbb{R}^d$ be a given set of points with generic coordinates. Assume that $|A| \geq \binom{d+c}{d}$. Then there exists an $a \in A$ such that (G, p') is infinitesimally rigid for the map $p' : V \to \mathbb{R}^d$ defined by $p'(v) := a$ and $p'(u) := p(u)$ for $u \in V - v$.*

Proof. By deleting some edges of G for which the corresponding row of the rigidity matrix $R(G, p)$ is linearly dependent from the other rows of $R(G, p)$, we can assume that (G, p) is minimally infinitesimally rigid. Hence $|E| = d|V| - \binom{d+1}{2}$ by Theorem 1.1.

Let us consider the rigidity matrix $R(G, p_v)$ of another realization p_v of G which arises by taking $p_v(u) := p(u)$ for each $u \in V - v$ and considering $p_v(v)$ as a vector with d variable entries $(x_1, \ldots, x_d)$. (G, p_v) is not infinitesimally rigid if and only if $\mathrm{rank}(R(G, p_v)) < d|V| - \binom{d+1}{2} = |E|$, that is, the determinant of every $|E| \times |E|$ submatrix of $R(G, p_v)$ is 0. Each such determinant is a polynomial with variables $x_1, \ldots, x_d$ of degree at most c (as $d_G(v) = c$). One can look at the polynomials over $\mathbb{R}$ with d variables and maximum degree at most c as a $\binom{d+c}{d}$-dimensional vector space over $\mathbb{R}$ whose bases are the monomials with d variables and maximum degree at most c. As (G, p) is infinitesimally rigid, at least one of the polynomials corresponding to the submatrices of $R(G, p_v)$, say P, must be not identically zero.

We claim that no choice of $\binom{d+c}{d}$ points from A makes P vanish on each of these points. To see this, put the coefficients of P into a vector $u \in \mathbb{R}^{\binom{d+c}{d}}$ where the jth coordinate corresponds to the coefficient of the jth monomial with d variables and maximum degree at most c in the lexicographical order of these monomials. Next consider the $\binom{d+c}{d} \times \binom{d+c}{d}$ matrix M where the jth entry in the ith row is the value of the jth monomial in the lexicographical order (which has coefficient u_j in P) computed on the coordinates of the ith point $(a^i_1, \ldots, a^i_d)$ in A. Since A is generic and the determinant of M is a not identically zero polynomial on the coordinates of the points in A with integer coefficients, $\det(M) \neq 0$. If P vanishes on each of our $\binom{d+c}{d}$ points, then it means that $Mu = 0$ and hence, as $\det(M) \neq 0$, $u = 0$ contradicting our assumption that P is not identically 0. Therefore, we can extend $p'|_{V-v} \equiv p|_{V-v}$ with $p'(v) \in A$ such that (G, p') is infinitesimally rigid. $\square$

We note that Lemma 2.3 immediately implies the following.

Theorem 2.4. *Let $G = (V, E)$ be a generically rigid graph in $\mathbb{R}^d$ with maximum degree Δ and let $A \subset \mathbb{R}^d$ be a given set of points with generic coordinates. Assume that $|A| \geq \binom{d+\Delta}{d}$. Then there exists a realization $p : V \to A$ such that (G, p) is infinitesimally rigid.* $\square$

Adiprasio and Nevo [1] used Lemma 2.3 and the fact that contraction of an edge uv maintains rigidity in $\mathbb{R}^3$ when u and v have two common neighbors (see [9]) to prove Theorem 1.4 by induction on $|V|$. Beside other ideas, to use Lemma 2.3, they first showed that the above contraction can be performed in triangulated planar graphs in such a way that one endvertex of the contracted edge has low degree and hence, when the reverse operation of a contraction is performed, the arising new vertex will have low degree.

We note that it is enough to prove Theorem 1.6 for planar Laman graphs. Such graphs always have at least two triangle faces and it is known that we always can contract an edge incident to a triangle face by maintaining rigidity and *vice versa* (see [3]). However, by repeatedly using the operation in Fig. 1 on the 4-faces neighboring the four vertices of the two central triangles, one can see that the degree of each vertex, which is incident with a triangle face, can be arbitrary large. This implies that, when we use the reverse operation to build up our graph we cannot guarantee any upper bound for the degree of the new vertex and hence we cannot use Lemma 2.3 for the induction.

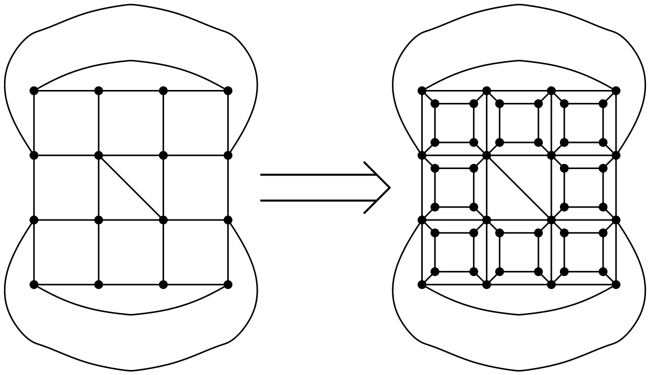

Fig. 1. Increasing the degree of vertices on triangle faces in planar Laman graphs.

