FINAL REPORT FOR OTKA GRANT PROPOSAL NO. 75126: AUTOMORPHIC FORMS AND L-FUNCTIONS

GERGELY HARCOS

In joint work with Valentin Blomer (Göttingen) and Nicolas Templier (Princeton) we made progress on three problems concerning the size of automorphic forms and L-functions. The results appear in the research papers [BH10, BH11, HT11a, HT11b] and in the survey article [BH09]. A detailed account of the results is provided below.

1. TWISTED MODULAR L-FUNCTIONS OVER NUMBER FIELDS

The first result within the project is a Burgess-like subconvex bound for twisted modular L-functions over totally real number fields. It is of similar quality as our earlier result was over $\mathbb{Q}$, see [BHM07].

Theorem 1 ([BH10])**.** *Let K be a totally real number field. Let π be an irreducible cuspidal representation of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)$ with unitary central character, and let χ be a Hecke character of K of conductor $\mathfrak{q}$. Then for any $\varepsilon>0$ one has*

$$L(1/2,\pi\otimes\chi)\ll_{\pi,\chi_\infty,K,\varepsilon}(\mathcal{N}\mathfrak{q})^{\frac{3}{8}+\frac{\theta}{4}+\varepsilon}.$$

We note that this result contains a bound for all values $L(1/2+it,\pi\otimes\chi)$ on the critical line, because replacing $1/2$ by $1/2+it$ has the same effect as replacing χ by $\chi\otimes|\cdot|^{it}$. The convexity bound in this context is $(\mathcal{N}\mathfrak{q})^{\frac{1}{2}+\varepsilon}$. The first subconvex bound over totally real number fields is due to Cogdell, Piatetski-Shapiro and Sarnak [CPSS, Co03], in which they obtained the subconvexity exponent $\frac{1}{2}-\frac{1-2\theta}{14+4\theta}$ for π induced by a holomorphic Hilbert cusp form. They used an effective spectral method based on bounds for triple products [Sa94]. As an application of a new geometric method, Venkatesh [Ve10, Theorem 6.1] achieved the subconvexity exponent $\frac{1}{2}-\frac{(1-2\theta)^2}{14-12\theta}$ over any number field and for all irreducible cuspidal representations. For the proof of Theorem 1 we devised a different spectral approach based on the Kirillov model and Sobolev norms of automorphic forms, generalizing our earlier work [BH08b].

Perhaps the most appealing application of Theorem 1 is to combine it with the formula of Waldspurger [Wa81] and its extensions by Shimura [Sh93], Khuri-Makdisi [KM96], Kojima [Ko04], Baruch–Mao [BM07] and others in order to bound the Fourier coefficients of half-integral weight Hilbert modular forms. For $K=\mathbb{Q}$, the original breakthrough was achieved by Iwaniec [Iw87], and the currently strongest bounds are given in [BH08a]. For a totally real number field K other than $\mathbb{Q}$ we could not find an explicit reference in the literature.

Corollary 1 ([BH10])**.** *Let $(\tilde\pi,V_{\tilde\pi})$ be an irreducible cuspidal representation of $\widetilde{\mathrm{SL}}_2(K)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A}_K)$, orthogonal to one-dimensional theta series, and let $r\in\mathscr{O}_K$ be a nonzero square-free integer. Then the r-th normalized Fourier coefficient $\rho_{\tilde\phi}(r)$ of a pure tensor $\tilde\phi=\otimes_v\tilde\phi_v\in V_{\tilde\pi}$ satisfies*

$$\sqrt{|\mathcal{N}r|}\rho_{\tilde\phi}(r)\ll_{\tilde\phi,K,\varepsilon}|\mathcal{N}r|^{\frac{1}{4}-\frac{1}{16}(1-2\theta)+\varepsilon}.$$

One particular situation where such bounds are needed, are asymptotic formulae for the number of representations of totally positive integers by ternary quadratic forms. Hilbert's eleventh problem asks more generally which integers are integrally represented by a given n-ary quadratic form Q over a number field K. If Q is a binary form, it corresponds to some element in the class group of a quadratic extension of K. If Q is indefinite at some archimedean place, Siegel [Si52] for $n\geqslant 4$ and Kneser [Kn61] and Hsia [Hs76] for $n = 3$ proved a local-to-global principle, so Siegel's mass formula [Si37] tells us exactly which integers are represented by Q. If Q is positive definite at every archimedean place and $n \geqslant 4$, again Siegel's mass formula [Si37] and bounds for Fourier coefficients of Hilbert modular forms give a complete answer (some care has to be taken in the case $n = 4$). The only remaining case of Q positive definite and $n = 3$ was solved by Duke and Schulze-Pillot [DSP90] for $K = \mathbb{Q}$. For arbitrary totally real K, the result was established by Cogdell, Piatetski-Shapiro and Sarnak [CPSS]; an account of the key ideas appeared in [Co03]. In fact, the systematic study of subconvexity over number fields was initiated by [CPSS] about a decade ago motivated by this striking application. The relevant subconvex bound was subsequently generalized over arbitrary number fields by Venkatesh [Ve10], while our Corollary 1 allows a better approximation for the number of representations.

