




A positive solution of asymptotically periodic Schrödinger equations with local superlinear nonlinearities

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Abstract. In this paper, we investigate the following Schrödinger equation

$$-\Delta u + V(x)u = \lambda f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, V is an asymptotically periodic potential and the nonlinearity term $f(u)$ is only locally defined for $|u|$ small and satisfies some mild conditions. By using Nehari manifold and Moser iteration, we obtain the existence of positive solutions for the equation with sufficiently large λ .

Keywords: Schrödinger equation, positive solution, locally defined nonlinearity, asymptotically periodic potential.

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1 Introduction

In recent years, many researchers consider the following Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $N \geq 3$, V is a given potential and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Knowledge of the solutions of Eq. (1.1) has a great importance for studying standing wave solutions for

$$ih \frac{\partial \Psi}{\partial t} = -h^2 \Delta \Psi + W(x)\Psi - f(x, \Psi), \quad \text{for all } x \in \Omega, \quad (\text{NLS})$$

where $h > 0$, W is the real-valued potential and Ω is a domain in \mathbb{R}^N . Eq. (NLS) is one of the main objects of the quantum physics, because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics.

Eq. (1.1) has been researched intensively, see [1,3,5,7,10,11,13,14,19,21,22,28] and references therein. In the above works, we observe that many interesting conditions on f have been studied. Notice that, it seems necessary that the condition can be assumed on f at infinity, that is, f is assumed to be subcritical (or critical) at infinity, i.e.,

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$$(f_0) \quad 0 \leq \lim_{|s| \rightarrow +\infty} \frac{f(x,s)s}{|s|^{2^*}} < +\infty \quad \text{uniformly for } x \in \mathbb{R}^N,$$

where the number 2^* is denoted by $\frac{2N}{N-2}$ and called the critical Sobolev exponent for the embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. This aim is to ensure that the associated energy functional would be well defined and of class C^1 on $H^1(\mathbb{R}^N)$, and then its critical points are precisely the solutions of Eq. (1.1) by using variational methods. Certainly, many researchers tried to seek some suitable conditions to replace (f_0) . If there does not exist an assumption on f at infinity, can it be proved that there exists a nontrivial solution for Eq. (1.1)? Mathematically this problem is interesting. Accordingly, Costa and Wang [9] have considered the following equation

$$-\Delta u = \lambda f(u), \quad \text{in } \Omega,$$

where $\lambda > 0$ is a parameter, Ω is a bounded smooth domain in \mathbb{R}^N ($N \geq 3$), and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 satisfying the following conditions:

$$(f_1) \quad f(-u) = -f(u) \text{ for any } |u| \leq \delta \text{ (for some } \delta > 0);$$

$$(f_2) \quad \text{there exists } \gamma \in (2, 2^*) \text{ such that } \limsup_{|s| \rightarrow 0} \frac{f(s)s}{|s|^\gamma} = 0;$$

$$(f_3) \quad \text{there exists } \beta \in (2, 2^*) \text{ such that } \liminf_{|s| \rightarrow 0} \frac{f(s)s}{|s|^\beta} > 0;$$

$$(f_4) \quad \text{there exists } \mu \in (2, 2^*) \text{ such that } sf(s) \geq \mu F(s) > 0 \text{ for all } |s| \text{ small, where } F(s) = \int_0^s f(t)dt.$$

Motivated by Costa and Wang [9], do Ó et al. [12] have studied the following equation

$$-\Delta u + V(x)u = \lambda f(u) \quad \text{in } \mathbb{R}^N, \quad (\mathcal{P})$$

where V satisfies (V_1) – $[(V_2)$ or $(V_3)]$,

$$(V_1) \quad V \in C(\mathbb{R}^N, \mathbb{R}) \text{ and } \inf_{x \in \mathbb{R}^N} V(x) > 0,$$

$$(V_2) \quad V(x) \rightarrow \infty \text{ as } |x| \rightarrow \infty, \text{ or more generally, for every } M > 0, \text{ meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty,$$

$$(V_3) \quad \text{the function } [V(x)]^{-1} \text{ belongs to } L^1(\mathbb{R}^N),$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 satisfying (f'_1) – (f'_2) and (f_4) ,

$$(f'_1) \quad \text{there exists } p \in (2, 2^*) \text{ such that } \limsup_{|s| \rightarrow 0} \frac{f(s)s}{|s|^p} < +\infty,$$

$$(f'_2) \quad \text{there exists } q \in (2, 2^*) \text{ such that } \liminf_{|s| \rightarrow 0} \frac{F(s)}{|s|^q} > 0, \text{ where } F(s) = \int_0^s f(t)dt.$$

Further results for related problems can be found in [8, 15, 23, 24] and references therein.