Hence, for our proof, we shall use some other operations that preserve the rigidity of frameworks. The **Henneberg-0 extension**, or simply **0-extension**, on G adds a new vertex and connects it to 2 distinct vertices of G. The **1-extension**, deletes an edge $uw \in E$, adds a new vertex v and connects it to u, w and one other vertex of G. The following two lemmas show that 0- and 1-extensions preserve rigidity.

Lemma 2.5 ([10]). *Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^2$ with $p(v_1) \neq p(v_2)$. Let G^+ be a 0-extension of G that arises by adding a new vertex v with two incident edges vv_1 and vv_2 and let us take $p(v) \in \mathbb{R}^2$ such that it is not on the line through $p(v_1)$ and $p(v_2)$. Then (G^+, p) is also infinitesimally rigid in $\mathbb{R}^2$.* $\square$

Lemma 2.6 ([10]). *Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^2$ where the set $\{p(v_1), p(v_2), p(v_3)\}$ affinely spans the plane and v_1v_2 forms an edge. Let G^+ be a 1-extension of G that arises by deleting v_1v_2 and adding a new vertex v with three incident edges vv_1, vv_2 and vv_3 and let us take $p(v) \in \mathbb{R}^2 - \{p(v_1), p(v_2)\}$ such that it is on the line through $p(v_1)$ and $p(v_2)$. Then (G^+, p) is also infinitesimally rigid in $\mathbb{R}^2$.* $\square$

The following well-known result was a key in the proof of Theorem 1.2 in [5,7].

Lemma 2.7 ([5,7]). *A graph is Laman if and only if it arises from K_2 by using 0- and 1-extensions.* $\square$

The inverse operation of 1-extension is called a **1-reduction**. The following lemma is also well-known.

Lemma 2.8 ([8]). *Let G be a Laman graph and v be a vertex of G with exactly 3 neighbors v_1, v_2 and v_3. Then there exists some $1 \leq i < j \leq 3$ such that the 1-reduction of G, $G - v + v_iv_j$ is Laman.* $\square$

Note that, to use Lemmas 2.5 and 2.6 in our inductive proof, some pairs of vertices must have different location in the realization of the reduced framework. To ensure this property, we introduce a set F of extra edges which denotes the pair of vertices which must have different locations. Since there are infinitely many Laman graphs with constant number of vertices which have degree at most three, we cannot guarantee that (after a sequence of reductions) our graph has a vertex of degree at most three with low "F-degree", that is, low degree when restricted to edges in F. Thus, although all Laman graphs can be constructed by using only 0- and 1-extensions, we will also need to use the following operation which is called an **X-replacement**. Let $G = (V, E)$ and $v_1v_2, v_3v_4 \in E$ be two vertex-disjoint edges. The X-replacement deletes v_1v_2, v_3v_4, adds a new vertex v and connects it to v_1, v_2, v_3 and v_4. The following lemma shows that X-replacement preserves rigidity.

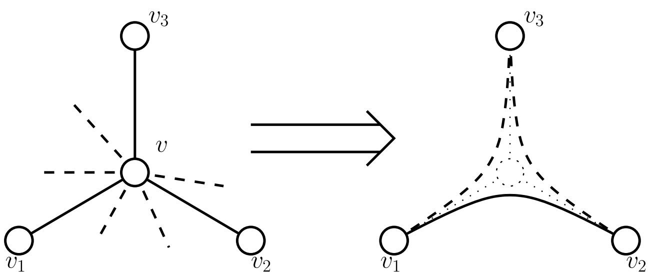

Fig. 2. A planar 1-reduction at v used in Case 2. Dashed edges are in F. The new edges are drawn "close" to the deleted ones hence they do not cross other edges.

Lemma 2.9 ([8]). *Let (G, p) be an infinitesimally rigid framework in $\mathbb{R}^2$ and let v_1v_2 and v_3v_4 be two edges of G such that any three element of the set $\{p(v_1), p(v_2), p(v_3), p(v_4)\}$ affinely span the plane, and the two lines through $p(v_1)$ and $p(v_2)$, and through $p(v_3)$ and $p(v_4)$ are intersecting in a point $y \in \mathbb{R}^2$. Let G^+ be an X-replacement of G that arises by deleting v_1v_2 and v_3v_4 and adding a new vertex v with four incident edges $vv_1,\ vv_2,\ vv_3$ and vv_4 and let $p(v) := y$. Then (G^+, p) is also infinitesimally rigid in $\mathbb{R}^2$.* □

Tay and Whiteley [8] showed that a degree 4 vertex can always be removed from a Laman graph along with adding two, possibly not independent, new edges between its neighbors such that the resulting graph is Laman. The following lemma shows when we can get a Laman graph after an inverse X-replacement.