Corollary 2 ([BH10])**.** *Let K be a totally real number field, and let Q be a positive integral ternary quadratic form over K. Then there is an ineffective constant $c > 0$ such that every totally positive square-free integer $r \in \mathscr{O}_K$ with $\mathscr{N}r \geqslant c$ is represented integrally by Q if and only if it is integrally represented over every completion of K. More precisely, the number of representations for such r equals $(\mathscr{N}r)^{\frac{1}{2}+o(1)} + O((\mathscr{N}r)^{\frac{7}{16}+\frac{\theta}{8}+o(1)})$, where the main term is the product of local densities given by Siegel's mass formula.*

2. Second moment of Rankin–Selberg L-functions

The second result within the project is concerned with the asymptotic behavior of the hybrid second moment

$$\mathscr{I}(T,K) := \int_0^\infty W_1\left(\frac{t}{T}\right) \sum_{k\equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right) \sum_{j=1}^{\theta_k(N,\chi)} \rho(f_{j,k})|L(1/2+it, f_{j,k}\otimes g)|^2\,dt$$

over bases $(f_{j,k})$ of holomorphic cuspidal newforms of large even weights k, fixed level N and fixed primitive nebentypus. Here g is a fixed automorphic form of full level (including the "limit Eisenstein series"), $\rho(f_{j,k}) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f_{j,k}\|^2}$ are the usual harmonic weights, $W_{1,2} : (0,\infty) \to [0,\infty)$ are fixed smooth functions with nonempty support in $[1,2]$, and $T, K \geqslant 1$ are two large parameters. Motivated by earlier results of Duke [Du88], Jutila–Motohashi [JM05], Kim–Zhang [KZ09] and Sarnak [Sa85], we established the following asymptotic formula for $\mathscr{I}(T,K)$ with a power saving error term.

Theorem 2 ([BH11])**.** *Define the analytic conductor*

$$\mathscr{C}(t,k) := \frac{N^2}{(2\pi)^4}\left(t^2 + \frac{k^2}{4}\right)^2,$$

and its smooth averages

$$\mathscr{L}_j(T,K) := \frac{1}{TK}\int_0^\infty\int_0^\infty W_1\left(\frac{t}{T}\right)W_2\left(\frac{x}{K}\right)\log^j\mathscr{C}(t,x)\,dt\,dx,$$
$$\mathscr{M}_{ir}(T,K) := \frac{1}{TK}\int_0^\infty\int_0^\infty W_1\left(\frac{t}{T}\right)W_2\left(\frac{x}{K}\right)\mathscr{C}(t,x)^{ir}\,dt\,dx.$$

For g cuspidal there are constants $a_0, a_1 \in \mathbb{R}$ depending only on N and g such that

$$\mathscr{I}(T,K) = TK\big(a_1\mathscr{L}_1(T,K) + a_0\mathscr{L}_0(T,K)\big) + O\left((TK)^{1+\varepsilon}(T^4K^{-5} + T^{-4}K^3)\right).$$

For $g = E(\cdot, 1/2+ir)$ with $r \in \mathbb{R}\setminus\{0\}$ there are constants $a_0, a_1, a_2 \in \mathbb{R}$ and $b_\pm \in \mathbb{C}$ depending only on N and r such that

$$\mathscr{I}(T,K) = TK\left(\sum_{j=0}^{2} a_j\mathscr{L}_j(T,K) + b_+\mathscr{M}_{ir}(T,K) + b_-\mathscr{M}_{-ir}(T,K)\right) + O\left((TK)^{1+\varepsilon}(T^4K^{-5} + T^{-4}K^3)\right).$$

For $g = \frac{\partial}{\partial s}E(\cdot,s)|_{s=1/2}$ there are constants $a_0,a_1,a_2,a_3,a_4 \in \mathbb{R}$ depending only on N such that

$$\mathscr{I}(T,K) = TK\left(\sum_{j=0}^{4} a_j \mathscr{L}_j(T,K)\right) + O\left((TK)^{1+\varepsilon}(T^4K^{-5}+T^{-4}K^3)\right).$$

The implied constants depend only on N, g, W_1, W_2, ε.

