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Existence, uniqueness and qualitative properties of heteroclinic solutions to nonlinear second-order ordinary differential equations

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Abstract. By means of the shooting method together with the maximum principle and the Kneser–Hukahara continuum theorem, the authors present the existence, uniqueness and qualitative properties of solutions to nonlinear second-order boundary value problem on the semi-infinite interval of the following type:

$$\begin{cases} y'' = f(x,y,y'), & x \in [0,\infty), \\ y'(0) = A, & y(\infty) = B \end{cases}$$

and

$$\begin{cases} y'' = f(x,y,y'), & x \in [0,\infty), \\ y(0) = A, & y(\infty) = B, \end{cases}$$

where $A, B \in \mathbb{R}$, $f(x,y,z)$ is continuous on $[0,\infty) \times \mathbb{R}^2$. These results and the matching method are then applied to the search of solutions to the nonlinear second-order non-autonomous boundary value problem on the real line

$$\begin{cases} y'' = f(x,y,y'), & x \in \mathbb{R}, \\ y(-\infty) = A, & y(\infty) = B, \end{cases}$$

where $A \neq B$, $f(x,y,z)$ is continuous on $\mathbb{R}^3$. Moreover, some examples are given to illustrate the main results, in which a problem arising in the unsteady flow of power-law fluids is included.

Keywords: semi-infinite interval, heteroclinic solution, shooting method, maximum principle, Kneser–Hukahara continuum theorem, matching method.

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1 Introduction

The study of heteroclinic solutions for second-order ordinary differential equations can be applied to various biological, physical, and chemical models, for instance, phase-transition,

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physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations, and has been intensively studied by many authors, see [6, 16, 28–31, 34, 38, 42, 44] and references therein. In particular, we mention that in [29], by means of a suitable fixed point technique, Malaguti and Marcelli proved the existence of a one-parameter family of solutions of the nonautonomous problem

$$\begin{cases} u'' = h(t,u,u') & \text{on } \mathbb{R}, \\ u(-\infty) = 0, & u(\infty) = 1, \end{cases}$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is continuous, and $h(t,u,v)/v$ is monotone nondecreasing in v for each $(t,u) \in \mathbb{R} \times (0,1)$.

In [34], Marcelli and Papalini considered the following problem

$$\begin{cases} u'' = f(t,u,u'), & \text{a.e. on } \mathbb{R}, \\ u(-\infty) = 0, & u(\infty) = 1, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function satisfying the condition $f(t,0,0) = f(t,1,0) = 0$ for a.e. $t \in \mathbb{R}$. Under suitable assumptions on f, the authors proved some existence and non-existence results for the problem which become operative criteria in the case that the function $f(t,u,u')$ has a product structure.

In [31], deriving from the comparison-type theory, Malaguti et al. obtained the expressive sufficient conditions for the solvability of the following problem

$$\begin{cases} u'' = f(t,u,u') & \text{on } \mathbb{R}, \\ x(-\infty) = 0, & x(\infty) = 1, \\ 0 \le u(t) \le 1 & \text{for } t \in \mathbb{R}, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous, $f(t,0,0) = f(t,1,0) = 0$ for $t \in \mathbb{R}$.

In recent years, due to the applications in various sciences, heteroclinic solutions of second-order ordinary differential equations governed by nonlinear differential operators, such as the classical p-Laplacian, Φ-Laplacian, singular Φ-Laplacian and some mixed differential operators, received more attractions see [8–11, 13, 14, 25, 32, 33, 35] and references therein. The main tools used in these works are the upper and lower solution method together with diagonalization process, and the fixed point theorem in cone.

Inspired by the above works and [19, 39], the main aim of the present paper is to establish the new results on the existence, uniqueness, and qualitative properties of heteroclinic solutions to nonlinear second-order ordinary differential equations

$$y'' = f(x,y,y') \quad \text{on } \mathbb{R} \tag{1.1}$$

by the matching method, where $f(x,y,z)$ is continuous on $\mathbb{R}^3$. To this end, we needs to consider the following second-order semi-infinite interval problems

$$\begin{cases} y'' = f(x,y,y') & \text{on } [0,\infty), \\ y'(0) = A, & y(\infty) = B, \end{cases} \tag{1.2}$$

and

$$\begin{cases} y'' = f(x,y,y') & \text{on } [0,\infty), \\ y(0) = A, & y(\infty) = B, \end{cases} \tag{1.3}$$

where $A, B \in \mathbb{R}$, $f(x, y, z)$ is continuous on $[0, \infty) \times \mathbb{R}^2$.

Second-order semi-infinite interval problems arise in the modeling of a great variety of physical phenomena such as the unsteady flow of a gas through semi-infinite porous medium, the heat transfer in radial flow between circular disks, plasma physics, the mass transfer on a rotating disk in a non-Newtonian fluid, the travelling waves in reaction-diffusion equations, et cetera [1, 36], and have been studied by many papers, for instance, see [2–5, 7, 9, 12, 15, 17, 18, 21–24, 26, 27, 37, 40, 43, 45, 46] and references therein. Among the above references, the main research methods they used are the fixed point theorems in cones [15, 21, 24, 27, 46], fixed point index theorems in cones [23, 37], upper and lower solutions method [2, 5, 22, 43], diagonalization process [3, 4, 26], variational methods [17, 18], Banach contraction mapping principle [40, 45], shooting method [7], etc.

The paper is organized as follows. In Section 2, we give some preparatory lemmas, including maximum principle, Kneser–Hukahara continuum theorem, comparison principle, continuum result and global existence of initial value problems for equation (1.1). In Section 3, using shooting method together with maximum principle and Kneser–Hukahara continuum theorem, we obtain the existence, uniqueness and qualitative properties of solutions to semi-infinite interval problems (1.2) and (1.3). In Section 4, by matching techniques we establish new results on existence, uniqueness and qualitative properties of solutions of full-infinite interval problem

$$\begin{cases} y'' = f(x, y, y') & \text{on } \mathbb{R}, \\ y(-\infty) = A, & y(\infty) = B, \end{cases} \tag{1.4}$$

where $A \neq B$. In Section 5, we demonstrate the importance of our results through some illustrative examples, which contain a problem that arises in the unsteady flow of power-law fluids.

To the best of our knowledge, the results presented in this paper are new. Compared with the recent results, we obtain not only the existence and uniqueness of the heteroclinic solutions, but also the monotonicity, convex-concave property, and asymptotic properties of the heteroclinic solutions, which are rarely considered in the literature. Moreover, the hypotheses used in this paper are different from those in recent literature, for instance, our monotonicity condition is different from those in [28, 29]. It is worth to note that one important feature of our work is that the nonlinearity $f(x, y, z)$ in Theorem 4.5 may be super-quadratic with respect to z, which are not studied by [13, 14, 32, 33, 35]. In addition, our Theorem 3.4 for problem (1.2) complements theorem 4.2 in [7].

2 Some preliminaries

In this section, as preliminaries we shall present some lemmas, which are useful in the proof of our main results.