Inspired by the above works, we are concerned with the existence of positive solutions for asymptotically periodic Eq. (\mathcal{P}) with a locally defined nonlinearity term, namely V satisfies (V_4) ,

$$(V_4) \quad \text{there exists a 1-periodic function } V_\infty(x) \in L^\infty(\mathbb{R}^N) \text{ such that } 0 \leq V(x) \leq V_\infty(x), \text{ } \inf_{x \in \mathbb{R}^N} V_\infty(x) > 0 \text{ and } V(x) - V_\infty(x) \in \mathcal{F}_1, \text{ where}$$

$$\mathcal{F}_1 := \{h(x) : \text{for any } \varepsilon > 0, \text{ meas}\{x \in B_1(y) : |h(x)| \geq \varepsilon\} \rightarrow 0 \text{ as } |y| \rightarrow \infty\},$$

and f satisfies (f_5) – (f_6) ,

(f_5) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $p > 2, \delta \in (0, 1)$ such that the function $s \mapsto \frac{f(s)}{s^{p-1}}$ is nondecreasing and $f(s) > 0$ on $(0, \delta]$,

(f_6) there exists $q \in (2, 2^*)$ such that $\liminf_{s \rightarrow 0^+} \frac{F(s)}{s^q} > 0$, where $F(s) = \int_0^s f(t)dt$.

As is well known, if f were assumed to be superlinear and subcritical (or critical) at infinity, then the associated energy functional

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \lambda \int_{\mathbb{R}^N} F(u)dx$$

would be of class C^1 on $H^1(\mathbb{R}^N)$ and has the mountain pass geometry. Classically, it is a minimax principle that shows the mountain pass level is a critical level of the functional (see [4, 5, 26]). Here, the assumptions (f_5) – (f_6) we make on the nonlinearity $f(u)$ refer solely to its behavior in a neighborhood of $u = 0$, and we will show that they suffice for the existence of a positive solution of Eq. (\mathcal{P}) when λ is large enough. Exactly we give our main result.

Theorem 1.1. *Assume that $N \geq 3$, (V_4) and (f_5) – (f_6) hold. Then there exists $\lambda_1 > 0$ such that Eq. (\mathcal{P}) has a positive solution for $\lambda \geq \lambda_1$.*

Remark 1.2. In this paper, we study the existence of positive solutions for Schrödinger equations with the assumptions of Theorem 1.1 that has never been investigated. For the case where the nonlinear term is only locally defined for $|u|$ small, we should point out that we refer [8, 9, 12, 15] for references in this direction. Costa and Wang [9] considered Eq. (\mathcal{P}) in bound domain. do Ó et al. [12] considered Eq. (\mathcal{P}) when V was coercive potential or satisfied that $[V(x)]^{-1}$ belongs to $L^1(\mathbb{R}^N)$. Li and Zhong [15] studied the Kirchhoff equation when the nonlinearity term was sub-linear growth. Chu and Liu [8] investigated quasi-linear Schrödinger equations in the radial space. In these papers, they have the compactness and get certain solutions easily. However, in our cases we do not have compact embedding, which is the main difficulty in this paper. Due to this difficult, the methods in [8, 9, 12, 15] fail in our case, so we will use a different way to overcome the lack of compactness.

We now make some comments on the key ingredients of the analysis in this paper. Following the idea of [8, 9, 12, 15], we first extend the nonlinear term f and introduce a modified nonlinear Schrödinger equation. Next, we show by variational methods that the modified nonlinear Schrödinger equation possesses a positive ground state solution. Finally, our approach is inspired by the results of [2, 6, 9, 12, 26] and is based on the fact that we can show a priori bound of the form

$$|u|_\infty < C\lambda^{-\beta}, \quad \beta > 0,$$

for a class of solutions for the modified nonlinear Schrödinger equation.

The organization of this paper is as follows. In the next section we reserve for setting the framework and establishing some preliminary results. Theorem 1.1 is proved in Section 3.

2 Preliminaries

From now on, we will use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the usual norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx.$$

- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$|u|_p^p = \int_{\mathbb{R}^N} |u|^p dx \quad \text{and} \quad |u|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |u(x)| \quad \text{for all } p \in [1, +\infty).$$

- $E := \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\}$ has the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx.$$

- $\operatorname{meas} \Omega$ denotes the Lebesgue measure of the set Ω .
- $u^\pm := \max\{\pm u, 0\}$ and $K := \{u \in E : u^+ \neq 0\}$.
- $\langle \cdot, \cdot \rangle$ denotes action of dual.
- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| \leq r\}$ and $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.
- C denotes a positive constant and is possibly various in different places.

We work in the space E and recall some facts that the norms $\|\cdot\|$ and $\|\cdot\|_H$ are equivalent and $E \hookrightarrow L^s(\mathbb{R}^N)$ for any $s \in [2, 2^*]$ is continuous. The proof can be done similarly to that in [19] and details are omitted here. We start by observing that $(f_5) - (f_6)$ imply that $p \leq q$ and

$$|f(s)s| \leq C|s|^p, \quad \text{for any } |s| \leq \delta.$$

In order to prove our main result via variational methods, we need to modify and extend $f(u)$ for outside a neighborhood of $u = 0$ to get $\tilde{f}(u)$. We set

$$\tilde{f}(s) := \begin{cases} 0, & s \leq 0, \\ f(s), & 0 < s \leq \delta, \\ C_1 s^{p-1}, & \delta < s, \end{cases}$$

and fix $C_1 > 0$ such that $\tilde{f} \in C(\mathbb{R}, \mathbb{R}^+)$. Combining with the definition of \tilde{f} , one can easily obtain the following lemma.