We note that for $T=K$ and fixed $W_{1,2}$ the above asymptotic formulae take a particularly simple shape as $\mathscr{L}_j(K,K)$ is a polynomial in $\log K$ of degree j and $\mathscr{M}_{ir}(K,K)$ is proportional to K^{4ir}. The presence of the oscillating secondary terms $\mathscr{M}_{\pm ir}(T,K)$ is rather interesting as it seems to display a new feature of moments of automorphic L-functions. In all cases we calculated the leading coefficient a_i and the coefficients $b_\pm$ explicitly, and they turned out to be nonzero. In particular,

$$b_\pm = \frac{1}{2}\zeta(1\pm 2ir)^4 \prod_{p|N}(1-p^{-2\mp 4ir}).$$

For the proof of Theorem 2 we develop a precise uniform approximate functional equation with explicit dependence on the archimedean parameters [BH11, Proposition 1], and a Voronoi summation formula for Eisenstein series [BH11, Proposition 2].

3. SUP-NORM OF MAASS CUSP FORMS

The third result within the project establishes a new upper bound for the sup-norm $\|f\|_\infty$ of a Hecke–Maass cuspidal newform f. The two basic parameters are the Laplacian eigenvalue λ and the level N. The form f is assumed to have L^2-norm 1 with respect to the hyperbolic measure $dxdy/y^2$ on $\Gamma_0(N)\backslash\mathscr{H}$. In the λ-aspect the first nontrivial (and so far unsurpassed) bound is due to Iwaniec and Sarnak [IS95] who established $\|f\|_\infty \ll_{N,\varepsilon} \lambda^{\frac{5}{24}+\varepsilon}$ for any $\varepsilon>0$, improving on $\|f\|_\infty \ll \lambda^{\frac{1}{4}}$ which is valid on any Riemannian surface (see Seeger–Sogge [SS89]). In the N-aspect the "trivial" bound is $\|f\|_\infty \ll_{\lambda,\varepsilon} N^\varepsilon$ (see [AU95, MU98, BlHo10]), while the most optimistic bound would be $\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-\frac{1}{2}+\varepsilon}$. Here and later the dependence on λ is understood continuous. The breakthrough in the N-aspect was recently achieved by Blomer–Holowinsky [BlHo10, p. 673] who proved $\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-\frac{25}{914}+\varepsilon}$, at least for square-free N. The restriction on N seems difficult to remove: it is needed for a certain application of Atkin–Lehner theory. Templier [Te10] revisited the proof by making a systematic use of geometric arguments, and derived a stronger exponent: $\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-\frac{1}{22}+\varepsilon}$. Helfgott–Ricotta (unpublished) improved some of the estimates in [Te10] and obtained $\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-\frac{1}{20}+\varepsilon}$. Using Atkin–Lehner theory we developed a more efficient treatment of the counting problem at the heart of the argument. This way we proved

Theorem 3 ([HT11b])**.** *Let f be an L^2-normalized Hecke–Maass cuspidal newform of square-free level N, trivial nebentypus, and Laplacian eigenvalue λ. Then for any $\varepsilon>0$ one has*

$$\|f\|_\infty \ll_{\lambda,\varepsilon} N^{-\frac{1}{6}+\varepsilon},$$

where the implied constant depends continuously on λ.

It seems that $-\frac{1}{6}$ is the natural exponent for the sup-norm problem in the level aspect. Examples of such exponents are the Weyl exponent $\frac{1}{6}$ (resp. Burgess exponent $\frac{3}{16}$) in the subconvexity problem for GL_1 in the archimedean (resp. non-archimedean) aspect, or their doubles in the GL_2-setting. For comparison, Blomer–Michel [BM11] obtained a bound of the same quality for Hecke eigenforms on certain compact arithmetic surfaces.

The key new idea in the proof of Theorem 3 is the observation that for Atkin–Lehner operators of square-free level N there is a fundamental domain consisting of points $z\in\mathscr{H}$ with very good diophantine properties [HT11a, Lemma 2.2].

REFERENCES

[AU95] A. ABBES, E. ULLMO, Comparaison des métriques d'Arakelov et de Poincaré sur $X_0(N)$, *Duke Math. J.* **80** (1995), 295–307.

[BM07] E. M. BARUCH, Z. MAO, Central value of automorphic L-functions, *Geom. Funct. Anal.* **17** (2007), 333–384.

[BH08a] V. BLOMER, G. HARCOS, Hybrid bounds for twisted L-functions, *J. Reine Angew. Math.* **621** (2008), 53–79.

[BH08b] V. BLOMER, G. HARCOS, The spectral decomposition of shifted convolution sums, *Duke Math. J.* **144** (2008), 321–339.

[BH09] V. BLOMER, G. HARCOS, L-functions, automorphic forms, and arithmetic, In: Symmetries in Algebra and Number Theory (I. Kersten, R. Meyer eds.), 11–25., Universitätsverlag Göttingen, 2009.

[BH10] V. BLOMER, G. HARCOS, Twisted L-functions over number fields and Hilbert's eleventh problem, *Geom. Funct. Anal.* **20** (2010), 1–52.