Throughout this paper we shall use the following conditions:

(H_1) $f(x, y, z)$ is continuous on $I \times \mathbb{R}^2$;

(H_2) $f(x, y, z)$ is nondecreasing in y for each fixed pair $(x, z) \in I \times \mathbb{R}$;

(H_3) $f(x, y, z)$ satisfies a uniform Lipschitz condition on each compact subset of $I \times \mathbb{R}^2$ with respect to z, i.e., for each compact subset $E \subset I \times \mathbb{R}^2$, there exists a constant $L_E > 0$ such that

$$|f(x, y, z_1) - f(x, y, z_2)| \leq L_E |z_1 - z_2|, \quad \forall (x, y, z_1), (x, y, z_2) \in E;$$

(H_4) $zf(x,y,z) \leq 0$ for $(x,y,z) \in I \times \mathbb{R}^2$,

where $I = [0,b](b > 0)$ or $[0,\infty)$.

Lemma 2.1 (Maximum principle [41]). *Let $u = u(x)$ be a nonconstant solution of the differential inequality*

$$u'' + \alpha(x)u' + \beta(x)u \geq 0 \quad \text{in } J = (a,b),$$

where $\alpha(x)$ and $\beta(x)$ are bounded function in J, and $\beta(x) \leq 0$ in J. Then a nonnegative maximum of $u = u(x)$ can only occur on ∂J, and the outward derivative $\frac{du}{dn} > 0$ there.

Lemma 2.2 ([7]). *Assume f satisfies assumptions (H_1), (H_2) and (H_3) with $I = [0,b]$. Suppose $\phi_1(x), \phi_2(x)$ have continuous second derivatives on an interval $[a_1, b_1) \subset I$ and satisfy*

$$\begin{aligned} \phi_1''(x) &\leq f(x, \phi_1(x), \phi_1'(x)), \quad a_1 \leq x < b_1; \\ \phi_2''(x) &\geq f(x, \phi_2(x), \phi_2'(x)), \quad a_1 \leq x < b_1. \end{aligned}$$

Suppose further that

$$\phi_1(a_1) \leq \phi_2(a_1), \quad \phi_1'(a_1) \leq \phi_2'(a_1)$$

and

$$\phi_1(a_1) + \phi_1'(a_1) < \phi_2(a_1) + \phi_2'(a_1).$$

Then

$$\phi_1'(x) \leq \phi_2'(x), \quad \phi_1(x) \leq \phi_2(x) \quad \text{for } a_1 \leq x < b_1.$$

Lemma 2.3 ([7]). *Suppose f satisfies assumptions (H_1), (H_2), (H_3) and (H_4) with $I = [0,b]$. Then every solution $\phi(x)$ of the initial value problem*

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y(0) = y_0, & y'(0) = y_1 \end{cases}$$

can be continued to the entire interval $[0,b]$.

Lemma 2.4 (Kneser–Hukahara Continuum Theorem [20]). *Consider the system $y' = f(x,y), y \in \mathbb{R}^n$. Suppose that the function $f(x,y)$ is continuous and bounded on $D = \{(x,y) : a \leq x \leq b, y \in \mathbb{R}^n\}$. Let C be a compact and connected subset of D and $\mathfrak{F}(C)$ be the set of solutions which start in C. Then $\mathfrak{F}(C)$ is a compact and connected subset of $C([a,b], \mathbb{R}^n)$.*

Consider the following initial value problems

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y(0) = \lambda, & y'(0) = A \end{cases} \qquad \mathrm{IVP}_0(\lambda)$$

and

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y(0) = A, & y'(0) = \lambda. \end{cases} \qquad \mathrm{IVP}_1(\lambda)$$

Now, we introduce some notations:

$$\mathfrak{F}_0 := \{\phi : \phi(x) \text{ is a solution of } \mathrm{IVP}_0(\lambda), \lambda \in \mathbb{R}\}$$

and

$$\mathfrak{F}_1 := \{\phi : \phi(x) \text{ is a solution of } \mathrm{IVP}_1(\lambda), \lambda \in \mathbb{R}\}.$$

Lemma 2.5. *Suppose that* (H_1), (H_2), (H_3) *and* (H_4) *with* $I = [0, b]$ *hold. Let* $\lambda_1, \lambda_2 \in \mathbb{R}$ *with* $\lambda_1 < \lambda_2$. *Then*

$$F_0 = \{\phi \in \mathfrak{F}_0 : \lambda_1 \leq \lambda \leq \lambda_2\}$$

is a compact and connected subset of $C^1[0, b]$.

Proof. Let $y_0 = y, y_1 = y_0'$. Then $\mathrm{IVP}_0(\lambda)$ is equivalent to the following initial value problem of system

$$\begin{cases} \dfrac{dY}{dx} = G(x, y_0, y_1), \\ Y(0) = (\lambda, A), \end{cases} \tag{2.1}$$

where $Y = (y_0, y_1)$, $G(x, y_0, y_1) = (y_1, f(x, y_0, y_1))$. Consider a set of solutions of (2.1), denoted by S as follows:

$$S := \{(y_0(x, \lambda), y_1(x, \lambda)) : \lambda_1 \leq \lambda \leq \lambda_2\}.$$

From Lemma 2.2 and 2.3, for $\lambda_1 \leq \lambda \leq \lambda_2$ and $i = 0, 1$, we have

$$y_i(x, \lambda_1 - 1) \leq y_i(x, \lambda) \leq y_i(x, \lambda_2 + 1) \quad \text{on } [0, b],$$

then there exists $M > 0$ such that

$$|y_i(x, \lambda)| \leq M, \quad i = 0, 1, \ (x, \lambda) \in [0, b] \times [\lambda_1, \lambda_2].$$

Let

$$H := \{(x, y_0, y_1) : 0 \leq x \leq b, |y_i| \leq M + 1, i = 0, 1\}.$$

Then $G(x, y_0, y_1)$ is continuous and bounded on H, and can be extended to a bounded continuous function $G^*(x, y_0, y_1)$ on $D = [0, b] \times \mathbb{R}^2$ such that

$$G^*(x, y_0, y_1) \equiv G(x, y_0, y_1) \quad \text{for } (x, y_0, y_1) \in H.$$

Now, we consider an initial value problem of system

$$\begin{cases} \dfrac{dY}{dx} = G^*(x, y_0, y_1), \\ Y(0) = (\lambda, A). \end{cases} \tag{2.2}$$

We note that

$$C := \{(0, \lambda, A) : \lambda_1 \leq \lambda \leq \lambda_2\}$$

is a compact and connected subset of D, and then by Lemma 2.4 the set of solutions of initial value problem of system (2.2)

$$\mathfrak{F}_0(C) := \{(y_0(x, \lambda), y_1(x, \lambda)) : \lambda_1 \leq \lambda \leq \lambda_2\}$$

is a compact and connected subset of $C([0, b], \mathbb{R}^2)$. Since $\mathfrak{F}_0(C) = S$, it follows that F_0 is a compact and connected subset of $C^1[0, b]$. This completes the proof of the lemma. □

The following lemma can be readily obtained by using Lemma 2.2 and 2.4.

Lemma 2.6. *Suppose that* (H_1), (H_2), (H_3) *and* (H_4) *with* $I = [0, b]$ *hold. Let* $\lambda_1, \lambda_2 \in \mathbb{R}$ *with* $\lambda_1 < \lambda_2$. *Then*

$$F_1 = \{\phi \in \mathfrak{F}_1 : \lambda_1 \leq \lambda \leq \lambda_2\}$$

is a compact and connected subset of $C^1[0, b]$.