Lemma 2.1. *Suppose that (f_5) hold. Then*

- $\lim_{s \rightarrow +\infty} \frac{\tilde{F}(s)}{s^2} = +\infty$, where $\tilde{F}(s) = \int_0^s \tilde{f}(t) dt$,
- there exists $C > 0$ such that $|\tilde{f}(s)s| \leq C|s|^p$ and $|\tilde{F}(s)| \leq C|s|^p$ for all $s \in \mathbb{R}$,
- there exists $\mu \in (2, p)$ such that the function $s \mapsto \frac{\tilde{f}(s)}{s^{\mu-1}}$ is strictly increasing on $(0, +\infty)$,

Now let us consider the modified equation of Eq. (P) given by

$$\begin{cases} -\Delta u + V(x)u = \lambda \tilde{f}(u), \\ u \in E. \end{cases} \quad (\tilde{\mathcal{P}})$$

The corresponding energy functional

$$\tilde{\mathcal{I}}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \lambda \int_{\mathbb{R}^N} \tilde{F}(u) dx$$

is of class C^1 by a standard argument and whose derivative is given by

$$\langle \tilde{\mathcal{I}}'(u), v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) dx - \lambda \int_{\mathbb{R}^N} \tilde{f}(u)v dx, \quad v \in E.$$

Formally, critical points of $\tilde{\mathcal{I}}$ are solutions of Eq. ($\tilde{\mathcal{P}}$). We note that critical points of $\tilde{\mathcal{I}}$ with L^∞ -norm less than or equal to δ are also solutions of the original Eq. (P). We recall the Nehari manifold

$$\mathcal{N} := \left\{ u \in E \setminus \{0\} : \langle \tilde{\mathcal{I}}'(u), u \rangle = 0 \right\} = \left\{ u \in K : \langle \tilde{\mathcal{I}}'(u), u \rangle = 0 \right\},$$

and set

$$c := \inf_{u \in \mathcal{N}} \tilde{\mathcal{I}}(u).$$

Lemma 2.2. *Suppose that (V_4) and (f_5) hold. Then*

- (a) *for any $u \in K$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, the maximum of $\tilde{\mathcal{I}}(tu)$ for $t > 0$ is achieved at t_u ,*
- (b) *there exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{N}$,*
- (c) *the functional \mathcal{I} is bounded from below on \mathcal{N} by a positive constant.*

Proof. (a) For any $u \in K$, we define

$$\Psi(t) := \tilde{\mathcal{I}}(tu) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \lambda \int_{\mathbb{R}^N} \tilde{F}(tu) dx, \quad t \in (0, +\infty).$$

It follows from (b) of Lemma 2.1 and the Sobolev inequality that

$$\int_{\mathbb{R}^N} \tilde{F}(tu) dx \leq C \int_{\mathbb{R}^N} |tu|^p dx \leq Ct^p \|u\|^p.$$

Thus one has

$$\Psi(t) \geq \frac{t^2}{2} \|u\|^2 - \lambda Ct^p \|u\|^p.$$

Then there exists $t_0 > 0$ such that $\Psi(t_0) > 0$. We set $\Omega = \{x \in \mathbb{R}^N : u(x) > 0\}$. Combining (a) in Lemma 2.1 with Fatou's lemma, we have

$$\liminf_{t \rightarrow \infty} \int_{\Omega} \frac{\tilde{F}(tu)}{(tu)^2} u^2 dx = +\infty.$$

Hence

$$\limsup_{t \rightarrow \infty} \frac{\Psi(t)}{t^2} = \frac{1}{2} \|u\|^2 - \lambda \liminf_{t \rightarrow \infty} \int_{\Omega} \frac{\tilde{F}(tu)}{t^2} dx = \frac{1}{2} \|u\|^2 - \lambda \liminf_{t \rightarrow \infty} \int_{\Omega} \frac{\tilde{F}(tu)}{(tu)^2} u^2 dx = -\infty.$$

One could deduce $\Psi(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. So there exists $t_u > 0$ such that $\Psi(t_u) = \max_{t>0} \Psi(t)$ and $\Psi'(t_u) = 0$, i.e., $\tilde{\mathcal{I}}(t_u u) = \max_{t>0} \tilde{\mathcal{I}}(tu)$ and $t_u u \in \mathcal{N}$. Suppose that there exists $t_1 > t_2 > 0$ such that $t_i u \in \mathcal{N}, i = 1, 2$, one has

$$\int_{\Omega} \frac{\tilde{f}(t_1 u) u^2}{t_1 u} dx = \int_{\Omega} \frac{\tilde{f}(t_2 u) u^2}{t_2 u} dx,$$

which contradicts (c) of Lemma 2.1. Thus we can conclude that t_u is unique.