Lemma 2.10. *Let $G = (V, E)$ be a Laman graph and v be a vertex in G with exactly four neighbors $v_1,\ v_2,\ v_3$ and v_4. Then $G' = G - v + v_1v_2 + v_3v_4$ is Laman if and only if there is no tight set $X \subseteq V - v$ in G with $v_1, v_2 \in X$ or $v_3, v_4 \in X$.*

Proof. Since the necessity of the condition is obvious, we only prove its sufficiency. Observe that G' has $2|V| - 3 - 4 + 2 = 2|V - v| - 3$ edges, hence we only need to prove its sparsity. Assume for a contradiction that there is a set $X \subseteq V - v$ such that $i_{G'}(X) > 2|X| - 3$. If $\{v_1, v_2, v_3, v_4\} \subseteq X$, then $i_G(X \cup \{v\}) > 2|X \cup \{v\}| - 3$, a contradiction. If $\{v_1, v_2\} \not\subseteq X$ and $\{v_3, v_4\} \not\subseteq X$ both hold, then $i_G(X) = i_{G'}(X) > 2|X| - 3$, a contradiction. Hence, by relabeling the neighbors of v, we can assume that $\{v_1, v_2\} \subseteq X$ and $v_4 \notin X$. Thus $2|X| - 2 \le i_{G'}(X) = i_G(X) + 1 \le 2|X| - 2$. Therefore, equality holds in the last inequality, implying that X is tight in G, contradicting the assumption. □

As we have seen before the introduction of X-replacements, the problem with using only 0- and 1-extension in our proof is that it is possible that there are just a constant number of vertices of degree at most three in a Laman graph. The following lemma shows that the number of vertices with degree at most four is much higher.

Lemma 2.11. *Let $G = (V, E)$ be a Laman graph on $n \ge 6$ vertices. Then it has at least $n/3 + 2$ vertices of degree at most 4.*

Proof. Let n_i ($n_{\le i},\ n_{\ge i}$, respectively) denote the number of vertices in G with degree i (at most i, at least i, respectively). Then

$$2n_{\le 4} + 5n_{\ge 5} \le \sum_{i=2}^{n-1} in_i = 2|E| = 4n - 6 = 4n_{\le 4} + 4n_{\ge 5} - 6.$$

Hence, $n = n_{\le 4} + n_{\ge 5} \le 3n_{\le 4} - 6$. Therefore, $n/3 + 2 \le n_{\le 4}$. □

It is easy to see that we can maintain the planarity of our graph extended with the extra edges in F while we delete degree-two vertices and perform 1-reductions (see Fig. 2 for an example of

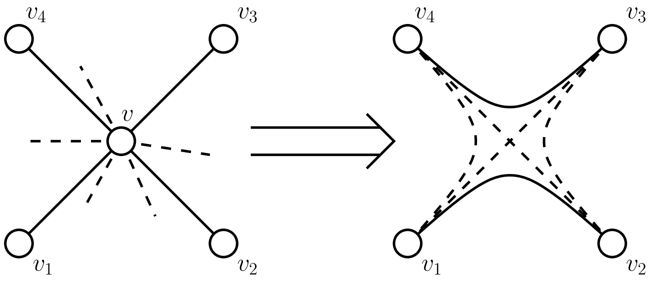

Fig. 3. A planar inverse X-replacement at v used in Subcase 3.1. Dashed edges are in F. The new edges are drawn "close" to the deleted ones hence they do not cross other edges except v_1v_3 and v_2v_4 which cross each other.

a 1-reduction). However, if we need to perform the inverse of an X-replacement we may need to add crossing edges to F to ensure condition of Lemma 2.9 that any three element of the set $\{p(v_1), p(v_2), p(v_3), p(v_4)\}$ affinely span the plane (see Fig. 3 for an example). To guarantee the low number of the edges in F, we need the following definition.

We say that a graph $G' = (V, E \cup F)$ is F-**crossing** if $E \cap F = \emptyset$ and G' can be drawn with continuous curves in the plane such that only edges in F can cross each other and each edge in F can cross at most one other edge in F. It is easy to observe the following property of F-crossing graphs.

Proposition 2.12. *If $G' = (V, E \cup F)$ is F-crossing, then there exists a partition of F into two sets F_1 and F_2 such that both of $G'_1 = (V, E \cup F_1)$ and $G'_2 = (V, E \cup F_2)$ are planar.* □

To guarantee the existence of a vertex with maximum degree four and with low F-degree we will need the following property.

Lemma 2.13. *Let $G = (V, E)$ be a Laman graph on n vertices and let $G' = (V, E \cup F)$ be F-crossing and simple. Then there exists at most $n/3 - 1$ vertices $v \in V$ such that $d_F(v) \geq 12$.*

Proof. Since G is Laman, $|E| = 2n-3$. By Proposition 2.12, there exists a partition of F into two sets F_1 and F_2 such that both of $G'_1 = (V, E \cup F_1)$ and $G'_2 = (V, E \cup F_2)$ are planar. As G'_i is simple planar, we get $|E \cup F_i| \leq 3n - 6$ for $i = 1, 2$. Hence $|F_1| \leq n - 3$ and $|F_2| \leq n - 3$ and thus $|F| \leq 2n - 6$.

Let $n'_{\geq 12}$ denote the number of vertices $v \in V$ for which $d_F(v) \geq 12$. Now, $12n'_{\geq 12} \leq 2|F| \leq 4n - 12$. Hence $n'_{\geq 12} \leq n/3 - 1$. □

3. Rigid planar graphs with few locations

In this section we prove Theorem 1.6. As we observed in Section 2, it is enough to prove Theorem 1.6 for planar Laman graphs. In fact, we will prove a slightly stronger result, as follows.

Theorem 3.1. *Let $G = (V, E)$ be a Laman graph and let us assume that $G' = (V, E \cup F)$ is an F-crossing graph. Let A be a set of generic points in the plane with $|A| = 26$. Then there exists a map $p : V \to A$ such that the framework (G, p) is infinitesimally rigid in the plane and $p(u) \neq p(v)$ holds for every edge $uv \in E \cup F$.*

Proof. The proof is by induction on $|V|$. By Theorem 1.2 the statement is true when $|V| \leq 26$.