[BH11] V. BLOMER, G. HARCOS, A hybrid asymptotic formula for the second moment of Rankin–Selberg L-functions, *Proc. London Math. Soc.*, to appear.

[BHM07] V. BLOMER, G. HARCOS, P. MICHEL, A Burgess-like subconvex bound for twisted L-functions (with Appendix 2 by Z. Mao), *Forum Math.* **19** (2007), 61–105.

[BlHo10] V. BLOMER, R. HOLOWINSKY, Bounding sup-norms of cusp forms of large level, *Invent. Math.* **179** (2010), 645–681.

[BM11] V. BLOMER, P. MICHEL, Hybrid bounds for automorphic forms on ellipsoids over number fields, submitted.

[Co03] J. COGDELL, On sums of three squares, *J. Théor. Nombres Bordeaux* **15** (2003), 33–44.

[CPSS] J. COGDELL, I. PIATETSKI-SHAPIRO, P. SARNAK, Estimates on the critical line for Hilbert modular L-functions and applications, preprint.

[Du88] W. DUKE, Fourth moments of L-functions attached to newforms, *Comm. Pure Appl. Math.* **41** (1988), 815–831.

[DSP90] W. DUKE, R. SCHULZE-PILLOT, Representations of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids, *Invent. Math.* **89** (1990), 49–57.

[HT11a] G. HARCOS, N. TEMPLIER, On the sup-norm of Maass cusp forms of large level: II, *Int. Math. Res. Not.*, Art. ID rnr202, 11 pp.

[HT11b] G. HARCOS, N. TEMPLIER, On the sup-norm of Maass cusp forms of large level. III, submitted

[Hs76] J. S. HSIA, Representations by spinor genera, *Pacific J. Math.* **63** (1976), 147–152.

[Iw87] H. IWANIEC, Fourier coefficients of modular forms of half-integral weight, *Invent. Math.* **87** (1987), 385–401.

[IS95] H. IWANIEC, P. SARNAK, L^∞ norms of eigenfunctions of arithmetic surfaces, *Ann. of Math.* **141** (2000), 301–320.

[JM05] M. JUTILA, Y. MOTOHASHI, Uniform bounds for Hecke L-functions, *Acta Math.* **195** (2005), 61–115.

[KM96] K. KHURI-MAKDISI, On the Fourier coefficients of nonholomorphic modular Hilbert modular forms of half-integral weight, *Duke Math. J.* **84** (1996), 399–452.

[Kn61] M. KNESER, Darstellungsmaße indefiniter quadratischer Formen, *Math. Z.* **77** (1961), 188–194.

[Ko04] H. KOJIMA, On the Fourier coefficients of Hilbert-Maass wave of half-integral weight over arbitrary algebraic number fields, *J. Number Theory* **107** (2004), 25–62.

[KZ09] H. H. KIM, Y. ZHANG, Divisor function for quaternion algebras and application to fourth moments of L-functions, *J. Number Theory* **129** (2009), 3000–3019.

[MU98] P. MICHEL, E. ULLMO, Points de petite hauteur sur les courbes modulaires $X_0(N)$, *Invent. Math.* **131** (1998), 645–674.

[Sa85] P. SARNAK, Fourth moments of Grössencharakteren zeta functions, *Comm. Pure Appl. Math.* **38** (1985), 167–178.

[Sa94] P. SARNAK, Integrals of products of eigenfunctions, *Int. Math. Res. Not.* **1994**, 251–260.

[SS89] A. SEEGER, C. D. SOGGE, Bounds for eigenfunctions of differential operators, *Indiana Univ. Math. J.* **38** (1989), 669–682.

[Sh93] G. SHIMURA, On the Fourier coefficients of Hilbert modular forms of half-integral weight, *Duke Math. J.* **71** (1993), 501–557.

[Si37] C. L. SIEGEL, Über die analytische Theorie der quadratischen Formen, *Ann. of Math.* **36** (1935), 527–606.; II, *ibid.* **37** (1936), 230–263.; III, *ibid.* **38** (1937), 212–291.

[Si52] C. L. SIEGEL, Indefinite quadratische Formen und Funktionentheorie. I, *Math. Ann.* **124** (1951), 17–54.; II, *ibid.* **124** (1952), 364–387.

[Te10] N. TEMPLIER, On the sup-norm of Maass cusp forms of large level, *Selecta Math. (N.S.)* **16** (2010), 501–531.

[Ve10] A. VENKATESH, Sparse equidistribution problems, period bounds and subconvexity, *Ann. of Math.* **172** (2010), 989–1094.

[Wa81] J.-L. WALDSPURGER, Sur les coefficients de Fourier des formes moulaires de poids demientier, *J. Math. Pures Appl.* **60** (1981), 375–484.