(b) For any $u \in \mathcal{N}$, combining the Sobolev embedding and (b) of Lemma 2.1, one obtains

$$\|u\|^2 = \lambda \int_{\mathbb{R}^N} \tilde{f}(u) u dx \leq C\lambda \int_{\mathbb{R}^N} |u|^p dx \leq C\lambda \|u\|^p. \quad (2.1)$$

It follows from (2.1) that there exists $\rho > 0$ independent of u such that

$$\rho \leq \|u\|.$$

(c) Also from (b) of Lemma 2.1 and the Sobolev inequality, we have

$$\tilde{\mathcal{I}}(u) \geq \frac{1}{2} \|u\|^2 - C\lambda \|u\|^p.$$

Since $p > 2$, there exists $\sigma > 0$ such that $\tilde{\mathcal{I}}(u) \geq \frac{\sigma^2}{4} > 0$ for $\|u\| = \sigma > 0$. For any $v \in \mathcal{N}$, there exists $t' > 0$ such that $t' \|v\| = \sigma$. Combining with (a)-(b) of Lemma 2.2, one obtains

$$\tilde{\mathcal{I}}(v) \geq \tilde{\mathcal{I}}(t'v) \geq \frac{\sigma^2}{4}.$$

This completes the proof. \square

From Lemmas 2.1–2.2, one can easily know (see also [19, 26])

$$c = \inf_{u \in \mathcal{N}} \tilde{\mathcal{I}}(u) = \inf_{u \in K} \sup_{t > 0} \tilde{\mathcal{I}}(tu) = \min_{\gamma \in \Gamma} \max_{t \in [0, 1]} \tilde{\mathcal{I}}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \tilde{\mathcal{I}}(\gamma(t)) < 0\}$. Notice that, $c > 0$ from (c) of Lemma 2.2. In order to prove our results, we introduce the following equation

$$-\Delta u + V_{\infty}(x)u = \lambda \tilde{f}(u), \quad (\mathcal{P}_{\infty})$$

and it follows from [16, 19, 26, 27] that Eq. (\mathcal{P}_{∞}) has a positive ground state solution ω . From Lemma 2.2 and (V_4) , there exists a unique $t_{\omega} > 0$ such that $t_{\omega} \omega \in \mathcal{N}$ and

$$c \leq \tilde{\mathcal{I}}(t_{\omega} \omega) \leq \tilde{\mathcal{I}}_{\infty}(t_{\omega} \omega) \leq \tilde{\mathcal{I}}_{\infty}(\omega) := c_{\infty}, \quad (2.2)$$

where $\tilde{\mathcal{I}}_{\infty}$ is the energy functional associated with Eq. (\mathcal{P}_{∞}) .

Lemma 2.3. *Suppose that (V_4) and (f_5) hold. If $u \in \mathcal{N}$ and $\tilde{\mathcal{I}}(u) = c$, then u is a nontrivial solution of Eq. $(\tilde{\mathcal{P}})$.*

Proof. Inspired by the method in [18], one supposes by contradiction that u is not a nontrivial solution of Eq. $(\tilde{\mathcal{P}})$. Then there exists $\phi \in E$ such that

$$\langle \tilde{\mathcal{I}}'(u), \phi \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + V(x)u\phi) dx - \lambda \int_{\mathbb{R}^N} \tilde{f}(u)\phi dx < -1.$$

Let $\varepsilon \in (0, 1)$ be small enough. Then

$$\langle \tilde{\mathcal{I}}'(tu + s\phi), \phi \rangle \leq -\frac{1}{2}, \quad \text{for any } |t - 1| \leq \varepsilon, |s| \leq \varepsilon. \quad (2.3)$$

We set a curve

$$\gamma(t) = tu + s\tau(t)\phi, \quad t > 0,$$

where $\tau \in C(\mathbb{R}, [0, 1])$ is a smooth cut-off function such that $\tau(t) = 1$ for $|t - 1| \leq \frac{\varepsilon}{2}$, $\tau(t) = 0$ for $|t - 1| \geq \varepsilon$. Obviously, γ is a continuous. We can claim that $\tilde{\mathcal{I}}(\gamma(t)) < c$ for any $t \in (0, +\infty)$.

Indeed, it follows from Lemma 2.2 that $\tilde{\mathcal{I}}(\gamma(t)) = \tilde{\mathcal{I}}(tu) < \tilde{\mathcal{I}}(u) = c$ for $|t - 1| \geq \varepsilon$. When $|t - 1| < \varepsilon$, owing to $\Phi(s) := \tilde{\mathcal{I}}(tu + s\tau(t)\phi)$ is of C^1 on $[0, \varepsilon]$, there exists $\bar{s} \in (0, \varepsilon)$ such that

$$\tilde{\mathcal{I}}(tu + s\tau(t)\phi) = \tilde{\mathcal{I}}(tu) + \langle \tilde{\mathcal{I}}'(tu + \bar{s}\tau(t)\phi), \varepsilon\tau(t)\phi \rangle \leq \tilde{\mathcal{I}}(tu) - \frac{1}{2}\varepsilon\tau(t) < c,$$

where the inequality holds from (2.3). Hence $\tilde{\mathcal{I}}(\gamma(t)) < c$ for any $t \in (0, +\infty)$.