Note that a Laman graph is always simple. Moreover, without loss of generality, we may assume that G' is simple since deleting each edge of F which is parallel to an edge in E does not change our statement. By Lemmas 2.11 and 2.13, there exists a vertex $v \in V$ with $d_G(v) \leq 4$ and $d_F(v) \leq 11$.

Case 1: $d_G(v) = 2$. Let us denote the neighbors of v in G by v_1 and v_2. $G - v$ is a planar Laman graph by Lemma 2.7. Furthermore, $G'' = (V - v, E(V - v) \cup F')$ is F'-crossing for $F' = F(V - v) \cup \{v_1v_2\}$

since G' is F-crossing and G'' arises from G' by deleting v and adding the F'-edge v_1v_2 which can be drawn by joining the curves corresponding to the edges v_1v and vv_2. By induction, there exists an infinitesimally rigid realization p of $G-v$ in A such that the two endvertices of each edge in $E(V-v)\cup F(V-v)$ have different locations and $p(v_1)\neq p(v_2)$. By Lemma 2.5, choosing a location for v which is not on the line through $p(v_1)$ and $p(v_2)$ results in an infinitesimally rigid realization of G. Since A is generic, no member of $A-\{p(v_1), p(v_2)\}$ is on this line. Hence we can find an infinitesimally rigid realization of G on A such that the two endvertices of each edge in $E\cup F$ have different locations by choosing $p(v)$ out of the locations of the (at most 13) neighbors of v in G' by $|A|=26\geq 14$.

Case 2: $d_G(v) = 3$. Let us denote the neighbors of v in G by $v_1,\ v_2$ and v_3. By Lemma 2.8, we can perform a 1-reduction on v resulting in a Laman graph. By relabeling the neighbors of v, we can assume that $G-v+v_1v_2$ is Laman. It is easy to observe that $G-v+v_1v_2$ is also planar and $G''=(V-v, E(V-v)\cup\{v_1v_2\}\cup F')$ is F'-crossing for $F'=F(V-v)\cup\{v_1v_3, v_2v_3\}$ (see Fig. 2). By induction, there exists an infinitesimally rigid realization p of $G-v+v_1v_2$ on A such that the two endvertices of each edge in $E(V-v)\cup F(V-v)$ have different locations and $|\{p(v_1), p(v_2), p(v_3)\}| = 3$. Since A is generic, the latter statement implies that $p(v_1), p(v_2), p(v_3)$ affinely span the plane. Lemma 2.6 implies that we can define $p(v)$ in such a way that (G, p) is infinitesimally rigid. However, at this point we cannot guarantee that $p(v)\in A$, although, we have $p(u)\in A$ for every $u\in V-v$. Now, by Lemma 2.3, we can define a map $p' : V\to A$ such that $p'(u)=p(u)$ for $u\in V-v$, $p'(v)\in A$, (G, p') is infinitesimally rigid, and $p'(v)$ is not equal to the location of any of its F-neighbors since $|A| = 26\geq\binom{5}{2}+11$. Note that $p'(v)$ is not equal to the location of any of the neighbors of v in G' since otherwise one of the $2|V|-6$ rows of the rigidity would be 0, contradicting the infinitesimal rigidity of (G, p').

Case 3: $d_G(v) = 4$. Let us denote the neighbors of v in G by $v_1,\ v_2,\ v_3$ and v_4, such that this is the order of the outgoing edges in E from v in a fixed F-crossing drawing of G'. We have the following two subcases:

Subcase 3.1: $G-v+v_1v_2+v_3v_4$ or $G-v+v_1v_4+v_2v_3$ is Laman. By relabeling the neighbors of v we can assume that $G-v+v_1v_2+v_3v_4$ is Laman. It is easy to see that $G-v+v_1v_2+v_3v_4$ is planar and $G''=(V-v, E(V-v)\cup\{v_1v_2, v_3v_4\}\cup F')$ is F'-crossing for $F'=F(V-v)\cup\{v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$ (see Fig. 3). By induction, there exists an infinitesimally rigid realization p of $G-v+v_1v_2+v_3v_4$ on A such that the two endvertices of each edge in $E(V-v)\cup F(V-v)$ have different locations and $|\{p(v_1), p(v_2), p(v_3), p(v_4)\}| = 4$. Since A is generic, we can use Lemma 2.9 to prove that there exists a placement of $p(v)$ in $\mathbb{R}^2$ such that (G, p) is infinitesimally rigid. However, we need to take it from the set A. By Lemma 2.3, we can define a map $p' : V\to A$ such that $p'(u) = p(u)$ for $u\in V-v$, $p'(v)\in A$, (G, p') is infinitesimally rigid, and $p'(v)$ is not equal to the location of any of its F-neighbors since $|A| = 26\geq\binom{6}{2}+11$. Note that, as in Case 2, $p'(v)$ is not equal to the location of any of its neighbors in G'.