Lemma 2.7 ([7])**.** *Suppose f satisfies assumptions* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *with $I = [0, \infty)$. Suppose also that*

(H_5) *there exist constants γ, r, ρ, M_1, K for which $\gamma \geq 0$, $0 \leq r < \gamma + 1$, $\rho \geq 1$, $\gamma > \rho - 2$, $M_1 > 0$, $K > 0$, and*

$$|f(x, y, z)| \geq \frac{M_1 x^\gamma |z|^\rho}{|y|^r} \quad \textit{for } |y| \geq K, \ (x, z) \in [0, \infty) \times \mathbb{R}.$$

Then every solution of the initial value problem

$$\begin{cases} y'' = f(x, y, y'), & 0 \leq x < \infty, \\ y(0) = y_0, & y'(0) = y_1 \end{cases}$$

can be continued to the entire interval $[0, \infty)$. Moreover, this global solution $\phi(x)$ is bounded and monotone and hence $\lim_{x \to \infty} \phi(x)$ exists and is finite.

3 Semi-infinite interval problems

In this section, we begin with the study of the finite interval case for problem (1.2) and (1.3) by shooting method.

Theorem 3.1. *Suppose that* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *with $I = [0, b]$ hold. Then the finite interval problem*

$$\begin{cases} y'' = f(x, y, y'), & 0 \leq x \leq b, \\ y'(0) = A, & y(b) = B \end{cases} \tag{3.1}$$

has a unique solution.

Proof. **Existence.** Let $\phi(x, \lambda)$ be a solution of the initial value problem

$$\begin{cases} y'' = f(x, y, y'), & 0 \leq x \leq b, \\ y(0) = \lambda, & y'(0) = A. \end{cases}$$

Then, by Lemma 2.3, $\phi(x, \lambda)$ can be extended to the entire interval $[0, b]$. From Lemma 2.2, it follows that

$$\phi'(x, \lambda) \leq \phi'(x, 0) \quad \text{for } \lambda < 0$$

and

$$\phi(b, \lambda) - \phi(b, 0) = \lambda + \int_0^b (\phi'(x, \lambda) - \phi'(x, 0)) \mathrm{d}x \leq \lambda.$$

Therefore

$$\phi(b, \lambda) \to -\infty \quad \text{as } \lambda \to -\infty.$$

Hence, there exists $\lambda_1 < 0$ such that $\phi(b, \lambda_1) < B$. Similarly, there exists $\lambda_2 > 0$ such that $\phi(b, \lambda_2) > B$.

From Lemma 2.5, the set

$$F_0 = \{\phi(x, \lambda) \in \mathfrak{F}_0 : \lambda_1 \leq \lambda \leq \lambda_2\}$$

is a compact and connected subset of $C^1[0, b]$.

Now, we define a mapping $T : F_0 \to \mathbb{R}$ as follows:

$$T(\phi(x,\lambda)) = \phi(b,\lambda) - B, \quad \forall \phi(x,\lambda) \in F_0.$$

Then T is continuous on F_0. Since $T(\phi(b,\lambda_1)) < 0$ and $T(\phi(b,\lambda_2)) > 0$, from Bolzano's theorem there exists $\phi(x,\lambda^*) \in F_0$ such that

$$T(\phi(x,\lambda^*)) = \phi(b,\lambda^*) - B = 0,$$

that is, $\phi(b,\lambda^*) = B$. Obviously, $\phi(x,\lambda^*)$ is a solution of problem (3.1).

Uniqueness. Suppose $\phi_1(x), \phi_2(x)$ are solutions of problem (3.1). We consider two cases.

Case 1. $\phi_2(x) - \phi_1(x)$ is a constant on $[0,b]$. In this case, since $\phi_2(b) = \phi_1(b)$, we have $\phi_2(x) \equiv \phi_1(x)$ on $[0,b]$.

Case 2. $\phi_2(x) - \phi_1(x)$ is not a constant on $[0,b]$. In this case, since $\phi_2(b) = \phi_1(b)$, there exists $x_1 \in [0,b)$ such that $\phi_2(x_1) \neq \phi_1(x_1)$. Without loss of generality, we assume that $\phi_2(x_1) > \phi_1(x_1)$. Then there exists $x_2 \in [0,b)$ such that

$$\phi_2(x_2) - \phi_1(x_2) = \max_{x\in[0,b]} (\phi_2(x) - \phi_1(x)) > 0.$$

From the condition $\phi_2'(0) = \phi_1'(0)$, it follows that

$$\phi_2'(x_2) = \phi_1'(x_2).$$

Also since $\phi_2(b) = \phi_1(b)$, there exists $x_3 \in (x_2, b]$ such that

$$\phi_2(x_3) - \phi_1(x_3) = 0, \quad \phi_2(x) - \phi_1(x) > 0, \quad x \in [x_2, x_3).$$

Now, let $\psi(x) = \phi_2(x) - \phi_1(x)$. Then, it is easy to check that $\psi(x)$ is a solution of the differential inequality

$$u'' + \alpha(x)u' + \beta(x)u \geq 0 \quad \text{in } J = (x_2, x_3),$$

where

$$\alpha(x) = \begin{cases} -\dfrac{f(x,\phi_2(x),\phi_2'(x)) - f(x,\phi_2(x),\phi_1'(x))}{\phi_2'(x) - \phi_1'(x)}, & \phi_2'(x) \neq \phi_1'(x); \\ 0, & \phi_2'(x) = \phi_1'(x), \end{cases}$$

and

$$\beta(x) = -\frac{f(x,\phi_2(x),\phi_1'(x)) - f(x,\phi_1(x),\phi_1'(x))}{\phi_2(x) - \phi_1(x)}.$$

Obviously, assumptions (H_1), (H_2) and (H_3) guarantees that $\alpha(x), \beta(x)$ are bounded on (x_2, x_3) and $\beta(x) \leq 0$ on (x_2, x_3). Therefore, by Lemma 2.1 the positive maximum of $\psi(x)$ can only occur on $\partial J = \{x_2, x_3\}$ and $\frac{d\psi}{dn} > 0$ there. Since $\psi(x_3) = 0$, the maximum must occur at x_2 and $\frac{d\psi}{dn}\big|_{x=x_2} = -\psi'(x_2) > 0$, i.e., $\psi'(x_2) < 0$, which is a contradiction to $\psi'(x_2) = \phi_2'(x_2) - \phi_1'(x_2) = 0$.

In summary, $\phi_2(x) \equiv \phi_1(x)$ on $[0,b]$. This completes the proof of the theorem. □

Theorem 3.2. *Suppose that* (H_1), (H_2), (H_3) *and* (H_4) *with* $I = [0,b]$ *hold. Suppose also that*

(H$_6$) *f satisfies the uniform Nagumo condition on $[0,\infty)\times\mathbb{R}$, i.e., for each compact subset $E\subset[0,\infty)\times\mathbb{R}$, there exists a continuous function $h_E:[0,\infty)\to(0,\infty)$ with $\int_0^\infty \frac{s}{h_E(s)}\mathrm{d}s=\infty$ such that*

$$|f(x,y,z)|\le h_E(|z|),\quad \forall (x,y,z)\in E\times\mathbb{R}.$$

Then the finite interval problem

$$\begin{cases} y''=f(x,y,y'), & 0\le x\le b,\\ y(0)=A, & y(b)=B \end{cases}\tag{3.2}$$

has a unique solution.