We denote $\mathcal{J}(u) = \langle \tilde{\mathcal{I}}'(u), u \rangle$. According to Lemma 2.2 and the definition of γ , we have $\mathcal{J}(\gamma(1 - \varepsilon)) = \mathcal{J}((1 - \varepsilon)u) > 0$ and $\mathcal{J}(\gamma(1 + \varepsilon)) = \mathcal{J}((1 + \varepsilon)u) < 0$. By the continuity of $t \mapsto \mathcal{J}(\gamma(t))$ there exists $t' \in (1 - \varepsilon, 1 + \varepsilon)$ such that $\mathcal{J}(\gamma(t')) = 0$. Thus $\gamma(t') \in \mathcal{N}$ and $\tilde{\mathcal{I}}(\gamma(t')) < c$, which is a contradiction. This completes the proof. \square

Lemma 2.4. *Suppose that (V_4) and (f_5) hold. Then the Cerami sequence for $\tilde{\mathcal{I}}$ at level $m > 0$ (shortly: $(Ce)_m$ sequence) is bounded in E .*

Proof. We recall the $(Ce)_m$ sequence $\{u_n\}$, that is,

$$\tilde{\mathcal{I}}(u_n) \rightarrow m, \quad \|\tilde{\mathcal{I}}'(u_n)\|(1 + \|u_n\|) \rightarrow 0.$$

Then

$$o(1) = \langle \tilde{\mathcal{I}}'(u_n), u_n^- \rangle = -\|u_n^-\|^2.$$

Consequently we could deduce that $\{u_n^+\}$ is also a $(Ce)_m$ sequence. For the sake of convenience, we denote u_n^+ by u_n . By a contradiction, we assume that $\|u_n\| \rightarrow +\infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. Obviously up to a subsequence, there exists a nonnegative function $v \in E$ such that $v_n \rightarrow v \in E, v_n \rightarrow v \in L^2_{\text{loc}}(\mathbb{R}^N)$ and $v_n(x) \rightarrow v(x)$ a.e. in \mathbb{R}^N . We denote $\Omega_1 = \{x \in \mathbb{R}^N : v(x) > 0\}$. If $\text{meas } \Omega_1 > 0$, Fatou's lemma and (a) of Lemma 2.1 imply

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\tilde{F}(u_n)}{u_n^2} v_n^2 dx \geq \liminf_{n \rightarrow \infty} \int_{\Omega_1} \frac{\tilde{F}(u_n)}{u_n^2} v_n^2 dx = +\infty.$$

Then

$$0 = \limsup_{n \rightarrow \infty} \frac{\tilde{\mathcal{I}}(u_n)}{\|u_n\|^2} = \frac{1}{2} - \lambda \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\tilde{F}(u_n)}{u_n^2} v_n^2 dx = -\infty,$$

which is a contradiction. Thus $v = 0$. We denote

$$\alpha := \limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} v_n^2 dx. \quad (2.4)$$

If $\alpha = 0$, we have $v_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$ from the Lions lemma [17, 26]. Combining with (b) of Lemma 2.1, we obtain $\int_{\mathbb{R}^N} \tilde{F}(2\sqrt{m}v_n) dx = o(1)$. By the continuity of $\tilde{\mathcal{I}}$, there exists $t_n \in [0, 1]$

such that $\tilde{\mathcal{I}}(t_n u_n) = \max_{t \in [0,1]} \tilde{\mathcal{I}}(t u_n)$. Since $\|u_n\| \rightarrow +\infty$, one has $\frac{2\sqrt{m}}{\|u_n\|} \leq 1$ as n large enough. We observe that

$$\begin{aligned} \tilde{\mathcal{I}}(t_n u_n) + o(1) &\geq \tilde{\mathcal{I}}\left(\frac{2\sqrt{m}}{\|u_n\|} u_n\right) + o(1) = 2m\|v_n\|^2 - \lambda \int_{\mathbb{R}^N} \tilde{F}(2\sqrt{m}v_n) dx + o(1) \\ &= 2m + o(1). \end{aligned}$$

In view of $\tilde{\mathcal{I}}(u_n) \rightarrow m$ and (a) of Lemma 2.2, we can see that $t_n \in (0, 1)$ and $\langle \tilde{\mathcal{I}}'(t_n u_n), t_n u_n \rangle = 0$ as n large enough. Hence by Lemma 2.3 in [20], one has

$$\begin{aligned} m &= \tilde{\mathcal{I}}(u_n) + o(1) \\ &= \tilde{\mathcal{I}}(u_n) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'(u_n), u_n \rangle + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} \tilde{f}(u_n)u_n - \tilde{F}(u_n)\right) dx + o(1) \\ &\geq \frac{\mu-2}{2\mu} t_n^2 \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx + \lambda \int_{\mathbb{R}^N} \frac{1}{\mu} \tilde{f}(t_n u_n) t_n u_n - \tilde{F}(t_n u_n) dx + o(1) \\ &= \tilde{\mathcal{I}}(t_n u_n) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'(t_n u_n), t_n u_n \rangle + o(1) \\ &= \tilde{\mathcal{I}}(t_n u_n) + o(1) \\ &\geq 2m + o(1), \end{aligned}$$

which is a contradiction.