Subcase 3.2: If neither $G-v+v_1v_2+v_3v_4$ nor $G-v+v_1v_4+v_2v_3$ is Laman, then, by Lemma 2.10, there exists an $i\in\{1, 2, 3, 4\}$ such that there are tight sets $X, Y\subset V-v$ in G with $v_i, v_{i+1}\in X$ and $v_i, v_{i-1}\in Y$ where $v_0 := v_4$ and $v_5 := v_1$. By relabeling the vertices cyclically we can assume that $i=1$. Note that $v_3, v_4\notin X$ and $v_2, v_3\notin Y$ since otherwise $X\cup\{v\}$ (or $Y\cup\{v\}$, respectively) induces at least $2|X|-3+3 > 2|X\cup\{v\}|-3$ (or $2|Y|-3+3 > 2|Y\cup\{v\}|-3$, respectively) edges in G, contradicting the sparsity condition (L2). We will use the following two observations.

Claim 3.2. *There exists no tight set $Z\subset V-v$ in G with $v_2, v_4\in Z$.*

Proof. For the sake of contradiction, suppose that $Z\subset V-v$ is a tight set in G with $v_2, v_4\in Z$. Note that $v_1, v_3\notin Z$ since otherwise $Z\cup\{v\}$ induces at least $2|Z|-3+3 > 2|Z\cup\{v\}|-3$ edges in G, contradicting the sparsity condition (L2). Hence $v_1\in X\cap Y-Z,\ v_2\in X\cap Z-Y$, and $v_4\in Y\cap Z-Y$. Thus $X\cup Y\cup Z$ is tight in $G-v$ by Lemma 2.2. Since three of neighbors of v are in $X\cup Y\cup Z$, the tightness of $X\cup Y\cup Z$ implies $i_G(X\cup Y\cup Z\cup\{v\}) > 2|X\cup Y\cup Z\cup\{v\}|-3$, contradicting the sparsity condition. □

Claim 3.3. *There exists no set $Z'\subset V-v$ with $v_2, v_3, v_4\in Z'$ and $i_G(Z')\geq 2|Z'|-4$.*

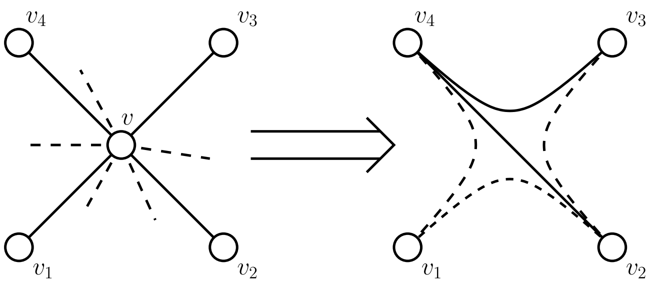

Fig. 4. The reduction at v used in Subcase 3.2. Dashed edges are in F. The new edges are drawn "close" to the deleted ones hence they do not cross other edges.

Proof. For the sake of contradiction, suppose that $Z' \subset V - v$ is a set with $v_2, v_3, v_4 \in Z'$ and $i_G(Z') \geq 2|Z'| - 4$. Then, in $G - vv_1$, v has exactly three neighbors in Z' and hence $Z' \cup \{v\}$ is tight in $G - vv_1$. Note that $v_1 \notin Z$ since otherwise $Z' \cup \{v\}$ induces at least $2|Z| - 4 + 4 > 2|Z \cup \{v\}| - 3$ edges in G, contradicting the sparsity condition (L2). Hence $v_1 \in X \cap Y - Z$, $v_2 \in X \cap Z - Y$, and $v_4 \in Y \cap Z - Y$. Thus $X \cup Y \cup (Z' \cup \{v\})$ is tight in $G - vv_1$ by Lemma 2.2. Since vv_1 is induced by $X \cup Y \cup Z' \cup \{v\}$ in G, this implies $i_G(X \cup Y \cup Z' \cup \{v\}) > 2|X \cup Y \cup Z' \cup \{v\}| - 3$, contradicting the sparsity condition. $\square$

Now, it is impossible to have two tight sets $Z_1, Z_2 \subset V - v$ with $v_2, v_3 \in Z_1$ and $v_3, v_4 \in Z_2$, since otherwise $i_G(Z_1 \cup Z_2) \geq 2|Z_1 \cup Z_2| - 4$ (by Lemma 2.1) and $v_2, v_3, v_4 \in Z_1 \cup Z_2$, contradicting Claim 3.3. By swapping v_2 and v_4, we can assume that there is no tight set $Z_2 \subset V - v$ with $v_3, v_4 \in Z_2$. This fact together with Claims 3.2 and 3.3 imply that $G - v \cup \{v_2v_4, v_3v_4\}$ is Laman. Furthermore, $G - v \cup \{v_2v_4, v_3v_4\}$ is planar and $G'' = (V - v, E(V - v) \cup \{v_2v_4, v_3v_4\} \cup F')$ is F'-crossing for $F' = F(V - v) \cup \{v_1v_2, v_1v_4, v_2v_3\}$ (see Fig. 4). By induction, there exists an infinitesimally rigid realization p of $G - v + v_2v_4 + v_3v_4$ on A such that the two endvertices of each edge in $E(V - v) \cup F(V - v)$ have different locations and either $|\{p(v_1), p(v_2), p(v_3), p(v_4)\}| = 4$, or $= 3$ and $p(v_1) = p(v_3)$.