Proof. If $A=B$, then from (H$_4$), $\phi(x)\equiv A$ is a solution of problem (3.2). Without loss of generality, we assume that $A<B$. Let $\phi(x,\lambda)$ be a solution of the initial value problem

$$\begin{cases} y''=f(x,y,y'), & 0\le x\le b,\\ y(0)=A, & y'(0)=\lambda. \end{cases}$$

Then, by Lemma 2.3, $\phi(x,\lambda)$ can be extended to the entire interval $[0,b]$. Furthermore, by Lemma 2.2, for each $\lambda>0$, $\phi(x,\lambda)$ is monotone nondecreasing on $[0,b]$. Let

$$\Sigma=\{\phi(b,\lambda):\lambda\in(0,\infty)\}.$$

We assert that $\sup\Sigma>B$. Indeed, suppose by contradiction, that $\sup\Sigma\le B$. Then there exists $R>0$ such that for each $\lambda\in(0,\infty)$,

$$\phi'(x,\lambda)\le R,\quad \forall x\in[0,b].$$

In fact, let $\eta=B-A>0$ and take $r>\eta/b$ such that

$$\int_{\eta/b}^{r}\frac{s}{h_E(s)}\mathrm{d}s\ge B-A,$$

where $E=[0,b]\times[A,B]$. If $\phi'(x,\lambda)>\eta/b$ on $[0,b]$, we get the following contradiction:

$$\eta\ge\phi(b,\lambda)-\phi(0,\lambda)=\int_0^b\phi'(x,\lambda)\mathrm{d}x>\eta.$$

Thus there exists $x_0\in[0,b]$ such that $\phi'(x_0,\lambda)\le\eta/b$. If $\phi'(x,\lambda)\le\eta/b$ on $[0,b]$, it is enough to take $R:=\eta/b$ to finish the proof. Suppose that there exist some $x\in[0,b]$ such that $\phi'(x,\lambda)>\eta/b$. Then by (H$_4$), for $\lambda>0$, $\phi''(x,\lambda)\le 0$ on $[0,b]$. Consider an interval $[x_2,x_1]$ such that $\phi'(x,\lambda)\ge\eta/b$ on $[x_2,x_1]$, $\phi'(x_1,\lambda)=\eta/b$ and $\phi'(x,\lambda)>\eta/b>0$ for every $x\in[x_2,x_1)$. Applying a convenient change of variable, by the fact that $\phi(x,\lambda)$ is monotone nondecreasing on $[0,b]$, we have

$$\begin{aligned}\int_{\phi'(x_1,\lambda)}^{\phi'(x_2,\lambda)}\frac{s}{h_E(s)}\mathrm{d}s&=\int_{x_1}^{x_2}\frac{\phi'(x,\lambda)}{h_E(\phi'(x,\lambda))}\phi''(x,\lambda)\mathrm{d}x\\ &=\int_{x_1}^{x_2}\frac{\phi'(x,\lambda)}{h_E(\phi'(x,\lambda))}f(x,\phi(x,\lambda),\phi'(x,\lambda))\mathrm{d}x\\ &\le\int_{x_2}^{x_1}\phi'(x,\lambda)\mathrm{d}x=\phi(x_1,\lambda)-\phi(x_2,\lambda)\\ &\le\sup\Sigma-A\le\int_{\eta/b}^{r}\frac{s}{h_E(s)}\mathrm{d}s.\end{aligned}$$

Then $\phi'(x_2, \lambda) \le r$ and, by the way as x_1 and x_2 were taken, we have

$$\phi'(x, \lambda) \le r =: R, \quad \forall x \in [0, b],$$

which contradicts $\phi'(0, \lambda) = \lambda \to \infty$ as $\lambda \to \infty$.

In summary, $\sup \Sigma > B$. Therefore there exists $\lambda_1 > 0$ such that $\phi(b, \lambda_1) > B$. Notice that $A < B$, it is clear from Lemma 2.2 that $\phi(b, \lambda_2) < B$ for each $\lambda_2 < 0$.

The remaining part is similar to the proof of Theorem 3.1, therefore it is omitted here. This completes the proof of the theorem. □

Remark 3.3. It is easy to see that if $f(x, y, z)$ satisfies a uniform σ-Lipschitz condition on each compact subset of $I \times \mathbb{R}$ with respect to z, that is, for each compact subset E of $[0, \infty) \times \mathbb{R}$, there exists $L_E > 0$ which depends only on E, such that

$$|f(x, y, z_1) - f(x, y, z_2)| \le L_E |z_1 - z_2|^\sigma, \quad \forall (x, y, z_1), (x, y, z_2) \in E \times \mathbb{R},$$

where $0 < \sigma \le 2$, then f satisfies the condition (H_6).

Now, using Theorem 3.1 and some lemmas in Section 2, we establish here our main results for semi-infinite interval problem (1.2).

Theorem 3.4. *Suppose that* (H_1), (H_2), (H_3), (H_4) *with* $I = [0, \infty)$ *and* (H_5) *hold. Then the semi-infinite interval problem* (1.2) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A \ge 0$, *then* $\phi(x)$ *is monotone nondecreasing, concave on* $[0, \infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonpositive on* $[0, \infty)$ *when* $B \le 0$;

(2) *if* $A \le 0$, *then* $\phi(x)$ *is monotone nonincreasing, convex on* $[0, \infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative on* $[0, \infty)$ *when* $B \ge 0$.

Proof. Firstly, we show the existence of solutions of problem (1.2). Clearly, if $A = 0$, then $\phi(x) \equiv B$ is the solution of problem (1.2). Without loss of generality, we assume that $A > 0$. Then by Theorem 3.1, the finite interval problem

$$\begin{cases} y'' = f(x, y, y'), & 0 \le x \le 1, \\ y'(0) = A, & y(1) = B + 1 \end{cases} \tag{3.3}$$

has a unique solution $y = \psi(x)$ on $[0, 1]$, and which by Lemma 2.7 can be continued to the entire interval $[0, \infty)$ as a monotone solution of (1.1). Since $\psi'(0) = A > 0$, it follows that $\psi(x)$ is monotone nondecreasing on $[0, \infty)$. Thus from Lemma 2.7, we know that $\psi(\infty) := \lim_{x\to\infty} \psi(x)$ exists, and $\psi(\infty) > B$.

Suppose by contradiction, that problem (1.2) has no solution. Let

$$G = \{\phi \in C^2[0, \infty) : \phi(x) \text{ is solution of (1.1) with } \phi'(0) = A,\ \phi(\infty) < B\}.$$

Then $G \ne \emptyset$. In fact, let $\phi(x, \lambda)$ be a solution of initial value problem

$$\begin{cases} y'' = f(x, y, y'), \\ y(0) = \lambda, \quad y'(0) = A. \end{cases}$$

Then, by Lemma 2.7, $\phi(x, \lambda)$ can be continued to the entire interval $[0, \infty)$ and

$$\phi(\infty, \lambda) = \lim_{x\to\infty} \phi(x, \lambda) < \infty.$$

If $\phi(\infty,0) < B$, then $\phi(x,0) \in G$, and thus $G \neq \emptyset$. If $\phi(\infty,0) > B$, then it follows from Lemma 2.2 that for $\lambda < 0$,

$$\phi'(x,\lambda) \leq \phi'(x,0), \quad \phi(x,\lambda) \leq \phi(x,0), \quad x \in [0,\infty).$$

Hence for each $\lambda < 0$ we have

$$\phi(x,\lambda) - \phi(x,0) = \lambda + \int_0^x (\phi'(t,\lambda) - \phi'(t,0))\mathrm{d}t \leq \lambda, \quad x \in [0,\infty).$$

At the limit, as $x \to \infty$, we obtain

$$\phi(\infty,\lambda) - \phi(\infty,0) \leq \lambda, \quad \lambda < 0,$$

i.e.,

$$\phi(\infty,\lambda) \leq \phi(\infty,0) + \lambda, \quad \lambda < 0.$$

Since $\phi(\infty,0) + \lambda \to -\infty$ as $\lambda \to -\infty$, it follows that there exists $\bar{\lambda} < 0$ such that

$$\phi(\infty,\bar{\lambda}) \leq \phi(\infty,0) + \bar{\lambda} < B.$$

Therefore $\phi(x,\bar{\lambda}) \in G$, and thus $G \neq \emptyset$.