If $\alpha > 0$, there exists $\{z_n\} \subset \mathbb{R}^N$ such that

$$\frac{\alpha}{2} \leq \int_{B_1(z_n)} v_n^2 dx.$$

If $\{z_n\}$ is bounded, there exists $R > 0$ such that

$$\frac{\alpha}{2} \leq \int_{B_R} v_n^2 dx.$$

which is a contradiction with $v_n \rightarrow 0$ in $L_{loc}^2(\mathbb{R}^N)$. Then $\{z_n\}$ is unbounded, up to a subsequence, $|z_n| \rightarrow \infty$. We set $w_n(x) := v_n(x + z_n)$, where w_n satisfies

$$\frac{\alpha}{2} \leq \int_{B_1} w_n^2 dx,$$

up to a subsequence, there exists $w \in E$ such that $w_n \rightharpoonup w$ in E , $w_n \rightarrow w$ in $L_{loc}^2(\mathbb{R}^N)$ and $w_n(x) \rightarrow w(x)$ a.e. in \mathbb{R}^N . Evidently, $\text{meas } \Omega_2 > 0$ where $\Omega_2 = \{x \in \mathbb{R}^N : w(x) > 0\}$. In fact $w_n(x) = \frac{u_n(x+z_n)}{\|u_n\|}$. Also from Fatou's lemma and (a) of Lemma 2.1, one obtains

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left[\frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \tilde{F}(u_n) dx \right] &= \liminf_{n \rightarrow \infty} \left[\frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \tilde{F}(u_n(x+z_n)) dx \right] \\ &\geq \liminf \int_{\Omega_2} \frac{\tilde{F}(u_n(x+z_n))}{[u_n(x+z_n)]^2} w_n^2 dx \\ &= +\infty. \end{aligned}$$

Hence

$$0 = \limsup_{n \rightarrow \infty} \frac{\tilde{\mathcal{I}}(u_n)}{\|u_n\|} = \frac{1}{2} - \lambda \liminf_{n \rightarrow \infty} \left[\frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \tilde{F}(u_n) dx \right] = -\infty,$$

which is a contradiction. In a word, the $(Ce)_m$ sequence $\{u_n\}$ is bounded in E . \square

Proposition 2.5. *Suppose that (V_4) and (f_5) hold. Then Eq. $(\tilde{\mathcal{P}})$ has a positive ground state solution.*

Proof. Notice that $0 < c \leq c_\infty$. Therefore, one of the two cases occurs:

Case 1. $c = c_\infty$. It follows from (2.2) that

$$c_\infty \leq \tilde{\mathcal{I}}(t_\omega \omega) \leq \tilde{\mathcal{I}}_\infty(t_\omega \omega) \leq \tilde{\mathcal{I}}_\infty(\omega) = c_\infty.$$

Then ω is also a positive ground state solution of Eq. $(\tilde{\mathcal{P}})$ from Lemma 2.3.

Case 2. $0 < c < c_\infty$. We see easily $\tilde{\mathcal{I}}$ satisfies the mountain pass geometry. From the mountain pass theorem [25, 26] and Lemma 2.4, there exists a nonnegative and bounded sequence $\{u_n\} \in E$ such that

$$\tilde{\mathcal{I}}(u_n) \rightarrow c, \quad \|\tilde{\mathcal{I}}'(u_n)\|(1 + \|u_n\|) \rightarrow 0.$$

Then there exists a nonnegative function $u \in E$ such that up to a subsequence, $u_n \rightharpoonup u$ in E , $u_n \rightarrow u$ in $L^2_{loc}(\mathbb{R}^N)$ and $u_n(x) \rightarrow u(x)$ a.e. in \mathbb{R}^N . For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has $0 = \langle \tilde{\mathcal{I}}'(u_n), \varphi \rangle + o(1) = \langle \tilde{\mathcal{I}}'(u), \varphi \rangle$, i.e., u is a nonnegative solution of Eq. $(\tilde{\mathcal{P}})$. If $u \neq 0$ in E , combining Lemma 2.3 in [20] with Fatou's lemma one obtains

$$\begin{aligned} c &= \tilde{\mathcal{I}}(u_n) + o(1) \\ &= \tilde{\mathcal{I}}(u_n) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'(u_n), u_n \rangle + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} \tilde{f}(u_n)u_n - \tilde{F}(u_n) \right) dx + o(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} \tilde{f}(u)u - \tilde{F}(u) \right) dx + o(1) \\ &= \tilde{\mathcal{I}}(u) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'(u), u \rangle + o(1) \\ &= \tilde{\mathcal{I}}(u) + o(1). \end{aligned} \tag{2.5}$$

At the same time, one knows $c \leq \tilde{\mathcal{I}}(u)$ from the definition of c and $u \in \mathcal{N}$. Applying the strongly maximum principle, we could deduce that u is a positive ground state solution of Eq. $(\tilde{\mathcal{P}})$.

We assume that $u = 0$ (otherwise we complete the proof). Then there exists $\alpha \geq 0$ such that

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^2 dx = \alpha.$$

Indeed, if $\alpha = 0$, applying the Lions lemma [17, 26] we obtain

$$u_n \rightarrow 0 \quad \text{in } L^p(\mathbb{R}^N). \tag{2.6}$$

Hence $\tilde{\mathcal{I}}(u_n) \rightarrow 0$ as $n \rightarrow \infty$ from (b) in Lemma 2.1, which contradicts $c > 0$. Then there exists $\{z_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(z_n)} |u_n|^2 dx \geq \frac{\alpha}{2} > 0$.

If $\{z_n\}$ is bounded, there exists $R > 0$ such that $\int_{B_R(0)} |u_n|^2 dx \geq \frac{\alpha}{2} > 0$, which is a contradiction with $u_n \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^N)$. Then $\{z_n\}$ is unbounded. After extracting a subsequence if

necessary, we have

- (i) $|z_n| \rightarrow +\infty$,
- (ii) $u_n(\cdot + z_n) \rightharpoonup v \neq 0$ in E .