Next we add v to $G - v + v_2v_4 + v_3v_4$ by a 1-extension on v_2v_4 along with the edges vv_1, vv_2 and vv_4. By using Lemma 2.3 as in the proof of Case 2, we can see that from any 10 points in A we can find at least one, say a, for which the extension p^a of p with $p^a(v) := a$ is an infinitesimally rigid realization of $G - vv_3 + v_3v_4$. Furthermore, this also implies that from any 11 points in A we can find at least two, say a and b, for which the extensions p^a and p^b of p with $p^a(v) := a$ and $p^b(v) := b$ are both infinitesimally rigid realizations of $G - vv_3 + v_3v_4$. As $|A| = 26 \geq 11 + 11 + 4$, we can choose such a and b in such a way that $p(u) \neq a$ and $p(u) \neq b$ both hold for every $u \in V$ for which $uv \in E \cup F$. We shall show that (G, p^a) or (G, p^b) is infinitesimally rigid.

Note that, in an infinitesimally rigid realization of a Laman graph G^* on vertex set V, any tight set in G^* induces an infinitesimally rigid subframework (since otherwise the corresponding rows of the rigidity matrix are not linearly independent and hence the rigidity matrix has at most $2|V| - 4$ linearly independent rows contradicting to the infinitesimal rigidity of G^*). For the tight sets X and Y of G defined above, observe that $X \cup Y$ induces at least $2|X \cup Y| - 4$ edges in $G - vv_3 + v_3v_4$ by Lemma 2.1 and hence $X \cup Y \cup \{v\}$ is tight in $G - vv_3 + v_3v_4$ (as v has three neighbors in $X \cup Y$: v_1, v_2, and v_4). Thus both of $((G - vv_3 + v_3v_4)[X \cup Y \cup \{v\}], p^a)$ and $((G - vv_3 + v_3v_4)[X \cup Y \cup \{v\}], p^b)$ are infinitesimally rigid, since $G - vv_3 + v_3v_4$ is Laman and the set $X \cup Y \cup \{v\}$ is tight in $G - vv_3 + v_3v_4$. Note that $v_3 \notin X \cup Y$, hence $(G - vv_3 + v_3v_4)[X \cup Y \cup \{v\}] = (G - vv_3)[X \cup Y \cup \{v\}]$. Thus $((G - vv_3)[X \cup Y \cup \{v\}], p^a)$ and $((G - vv_3)[X \cup Y \cup \{v\}], p^b)$ are also infinitesimally rigid.

Observe that $G - vv_3$ has only $2|V| - 4$ edges and hence neither $(G - vv_3, p^a)$ nor $(G - vv_3, p^b)$ is infinitesimally rigid. However, the infinitesimal rigidity of $(G - vv_3 + v_3v_4, p^a)$ (and of $(G - vv_3 + v_3v_4, p^b)$, respectively) implies that the dimension of the space of the infinitesimal motions of $(G - vv_3, p^a)$ (and of $(G - vv_3, p^b)$, respectively) is four. Since $((G - vv_3)[X \cup Y \cup \{v\}], p^a)$ is infinitesimally rigid, we can add a trivial infinitesimal motion to any non-trivial infinitesimal motion of $(G - vv_3, p^a)$ in such a way that we get a non-trivial infinitesimal motion m_0 of $(G - vv_3, p^a)$ for which $m_0(u) = 0$ holds for each $u \in X \cup Y \cup \{v\}$. Observe that m_0 is also such a non-trivial infinitesimal motion of $(G - vv_3, p^b)$, since p^a and p^b only differ on the location of v and the value of m_0 on v and on all its neighbors is 0. The previous dimension constraint implies that such infinitesimal motion m_0 of $(G - vv_3, p^a)$ (or $(G - vv_3, p^b)$) is unique up to a constant multiplier. Note also that $m_0(v_3) \neq 0$ since otherwise m_0 is also an infinitesimal motion of the infinitesimally rigid framework $(G - vv_3 + v_3v_4, p^a)$ which contradicts its non-triviality.

Assume now that each of (G, p^a) and (G, p^b) has a non-trivial infinitesimal motion, say, m_a and m_b. Like for m_0, we may assume without loss of generality that $m_a(u) = m_b(u) = 0$ holds for each $u \in X \cup Y \cup \{v\}$. Since m_a and m_b are also infinitesimal motions of $(G - vv_3, p^a)$ and $(G - vv_3, p^b)$, respectively, $m_a = c_a m_0$ and $m_b = c_b m_0$ must hold for some constants $c_a, c_b \neq 0$. Furthermore, (1) implies that $0 = \langle p^a(v_3) - p^a(v), m_a(v_3) - m_a(v)\rangle = \langle p(v_3) - a, c_a m_0(v_3)\rangle = c_a\langle p(v_3) - a, m_0(v_3)\rangle$, and $\langle p(v_3) - b, m_0(v_3)\rangle = 0$. However, by the genericity of A and $a, b \neq p(v_3)$, $m_0(v_3) \neq 0$ cannot be orthogonal to both of $a - p(v_3)$ and $b - p(v_3)$, a contradiction. Therefore, at least one of (G, p^a) and (G, p^b) has no non-trivial infinitesimal motion, and hence it is infinitesimally rigid. This completes the proof of Theorem 3.1. □

4. Rigid graphs with few locations

In this section we show that Theorem 1.6 can be extended to the class of graphs that can be embedded in a fixed closed surface. Later we use this generalization of Theorem 1.6 to prove Theorem 1.7. We refer to the book of Mohar and Thomassen [6, Chapter 3] for an introduction to the topic of graph embeddings in surfaces.