Now, let

$$\Theta = \{\lambda = \phi(0) : \phi \in G\}.$$

Notice that for each $\phi \in G$,

$$\phi(x) \leq \psi(x), \quad \phi'(x) \leq \psi'(x), \quad x \in [0,\infty),$$

then Θ is upper bounded, and $\lambda^* := \sup \Theta < \infty$. Hence there exists $\{\lambda_n\} \subset \Theta$ such that $\lambda_n < \lambda_{n+1} < \lambda^*$ and $\lambda_n \to \lambda^*$ as $n \to \infty$. From Lemma 2.2, for $\phi(x,\lambda_n) \in G, n = 1,2,\ldots,$

$$\phi^{(i)}(x,\lambda_n) \leq \phi^{(i)}(x,\lambda_{n+1}) \leq \phi^{(i)}(x,\lambda^*), \quad i = 0,1,\ x \in [0,\infty).$$

Let $\hat{\phi}(x) = \sup_n \phi(x,\lambda_n)$. Since for each fixed positive number b, the sequence of functions $\{\phi^{(i)}(x,\lambda_n)\}$ $(i = 0,1)$ is equicontinuous on $[0,b]$, then

$$\phi^{(i)}(x,\lambda_n) \to \hat{\phi}^{(i)}(x) \quad (n \to \infty) \quad \text{uniformly on } [0,b],\ i = 0,1.$$

It follows that $\hat{\phi}(x)$ is a solution of (1.1) satisfying $\hat{\phi}'(0) = A$ and $\hat{\phi}(\infty) \leq B$. From the assumption that semi-infinite interval problem (1.2) has no solution, we have $\hat{\phi}(\infty) < B$.

Next, we show that there exists $\check{\phi} \in G$ such that

$$\hat{\phi}(\infty) < \check{\phi}(\infty) < B, \tag{3.4}$$

and thus obtain a contradiction. To do this, choose $b \geq 1$ sufficiently large such that

$$\psi(\infty) - \psi(b) < \frac{1}{2}(B - \hat{\phi}(\infty)).$$

Then by Theorem 3.1, the finite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y'(0) = A, & y(b) = (B + \hat{\phi}(\infty))/2 \end{cases} \tag{3.5}$$

has a unique solution $\check{\phi}(x)$, which by Lemma 2.7 can be continued to $[0,\infty)$ as a monotone nondecreasing solution of (1.1). Thus from (3.5) and (3.3) we obtain

$$\check{\phi}(1) \le \check{\phi}(b) < B < \psi(1).$$

It follows from Lemma 2.2 that

$$\check{\phi}'(x) \le \psi'(x), \quad \forall x \in [b,\infty) \subset [1,\infty).$$

Therefore

$$\begin{aligned}
\check{\phi}(\infty) &= \check{\phi}(b) + \int_b^\infty \check{\phi}'(x)\mathrm{d}x \\
&\le \check{\phi}(b) + \int_b^\infty \psi'(x)\mathrm{d}x \\
&= \frac{1}{2}(B + \hat{\phi}(\infty)) + \psi(\infty) - \psi(b) \\
&< \frac{1}{2}(B + \hat{\phi}(\infty)) + \frac{1}{2}(B - \hat{\phi}(\infty)) = B.
\end{aligned}$$

Also from (3.5) and $\hat{\phi}(\infty) < B$, it follows that $\check{\phi}(b) > \hat{\phi}(\infty)$, then by the monotonicity of $\check{\phi}(x)$ on $[0,\infty)$, we have $\check{\phi}(\infty) \ge \check{\phi}(b) > \hat{\phi}(\infty)$, and so $\check{\phi}(x)$ satisfies (3.4).

Secondly, we show the uniqueness of solutions of problem (1.2). To do this, let $\phi_1(x), \phi_2(x)$ be solutions of problem (1.2). We consider two cases to prove.

Case 1. $\phi_1(0) \ne \phi_2(0)$. Without loss of generality, we assume that $\phi_1(0) < \phi_2(0)$. Then by Lemma 2.2, $\phi_1'(x) \le \phi_2'(x)$ on $[0,\infty)$, and thus

$$\phi_2(\infty) - \phi_1(\infty) = \phi_2(0) - \phi_1(0) + \int_0^\infty (\phi_2'(x) - \phi_1'(x))\mathrm{d}x > 0,$$

which contradicts $\phi_2(\infty) = \phi_1(\infty)$.

Case 2. $\phi_1(0) = \phi_2(0)$. In this case, we have $\phi_1(x) \equiv \phi_2(x)$ on $[0,\infty)$. In fact, if not, there exists $x_0 \in (0,\infty)$ such that $\phi_1(x_0) \ne \phi_2(x_0)$. We can assume that $\phi_1(x_0) < \phi_2(x_0)$. Then there exists $x_1 \in [0, x_0)$ such that $\phi_1(x_1) = \phi_2(x_1)$ and $\phi_1(x) < \phi_2(x)$ on $(x_1, x_0]$, and so there exists $x_2 \in (x_1, x_0]$ such that $\phi_1'(x_2) < \phi_2'(x_2)$. It follows from Lemma 2.2 that $\phi_1'(x) \le \phi_2'(x)$ on $[x_2,\infty)$. Therefore

$$0 = \phi_2(\infty) - \phi_1(\infty) = \phi_2(x_2) - \phi_1(x_2) + \int_{x_2}^\infty (\phi_2'(x) - \phi_1'(x))\mathrm{d}x > 0,$$

which is a contradiction. In summary, $\phi_1(x) \equiv \phi_2(x)$ on $[0,\infty)$.

Finally, the qualitative properties of the unique solution is obvious by Lemma 2.7. This completes the proof of the theorem. □

Theorem 3.5. *Suppose that* (H_1), (H_2), (H_3), (H_4) *with* $I = [0,\infty)$, (H_5) *and* (H_6) *hold. Then the semi-infinite interval problem* (1.3) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A \le B$, *then* $\phi(x)$ *is monotone nondecreasing, concave on* $[0,\infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative or nonpositive on* $[0,\infty)$ *when* $A \ge 0$ *or* $B \le 0$, *respectively;*

(2) *if* $A \ge B$, *then* $\phi(x)$ *is monotone nonincreasing, convex on* $[0,\infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative or nonpositive on* $[0,\infty)$ *when* $B \ge 0$ *or* $A \le 0$, *respectively.*

Proof. The proof is the same as that for Theorem 3.4 except that Theorem 3.1 is used in place of Theorem 3.2, and we omitted here. This completes the proof of the theorem. □

4 Heteroclinic solutions

In order to obtain the existence, uniqueness and qualitative properties of solutions for full-infinite interval problem (1.4) via matching technique, we first discuss the existence, uniqueness and qualitative properties of solutions to the following semi-infinite interval problems

$$\begin{cases} y'' = f(x,y,y'), & -\infty < x \le 0, \\ y(-\infty) = A, & y'(0) = \eta \end{cases} \tag{4.1}$$

and

$$\begin{cases} y'' = f(x,y,y'), & -\infty < x \le 0, \\ y(-\infty) = A, & y(0) = \eta, \end{cases} \tag{4.2}$$

where $\eta \in \mathbb{R}$.