From Lemma 2.4 in [19], we have

$$\begin{aligned}
0 &= \langle \tilde{\mathcal{I}}'(u_n), \varphi(\cdot - z_n) \rangle + o(1) \\
&= \int_{\mathbb{R}^N} [\nabla u_n \cdot \nabla \varphi(\cdot - z_n) + V(x)u_n \varphi(x - z_n)] dx - \lambda \int_{\mathbb{R}^N} \tilde{f}(u_n) \varphi(x - z_n) dx + o(1) \\
&= \int_{\mathbb{R}^N} [\nabla u_n \cdot \nabla \varphi(\cdot - z_n) + V_\infty(x)u_n \varphi(x - z_n)] dx - \lambda \int_{\mathbb{R}^N} \tilde{f}(u_n) \varphi(x - z_n) dx + o(1) \\
&= \langle \tilde{\mathcal{I}}'_\infty(v), \varphi \rangle + o(1).
\end{aligned}$$

Then v is a nontrivial solution of Eq. (\mathcal{P}_∞) . Notice that, also from [19], we obtain

$$\begin{aligned}
c &= \tilde{\mathcal{I}}(u_n) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'(u_n), u_n \rangle + o(1) \\
&= \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} \tilde{f}(u_n)u_n - \tilde{F}(u_n) \right) dx + o(1) \\
&= \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_\infty(x)u_n^2) dx \\
&\quad + \lambda \int_{\mathbb{R}^N} \left[\frac{1}{\mu} \tilde{f}(u_n(\cdot + z_n))u_n(\cdot + z_n) - \tilde{F}(u_n(\cdot + z_n)) \right] dx + o(1) \\
&= \tilde{\mathcal{I}}_\infty(u_n(\cdot + z_n)) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'_\infty(u_n(\cdot + z_n)), u_n(\cdot + z_n) \rangle + o(1) \\
&= \tilde{\mathcal{I}}_\infty(v) + o(1) \\
&\geq c_\infty + o(1),
\end{aligned}$$

which is a contradiction.

In conclusion, whether Case 1 occurs or Case 2 occurs, we can prove Proposition 2.5. \square

3 Proof of Theorem 1.1

Lemma 3.1. *Suppose that (V_4) and (f_5) hold. If u is a critical point of $\tilde{\mathcal{I}}$, then $u \in L^\infty(\mathbb{R}^N)$. Furthermore, there exists a positive constant C independent of λ such that*

$$|u|_\infty \leq C \lambda^{\frac{1}{2^*-p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{2^*-2}{2(2^*-p)}}.$$

Proof. We prove the result by using the Moser iteration. For each $k > 0$, we define

$$u_k(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq k, \\ \pm k, & \text{if } \pm u(x) > k. \end{cases}$$

For $\beta > 1$, we use $\varphi_k = |u_k|^{2(\beta-1)}u$ as a test function in $\langle \tilde{\mathcal{I}}'(u), \varphi_k \rangle$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_k|^{2(\beta-1)} |\nabla u|^2 dx + 2(\beta-1) \int_{\mathbb{R}^N} |u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k dx \\
+ \int_{\mathbb{R}^N} V(x) |u_k|^{2(\beta-1)} u^2 dx = \lambda \int_{\mathbb{R}^N} \tilde{f}(u) |u_k|^{2(\beta-1)} u dx. \quad (3.1)
\end{aligned}$$

Then we use the Sobolev inequality to yield

$$\begin{aligned}
 & \beta^2 \int_{\mathbb{R}^N} \left(|u_k|^{2(\beta-1)} |\nabla u|^2 dx + 2(\beta-1) |u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k \right) dx \\
 & \geq \int_{\mathbb{R}^N} |u_k|^{2(\beta-1)} |\nabla u|^2 + (\beta-1)^2 |u_k|^{2(\beta-2)} u^2 |\nabla u_k|^2 + 2(\beta-1) |u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k dx \\
 & \geq \int_{\mathbb{R}^N} \left| \nabla \left(|u_k|^{\beta-1} u \right) \right|^2 dx \\
 & \geq C \left(\int_{\mathbb{R}^N} \left| |u_k|^{\beta-1} u \right|^{2^*} dx \right)^{\frac{2}{2^*}}, \tag{3.2}
 \end{aligned}$$

where we also have used the facts that $u^2 |\nabla u_k|^2 \leq u_k^2 |\nabla u|^2$ and $\beta > 1$. From (b) in Lemma 2.1, we deduce

$$\int_{\mathbb{R}^N} \tilde{f}(u) |u_k|^{2(\beta-1)} u dx \leq C \int_{\mathbb{R}^N} |u|^p |u_k|^{2(\beta-1)} dx. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned}
 \left(\int_{\mathbb{R}^N} \left| |u_k|^{\beta-1} u \right|^{2^*} dx \right)^{\frac{2}{2^*}} & \leq C \beta^2 \lambda \int_{\mathbb{R}^N} |u|^{p-2} |u_k|^{2(\beta-1)} u^2 dx \\
 & \leq C \beta^2 \lambda \left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{p-2}{2^*}} \left(\int_{\mathbb{R}^N} \left| |u_k|^{2(\beta-1)} u^2 \right|^{\frac{2^*}{2^*-p+2}} dx \right)^{\frac{2^*-p+2}{2^*}}.
 \end{aligned}$$