4.1. Graphs on surfaces

Note that in the proof of Theorem 1.6 we used planarity twice:

- In Lemma 2.13, we used the edge bound (which follows from Euler's formula) for planar graphs.
- In our reduction steps, we used planarity 'locally' to show that the reduced graphs are also F'-crossing (see Figs. 2–4).

Note that Euler's formula extend for graphs which can be embedded in a given closed surface (by using the Euler characteristic of the surface), furthermore, a closed surface is locally homeomorphic to the plane. Hence we get the following result with the same proof.

Theorem 4.1. *For every closed surface $\mathcal{C}$ with Euler characteristic $\chi_{\mathcal{C}} \leq 0$, there exists a constant $k_{\mathcal{C}} = O(\sqrt{-\chi_{\mathcal{C}}})$ such that for every graph $G = (V, E)$ which has an embedding into $\mathcal{C}$ and is rigid in $\mathbb{R}^2$ and for every set A of generic points in $\mathbb{R}^2$ with $|A| \geq k_{\mathcal{C}}$, there exists an infinitesimally rigid realization $p : V \to A$ of G.*

Proof of Sketch. Since the proof is just a copy of our proof for the planar case, we only show why $k_{\mathcal{C}} = O(\sqrt{-\chi_{\mathcal{C}}})$. In our proof for the planar case, we used Euler's formula in the proof of Lemma 2.13. As in the planar case, we say that $G' = (V, E \cup F)$ is ***F*-crossing on $\mathcal{C}$** for a closed surface $\mathcal{C}$ if $E \cap F = \emptyset$ and G' can be drawn with continuous curves on $\mathcal{C}$ such that only edges in F can cross each other and each edge in F can cross at most one other edge in F. Now Lemma 2.13 can be modified, as follows.

Lemma 4.2. *Let $G = (V, E)$ be a Laman graph on n vertices, let $\mathcal{C}$ be a closed surface with Euler characteristic $\chi_{\mathcal{C}}$, and let $G' = (V, E \cup F)$ be F-crossing on $\mathcal{C}$ and simple. Then it has less than $n/3$ vertices of F-degree more than $12 - \frac{36}{n}(\chi_{\mathcal{C}} - 1)$.*

Proof. Since G is Laman, $|E| = 2n - 3$. Like in the planar case, there exists a partition of F into to sets F_1 and F_2 such that both of $G'_1 = (V, E \cup F_1)$ and $G'_2 = (V, E \cup F_2)$ can be embedded into $\mathcal{C}$. As G'_i can be embedded into $\mathcal{C}$ that has Euler characteristic $\chi_{\mathcal{C}}$, $|E \cup F_i| = n + n^*_i - \chi_{\mathcal{C}}$ for $i = 1, 2$ where n^*_i is the number of faces of G'_i embedded into $\mathcal{C}$. Since $n^*_i \leq \frac{2}{3}|E \cup F_i|$ follows by the simplicity of G_i, we get $|E \cup F_i| \leq 3n - 3\chi_{\mathcal{C}}$ for $i = 1, 2$. Hence $|F_1| \leq n - 3\chi_{\mathcal{C}} + 3$ and $|F_2| \leq n - 3\chi_{\mathcal{C}} + 3$ and thus $|F| \leq 2n - 6\chi_{\mathcal{C}} + 6$.

For a constant $c \in \mathbb{R}^+$, let $n'_{>c}$ denote the number of vertices in G' of F-degree more than c. Now, $cn'_{>c} < 2|F| \leq 4n - 12\chi_{\mathcal{C}} + 12$. To prove that $n'_{>c} < n/3$, we need $4n - 12\chi_{\mathcal{C}} + 12 \leq cn/3$ and hence $12 + \frac{36}{n}(1 - \chi_{\mathcal{C}}) \leq c$. □

As in the planar case, it is enough to prove Theorem 4.1 for Laman graphs. We prove the following slightly stronger result.

Theorem 4.3. *Let $\mathcal{C}$ be a closed surface with Euler characteristic $\chi_{\mathcal{C}} \leq 0$, let $G = (V, E)$ be a Laman graph, and let us assume that $G' = (V, E \cup F)$ is an F-crossing graph on $\mathcal{C}$. Then there exist constants $c, c' \geq 1$ for which, for each set A of generic points in the plane with $|A| = c\sqrt{-\chi_{\mathcal{C}}} + c'$, there exists a map $p : V \to A$ such that the framework (G, p) is infinitesimally rigid in the plane and $p(u) \neq p(v)$ holds for every edge $uv \in E \cup F$.*

Proof of Sketch. For any $c, c' \geq 1$, the statement is obvious when $|V| \leq c\sqrt{-\chi_{\mathcal{C}}} + c'$. Hence we can assume that $|V| > c\sqrt{-\chi_{\mathcal{C}}} + c' \geq \sqrt{-\chi_{\mathcal{C}}} + 1$. By Lemmas 2.11 and 4.2, there exists a vertex $v \in V$ with $d_G(v) \leq 4$ and $d_F(v) \leq \left\lfloor 12 + \frac{36(1-\chi_{\mathcal{C}})}{\sqrt{-\chi_{\mathcal{C}}}+1} \right\rfloor$ since we only need to use the previous formula when $n > c\sqrt{-\chi_{\mathcal{C}}} + c' \geq \sqrt{-\chi_{\mathcal{C}}} + 1$. Now, by following the proof of Theorem 3.1, we obtain that $|A| \geq \left\lfloor 27 + \frac{36(1-\chi_{\mathcal{C}})}{\sqrt{-\chi_{\mathcal{C}}}+1} \right\rfloor$ suffices. □

This completes the proof of Theorem 4.1. □

4.2. Genus of Laman graphs

Next we show the following simple bound on the genus of Laman graphs.