Let us list the following conditions for convenience.

($\overline{\mathrm{H}}_1$) $f(x,y,z)$ is continuous on $\mathbb{R}^3$;

($\overline{\mathrm{H}}_2$) $f(x,y,z)$ is nondecreasing in y for each fixed $(x,z) \in \mathbb{R}^2$;

($\overline{\mathrm{H}}_3$) $f(x,y,z)$ satisfies a uniform Lipschitz condition on each compact subset of $\mathbb{R}^3$ with respect to z;

($\overline{\mathrm{H}}_4$) $zf(x,y,z) \ge 0$ for $(x,y,z) \in (-\infty,0] \times \mathbb{R}^2$, and $zf(x,y,z) \le 0$ for $(x,y,z) \in [0,\infty) \times \mathbb{R}^2$;

($\overline{\mathrm{H}}_5$) there exist constants γ, r, ρ, M_1, K for which $\gamma \ge 0$, $0 \le r < \gamma + 1$, $\rho \ge 1$, $\gamma > \rho - 2$, $M_1 > 0$, $K > 0$, and

$$|f(x,y,z)| \ge \frac{M_1 |x|^\gamma |z|^\rho}{|y|^r} \quad \text{for } |y| \ge K,\ (x,z) \in \mathbb{R}^2;$$

($\overline{\mathrm{H}}_6$) f satisfies the uniform Nagumo condition on $\mathbb{R}^2$, i.e., for each compact subset $E \subset \mathbb{R}^2$, there exists a continuous function $h_E : [0,\infty) \to (0,\infty)$ with $\int_0^\infty \frac{s}{h_E(s)} \mathrm{d}s = \infty$ such that

$$|f(x,y,z)| \le h_E(|z|) \quad \text{for } (x,y,z) \in E \times \mathbb{R};$$

($\overline{\mathrm{H}}_6'$) for each $b > 0$, there exists $M = M(b) > 0$ so that

$$|f(x,y,z)| \le M|x|^q |z|^p \quad \text{for } (x,y,z) \in [-b,b] \times \mathbb{R}^2,$$

where $q \ge 0, p \ge 1, q \ge p - 2$.

Theorem 4.1. *Suppose that* ($\overline{\mathrm{H}}_1$), ($\overline{\mathrm{H}}_2$), ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$) *and* ($\overline{\mathrm{H}}_5$) *hold. Then problem* (4.1) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $\eta \le 0$, *then* $\phi(x)$ *is monotone nonincreasing, concave on* $(-\infty,0]$ *and* $\lim_{x\to-\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonpositive on* $(-\infty,0]$ *when* $A \le 0$;

(2) *if* $\eta \ge 0$, *then* $\phi(x)$ *is monotone nondecreasing, convex on* $(-\infty,0]$ *and* $\lim_{x\to-\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative on* $(-\infty,0]$ *when* $A \ge 0$.

Proof. Let $x = -t$ and $y(x) = u(t)$. Then problem (4.1) is transformed into an equivalent problem

$$\begin{cases} u'' = F(t,u,u'), & 0 \le t < \infty, \\ u'(0) = -\eta, & u(\infty) = A, \end{cases} \tag{4.3}$$

where $F(t,y,z) = f(-t,y,-z)$. It is easy to check that conditions ($\overline{\mathrm{H}}_1$), ($\overline{\mathrm{H}}_2$), ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$) and ($\overline{\mathrm{H}}_5$) imply conditions (H_1), (H_2), (H_3), (H_4) with $I = [0,\infty)$ and (H_5) hold for problem (4.3). Hence by Theorem 3.4, problem (4.3) has a unique solution $u = \psi(t)$, and thus $\phi(x) = \psi(-x)$ is a unique solution of problem (4.1) and satisfies property (1) and (2). This completes the proof of the theorem. □

Applying Theorem 3.5, we can easily obtain the following.

Theorem 4.2. *Suppose that* ($\overline{\mathrm{H}}_1$), ($\overline{\mathrm{H}}_2$), ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$), ($\overline{\mathrm{H}}_5$) *and* ($\overline{\mathrm{H}}_6$) *hold. Then problem* (4.2) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $\eta \le A$, *then* $\phi(x)$ *is monotone nonincreasing, concave on* $(-\infty,0]$ *and* $\lim_{x\to-\infty}\phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative or nonpositive on* $(-\infty,0]$ *when* $\eta \ge 0$ *or* $A \le 0$, *respectively;*

(2) *if* $\eta \ge A$, *then* $\phi(x)$ *is monotone nondecreasing, convex on* $(-\infty,0]$ *and* $\lim_{x\to-\infty}\phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative or nonpositive on* $(-\infty,0]$ *when* $A \ge 0$ *or* $\eta \le 0$, *respectively.*

Proof. The proof is similar to that of Theorem 4.1, and is omitted. This completes the proof of the theorem. □

Remark 4.3. Due to Theorem 4.3 of [7], it is easy to see that with the same hypothesis as in Theorem 4.2, except now ($\overline{\mathrm{H}}_6$) is replaced by ($\overline{\mathrm{H}}_6'$), the conclusion of Theorem 4.2 is still true.

With the above theorems we may now establish our main result of this section on the existence, uniqueness and qualitative properties of solutions for the full-infinite interval problem (1.4).

Theorem 4.4. *Suppose that* ($\overline{\mathrm{H}}_1$), ($\overline{\mathrm{H}}_2$), ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$), ($\overline{\mathrm{H}}_5$) *and* ($\overline{\mathrm{H}}_6$) *hold. Then problem* (1.4) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A < B$, *then* $\phi(x)$ *is monotone nondecreasing on* $\mathbb{R}$, *convex on* $(-\infty,0]$ *concave on* $[0,\infty)$ *and* $\lim_{x\to\pm\infty}\phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative (nonpositive) on* $\mathbb{R}$ *when* $A \ge 0$ $(B \le 0)$;

(2) *if* $A > B$, *then* $\phi(x)$ *is monotone nonincreasing on* $\mathbb{R}$, *concave on* $(-\infty,0]$, *convex on* $[0,\infty)$ *and* $\lim_{x\to\pm\infty}\phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative (nonpositive) on* $\mathbb{R}$ *when* $B \ge 0$ $(A \le 0)$.

Proof. By Theorem 3.4, for any $\eta \in \mathbb{R}$, the following semi-infinite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \le x < \infty, \\ y'(0) = \eta, & y(\infty) = B \end{cases} \tag{4.4}$$

has a unique solution $\phi_1(x,\eta)$.

First it will be shown that $\phi_1(0,\eta)$ is a continuous and strictly decreasing function of η and its range is the set of all real numbers.