Letting $k \rightarrow \infty$, we have

$$|u|_{\beta \cdot 2^*} \leq (C \beta^2 \lambda)^{\frac{1}{2\beta}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{p-2}{4\beta}} |u|_{\frac{2 \cdot 2^* \beta}{2^* - p + 2}}. \tag{3.4}$$

To carry out an iteration process, we set

$$\beta_m = \left(\frac{2^* - p + 2}{2} \right)^{m+1}, \quad m = 0, 1, \dots$$

Then we have

$$\frac{2 \cdot 2^* \beta_m}{2^* - p + 2} = 2^* \beta_{m-1}.$$

By (3.4), one obtains

$$\begin{aligned}
 |u|_{\beta_m \cdot 2^*} & \leq (C \beta_m^2 \lambda)^{\frac{1}{2\beta_m}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{p-2}{4\beta_m}} |u|_{\frac{2 \cdot 2^* \beta_m}{2^* - p + 2}} \\
 & = (C \lambda)^{\frac{1}{2\beta_m}} \beta_m^{\frac{1}{\beta_m}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{p-2}{4\beta_m}} |u|_{\beta_{m-1} \cdot 2^*}.
 \end{aligned}$$

By the Moser iteration, we have

$$|u|_{\beta_m \cdot 2^*} \leq (C \lambda)^{\sum_{i=0}^m \frac{1}{2\beta_i}} \prod_{i=0}^m \beta_i^{\frac{1}{\beta_i}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{p-2}{4} \sum_{i=0}^m \frac{1}{\beta_i}} |u|_{2^*}. \tag{3.5}$$

Since $\beta_0 = \left(\frac{2^*-p+2}{2}\right) > 1$ and $\beta_i = \beta_0^{i+1}$, we observe that

$$\sum_{i=0}^m \frac{1}{\beta_i} = \sum_{i=0}^m \frac{1}{\beta_0^{i+1}}, \quad \prod_{i=0}^m \beta_i^{\frac{1}{\beta_i}} = \prod_{i=0}^m \left(\beta_0^{i+1}\right)^{\frac{1}{\beta_0^{i+1}}} = (\beta_0)^{\sum_{i=0}^m \frac{i+1}{\beta_0^{i+1}}}.$$

One can easily see

$$\sum_{i=0}^{\infty} \frac{i+1}{\beta_0^{i+1}} = \beta^* < +\infty, \quad \sum_{i=0}^{\infty} \frac{1}{\beta_0^{i+1}} = \frac{2}{2^*-p}.$$

Letting $m \rightarrow \infty$ in (3.5), we conclude that $u \in L^\infty(\mathbb{R}^N)$ and

$$|u|_\infty \leq C\lambda^{\frac{1}{2^*-p}} \beta_0^{\beta^*} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{p-2}{2(2^*-p)}} |u|_{2^*} \leq C\lambda^{\frac{1}{2^*-p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{2^*-2}{2(2^*-p)}}. \quad (3.6)$$

This completes the proof. \square

Proof of Theorem 1.1. By proposition 2.5, Eq. ($\tilde{\mathcal{P}}$) has a positive ground solution u . Combining the Sobolev embedding and (b) of Lemma 2.1, one obtains

$$c = \tilde{\mathcal{I}}(u) - \frac{1}{\mu} \langle \tilde{\mathcal{I}}'(u), u \rangle \geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2. \quad (3.7)$$

We can see that there exists $v \in K \cap L^\infty(\mathbb{R}^N)$ such that $|v|_\infty < 1$. Since (f_6), there exists $C > 0$ independent of λ such that

$$\tilde{F}(tv) \geq C|tv|^q, \quad t \in [0, 1].$$

At the same time there exists $\lambda_0 > 0$ such that $\tilde{\mathcal{I}}(v) < 0$ for $\lambda \geq \lambda_0$. Then from the definition of c , we have

$$\begin{aligned} c &\leq \max_{t \in [0, 1]} \tilde{\mathcal{I}}(tv) \\ &= \max_{t \in [0, 1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2) dx - \lambda \int_{\mathbb{R}^N} \tilde{F}(tv) dx \\ &\leq \max_{t \in [0, 1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2) dx - Ct^q \lambda \int_{\mathbb{R}^N} |v|^q dx \\ &\leq C\lambda^{-\frac{2}{q-2}}. \end{aligned} \quad (3.8)$$

Combining (3.6), (3.7) and (3.8), we have

$$|u|_\infty \leq C\lambda^{\frac{1}{2^*-p}} \|u\|^{\frac{2^*-2}{2^*-p}} \leq C\lambda^{\frac{1}{2^*-p}} \lambda^{\frac{1}{2^*-q} \cdot \frac{2^*-2}{2^*-p}}.$$

Since $p, q \in (2, 2^*)$, there exists $\lambda_1 \geq \lambda_0$ such that

$$|u|_\infty \leq C\lambda_1^{\frac{2^*-q}{(2^*-p)(2-q)}} \leq \delta.$$

Therefore, from the definition of \tilde{f} , we can conclude that u is also a positive solution of Eq. (\mathcal{P}) for $\lambda \geq \lambda_1$. This completes the proof of Theorem 1.1. \square

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