Lemma 4.4. *Let $G = (V, E)$ be a Laman graph. Then G can be embedded in an orientable closed surface which has genus* $\max(|V| - 5, 0)$.

Proof. It is easy to check that each Laman graph on at most 5 vertices is planar. By Lemma 2.7, each Laman graph can be constructed by 0- and 1-extensions from the complete graph on 2 vertices. If G has an embedding in an orientable closed surface $\mathcal{C}$ and G' is its 0-extension, then it is easy to see that we can add a new handle to $\mathcal{C}$ (with ends close to the location of the two neighbors of the new vertex in G') in such a way that G' is embeddable into this new surface. Similarly, when G' is a 1-extension of G, we can add a new handle to $\mathcal{C}$ (with ends close to the subdivided edge and to the third neighbor of the new vertex in G'). □

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Again it is enough to consider the case when $G = (V, E)$ is Laman and hence its genus is at most $\max(|V| - 5, 0)$ by Lemma 4.4. It is well-known that an orientable closed surface with genus g has Euler characteristic $2 - 2g$ (see [6]). Hence our statement follows from Theorem 4.1 (or Theorem 1.6 when $g = 0$). □

5. Rigid realizations on few integer points

Fekete and Jordán [2] showed that one can construct an infinitesimally rigid realization of a graph $G = (V, E)$ with integer coordinates from $\{1, \ldots, |V|\}$ by changing the coordinates one-by-one of an infinitesimally rigid realization of G, preserving infinitesimal rigidity. We prove Theorem 1.8 by showing that the coordinates of coincident vertices can be changed simultaneously.

Proof of Theorem 1.8. The statement is obvious when $|V|=1$ hence we can assume $|V|\geq 2$. Let x be a map which maps each point $a\in A$ to a d-dimensional vector with variables $(x_{a,1},\ldots,x_{a,d})$. Let us consider the matrix $R(G,x\circ p)$. Since (G,p) is rigid, $R(G,p)$ has a $\left(d|V|-\binom{d+1}{2}\right)\times\left(d|V|-\binom{d+1}{2}\right)$ non-singular submatrix $M(G,p)$. Now $M(G,x\circ p)$ is a $\left(d|V|-\binom{d+1}{2}\right)\times\left(d|V|-\binom{d+1}{2}\right)$ submatrix of $R(G,x\circ p)$ whose determinant is a polynomial $P\not\equiv 0$ as the substitution of a_i into $x_{a,i}$ gives the determinant of $M(G,p)$ which is nonzero. Note that no graph on at least two vertices has infinitesimally rigid realization with one location hence at most $|V|-1$ vertices have the same location in (G,p). Thus the variable $x_{a,i}$ is only included in at most $|V|-1$ columns of $R(G,x\circ p)$ for each $i\in\{1,\ldots,d\}$ and $a\in A$. Hence the degree of P is at most $|V|-1$ in each of its variables. Thus P vanishes on at most $|V|-1$ entries for each variable. Therefore, fixing $a_0\in A$ we can choose a value $\varphi_1(a_0)\in\{1,\ldots,|V|\}$ for $x_{a_0,1}$ such that $P|_{x_{a_0,1}=\varphi_1(a_0)}\not\equiv 0$. Next, we add values $\varphi_i(a)\in\{1,\ldots,|V|\}$ sequentially for each $a\in A$ and $i\in\{1,\ldots,d\}$ in such a way that finally we get a nonzero value for the constant polynomial $P|_{\{x_{a,i}=\varphi_i(a):a\in A,i\in\{1,\ldots,d\}\}}$. Therefore, the rigidity matrix $R(G,\varphi\circ p)$ has a $\left(d|V|-\binom{d+1}{2}\right)\times\left(d|V|-\binom{d+1}{2}\right)$ non singular submatrix $M(G,\varphi\circ p)$, that is $(G,\varphi\circ p)$ is rigid. Furthermore, $B=\varphi(A)\subseteq\{1,\ldots,|V|\}^d$ and $|B|\leq|A|$. □

The next corollary follows from Theorems 1.7 and 1.8.

Corollary 5.1. *There exists a constant $c>0$ such that, for every graph $G=(V,E)$ which is rigid in $\mathbb{R}^2$, there exists a set A of points in $\{1,\ldots,|V|\}^2$ with $|A|\leq c\sqrt{|V|}$ such that there exists an infinitesimally rigid realization $p:V\to A$ of G.* □

We obtain similar corollaries by combining Theorem 1.8 with Theorem 1.4, Theorem 1.6, Theorem 2.4, or Theorem 4.1, respectively.

Note that Corollary 5.1 states that every graph on n vertices, which is rigid in the plane, has an infinitesimally rigid realization with $O(\sqrt{n})$ integral locations with coordinates in $\{1,\ldots,n\}$. By contrast, we note that Fekete and Jordán [2] proved that such a graph has an infinitesimally rigid realization with integral locations with coordinates in $\{1,\ldots,\lceil\sqrt{n-1}\rceil+9\}$ in such a way that the locations are pairwise different.

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