Let $\eta_2 > \eta_1$, then $\phi_1(0,\eta_2) < \phi_1(0,\eta_1)$. Indeed, if $\phi_1(0,\eta_2) \geq \phi_1(0,\eta_1)$, then since $\phi_1'(0,\eta_2) = \eta_2 > \eta_1 = \phi_1'(0,\eta_1)$, it follows from Lemma 2.2 that $\phi_1'(x,\eta_2) \geq \phi_1'(x,\eta_1)$ on $[0,\infty)$. Notice that $\phi_1'(0,\eta_2) > \phi_1'(0,\eta_1)$ and $\phi_1(0,\eta_2) \geq \phi_1(0,\eta_1)$, there exists $x^* > 0$ such that $\phi_1(x^*,\eta_2) > \phi_1(x^*,\eta_1)$, and thus

$$\phi_1(x,\eta_2) - \phi_1(x,\eta_1) \geq \phi_1(x^*,\eta_2) - \phi_1(x^*,\eta_1) > 0 \quad \text{on } [x^*,\infty),$$

which contradicts $\phi_1(\infty,\eta_2) = B = \phi_1(\infty,\eta_1)$. Therefore $\phi_1(0,\eta)$ is a strictly decreasing function of η.

Suppose $\phi_1(0,\eta)$ has a jump discontinuity at $\eta = \eta_1$ such that

$$\phi_1(0,\eta_1^-) = \alpha, \quad \phi_1(0,\eta_1) = \beta, \quad \phi_1(0,\eta_1^+) = \gamma,$$

where the monotonicity asserts that $\alpha \geq \beta \geq \gamma$ and $\alpha > \gamma$. Let $\hat{\beta}$ be a real number different from β such that $\alpha \geq \hat{\beta} \geq \gamma$. Then by Theorem 3.5, the following semi-infinite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x < \infty, \\ y(0) = \hat{\beta}, & y(\infty) = B \end{cases}$$

has a unique solution $y = \phi(x)$. Let $\phi'(0) = \hat{\eta}$. Then by Theorem 3.4, $\phi(x) = \phi_1(x,\hat{\eta})$ for all $x \in [0,\infty)$, and thus

$$\phi_1(0,\hat{\eta}) = \phi(0) = \hat{\beta},$$

which is a contradiction. Thus $\phi_1(0,\eta)$ is a continuous function of η.

Suppose that for all real numbers η, $\phi_1(0,\eta)$ is bounded from above, that is, there exists $M_1 > 0$ such that $\phi_1(0,\eta) \leq M_1 < \infty$ for all $\eta \in \mathbb{R}$. By Theorem 3.5, the following semi-infinite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x < \infty, \\ y(0) = M_1 + 1, & y(\infty) = B \end{cases}$$

has a unique solution $y = \psi(x)$. Let $\psi'(0) = \check{\eta}$, then from Theorem 3.4 it follows that $\psi(x) = \phi_1(x,\check{\eta})$ for all $x \in [0,\infty)$, and thus

$$\phi_1(0,\check{\eta}) = \psi(0) = M_1 + 1,$$

which is a contradiction. Thus $\phi_1(0,\eta)$ is unbounded from above. Similarly, it can be shown that $\phi_1(0,\eta)$ is not bounded from below.

We now denote the unique solution of the semi-infinite interval problem (4.1) by $\phi_2(x,\eta)$. Using Theorem 4.1 and 4.2, it can be shown by the same arguments that $\phi_2(0,\eta)$ is a continuous and strictly increasing function of η and its range is the set of all real numbers. Consequently, there exists a unique $\eta^* \in \mathbb{R}$ such that $\phi_1(0,\eta^*) = \phi_2(0,\eta^*)$, and thus $\phi_1^{(i)}(0,\eta^*) = \phi_2^{(i)}(0,\eta^*)$, $i = 0,1$. Therefore $\phi(x)$ defined as

$$\phi(x) := \begin{cases} \phi_1(x,\eta^*), & x \in [0,\infty); \\ \phi_2(x,\eta^*), & x \in (-\infty,0] \end{cases}$$

is a solution of problem (1.4).

We now show the uniqueness. Suppose that $\bar{\phi}(x)$ is another solution of problem (1.4). Let the restrictions of $\bar{\phi}(x)$ to the subinterval $[0,\infty)$ and $(-\infty,0]$ be labeled as $\bar{\phi}_1(x)$ and $\bar{\phi}_2(x)$ respectively. Then from Theorem 3.4 and 4.1, it follows that

$$\bar{\phi}_1(x) \equiv \phi_1(x,\bar{\eta}) \quad \text{on } [0,\infty)$$

and

$$\bar{\phi}_2(x) \equiv \phi_2(x, \bar{\eta}) \quad \text{on } (-\infty, 0],$$

where $\bar{\eta} = \bar{\phi}'(0)$. Now, we assert that $\bar{\eta} = \eta^*$. Indeed, if $\bar{\eta} > \eta^*$, then

$$\bar{\phi}_1(0) = \phi_1(0, \bar{\eta}) < \phi_1(0, \eta^*) = \phi_2(0, \eta^*) < \phi_2(0, \bar{\eta}) = \bar{\phi}_2(0),$$

which is a contradiction, and hence $\bar{\eta} \leq \eta^*$. Similarly, $\bar{\eta} \geq \eta^*$. Thus $\bar{\eta} = \eta^*$. Therefore $\bar{\phi}(x) \equiv \phi(x)$ on $\mathbb{R}$, which proves the uniqueness of solution to problem (1.4).

Finally, we show the qualitative properties of the unique solution. We shall consider only the conclusion (1), since the other conclusion is somewhat tricky. Let $\phi(x)$ be the unique solution to problem (1.4), and let $A < B$. It suffices to show that $A \leq \phi(0) \leq B$. Suppose, by contradiction, that $\phi(0) > B$ or $\phi(0) < A$. To make sure, we can assume that $\phi(0) > B$. Then, by Theorem 3.5 and 4.2, $\phi(x)$ is monotone nonincreasing and monotone nondecreasing on $[0, \infty)$ and $(-\infty, 0]$, respectively, and thus $\phi'(0) = 0$. By the uniqueness results of solutions of Theorem 3.4, $\phi(x) \equiv B$ on $[0, \infty)$, and hence $\phi(0) = B$, which contradicts $\phi(0) > B$. In summary, $A \leq \phi(0) \leq B$. Consequently, the conclusion (1) holds. This completes the proof of the theorem. □

Theorem 4.5. *Suppose that* $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$, $(\overline{\mathrm{H}}_5)$ *and* $(\overline{\mathrm{H}}_6')$ *hold. Then problem* (1.4) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A < B$, *then* $\phi(x)$ *is monotone nondecreasing on* $\mathbb{R}$, *convex on* $(-\infty, 0]$ *concave on* $[0, \infty)$ *and* $\lim_{x \to \pm\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative (nonpositive) on* $\mathbb{R}$ *when* $A \geq 0$ $(B \leq 0)$;

(2) *if* $A > B$, *then* $\phi(x)$ *is monotone nonincreasing on* $\mathbb{R}$, *concave on* $(-\infty, 0]$, *convex on* $[0, \infty)$ *and* $\lim_{x \to \pm\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative (nonpositive) on* $\mathbb{R}$ *when* $B \geq 0$ $(A \leq 0)$.

Proof. The proof of this theorem is the same as that for Theorem 4.4 except that Theorem 4.3 of [7] and Remark 4.3 are used in place of Theorem 3.5 and Theorem 4.2, respectively. This completes the proof of the theorem. □

5 Some examples

In this section, as applications, we give five examples to demonstrate our main results.

Example 5.1. Consider nonlinear second-order semi-infinite interval problem

$$y'' + e^{-y} y' = 0, \quad 0 \leq x < \infty, \tag{5.1}$$

$$y'(0) = A, \quad y(\infty) = B, \tag{5.2}$$

where $A \geq 0$ and $B \leq 0$.

We put

$$f(x, y, z) = \begin{cases} -g(0)z, & \text{if } z < 0; \\ -g(y)z, & \text{if } z \geq 0, \end{cases}$$

where

$$g(y) = \begin{cases} e^{-y}, & \text{if } y \leq 0; \\ 1, & \text{if } y > 0. \end{cases}$$

It is easy to verify that f satisfies conditions (H_1), (H_2), (H_3), (H_4) with $I = [0, \infty)$. Also we have

$$|f(x, y, z)| \geq |z| \quad \text{for } (x, y, z) \in [0, \infty) \times \mathbb{R}^2.$$

Then the condition (H_5) is satisfied. Hence from Theorem 3.4, the modified semi-infinite interval problem consisting of

$$y'' = f(x, y, y'), \quad 0 \leq x < \infty$$

and (5.2) has a unique solution ϕ with $\phi'(x) \geq 0$ on $[0, \infty)$ and $\phi(x) \leq 0$ on $[0, \infty)$. Hence by the definitions of f and g, ϕ is the unique solution of problem (5.1), (5.2). Furthermore, ϕ is nonpositive, monotone nondecreasing, concave on $[0, \infty)$ and $\lim_{x\to\infty} \phi'(x) = 0$.

Example 5.2. Consider nonlinear second-order semi-infinite interval problem

$$y'' + xh(y)(y')^{2-q} = 0, \quad 0 \leq x < \infty, \tag{5.3}$$

$$y(0) = A, \quad y(\infty) = B, \tag{5.4}$$

where $0 \leq q \leq 1$, $0 \leq A < B$, $h(y)$ is nonincreasing, continuous and positive on $\mathbb{R}$ with $\inf_{\mathbb{R}} h(y) = m > 0$.

We set

$$f(x, y, z) = -xh(y)|z|^{2-q} \operatorname{sgn} z \quad \text{for } (x, y, z) \in [0, \infty) \times \mathbb{R}^2.$$

It is easy to see that f satisfies conditions (H_1)–(H_4) with $I = [0, \infty)$ and (H_6). Notice that

$$|f(x, y, z)| \geq mx|z|^{2-q} \quad \text{for } (x, y, z) \in [0, \infty) \times \mathbb{R}^2,$$

which implies the condition (H_5) is satisfied. Notice that $A < B$, hence from Theorem 3.5, the modified semi-infinite interval problem consisting of

$$y'' = f(x, y, y'), \quad 0 \leq x < \infty$$

and (5.4) has a unique solution ϕ with $\phi'(x) \geq 0$ on $[0, \infty)$. Therefore by the definition of f, ϕ is the unique solution of problem (5.3), (5.4). Morever, ϕ is positive, nondecreasing, concave on $(0, \infty)$ and $\lim_{x\to\infty} \phi'(x) = 0$.

Note that problem (5.3), (5.4) with $h(y) \equiv m > 0$ and $A = 0, B = 1$ models phenomena in the unsteady flow of power-law fluids (see [36]).

Example 5.3. Consider nonlinear second-order full-infinite interval problem

$$y'' + mx(y')^{2-q} = 0, \quad -\infty < x < \infty, \tag{5.5}$$

$$y(-\infty) = A, \quad y(\infty) = B, \tag{5.6}$$

where $0 \leq q \leq 1$, $m > 0$, $0 \leq A < B$.

We set

$$f(x, y, z) = -mx|z|^{2-q} \operatorname{sgn} z \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

It is easy to check that $f(x, y, z)$ satisfies conditions $(\overline{\mathrm{H}}_1)$–$(\overline{\mathrm{H}}_6)$. Hence from Theorem 4.4, the modified full-infinite interval problem consisting of

$$y'' = f(x, y, y'), \quad -\infty < x < \infty$$

and (5.6) has a unique solution ϕ which satisfies $\phi'(x) \geq 0$ on $\mathbb{R}$ since $A < B$. Therefore by the definition of f, $y = \phi(x)$ is the unique solution of problem (5.5), (5.6), which is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty, 0]$, concave on $[0, \infty)$ and $\lim_{x\to\pm\infty} \phi'(x) = 0$. Furthermore, $\phi(x)$ is positive on $\mathbb{R}$.

Example 5.4. Consider nonlinear second-order full-infinite interval problem

$$y'' + mx^3(y')^4 = 0, \quad -\infty < x < \infty, \tag{5.7}$$

$$y(-\infty) = A, \quad y(\infty) = B, \tag{5.8}$$

where $m > 0$, $0 \le A < B$.

We set

$$f(x,y,z) = -mx^3|z|^4 \operatorname{sgn} z \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

It is easy to check that $f(x,y,z)$ satisfies conditions $(\overline{\mathrm{H}}_1)$–$(\overline{\mathrm{H}}_5)$ and $(\overline{\mathrm{H}}_6')$. Similar to the discussion of Example 5.3, from Theorem 4.5, problem (5.7), (5.8) has a unique solution, which is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty, 0]$, concave on $[0,\infty)$ and $\lim_{x\to\pm\infty} \phi'(x) = 0$. Furthermore, $\phi(x)$ is positive on $\mathbb{R}$.

We note here that the results of [13,14,32,33,35] can not be applied to obtain the existence of solutions to problem (5.7), (5.8), since the nonlinearity of the equation (5.7) is super-quadratic with respect to z.

Example 5.5. Consider nonlinear second-order full-infinite interval problem

$$y'' + xy'(\pi - \arctan(xyy')) = 0, \quad -\infty < x < \infty, \tag{5.9}$$

$$y(-\infty) = A, \quad y(\infty) = B, \tag{5.10}$$

where $A, B \in \mathbb{R}$ and $A \ne B$.

We set

$$f(x,y,z) = -xz(\pi - \arctan(xyz)), \quad (x,y,z) \in \mathbb{R}^3.$$

It is easy to check that $f(x,y,z)$ satisfies $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$ and $(\overline{\mathrm{H}}_4)$. Also it is easily verified that

$$|f(x,y,z)| \ge \frac{\pi}{2}|x||z|, \quad (x,y,z) \in \mathbb{R}^3$$

and

$$|f(x,y,z)| \le \frac{3\pi}{2}|x||z|, \quad (x,y,z) \in \mathbb{R}^3.$$

Then $(\overline{\mathrm{H}}_5)$ and $(\overline{\mathrm{H}}_6)$ hold. Hence from Theorem 4.4, problem (5.9), (5.10) has a unique solution $y = \phi(x)$ satisfying

(1) if $A < B$, then $\phi(x)$ is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty, 0]$ and concave on $[0,\infty)$. Furthermore, $\phi(x)$ is nonnegative (nonpositive) on $\mathbb{R}$ when $A \ge 0$ $(B \le 0)$;

(2) if $A > B$, then $\phi(x)$ is monotone nonincreasing on $\mathbb{R}$, concave on $(-\infty, 0]$ and convex on $[0,\infty)$. Furthermore, $\phi(x)$ is nonnegative (nonpositive) on $\mathbb{R}$ when $B \ge 0$ $(A \le 0)$.

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