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Blow-up analysis in a quasilinear parabolic system coupled via nonlinear boundary flux

Pan Zheng✉ [1,2], **Zhonghua Xu** [1] and **Zhangqin Gao** [1]

[1]College of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, P.R. China
[2]College of Mathematics and Statistics, Yunnan University, Kunming 650091, P.R. China

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Abstract. This paper deals with the blow-up of the solution for a system of evolution p-Laplacian equations $u_{it} = \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i)$ $(i = 1, 2, \ldots, k)$ with nonlinear boundary flux. Under certain conditions on the nonlinearities and data, it is shown that blow-up will occur at some finite time. Moreover, when blow-up does occur, we obtain the upper and lower bounds for the blow-up time. This paper generalizes the previous results.

Keywords: blow-up, quasilinear parabolic system, nonlinear boundary flux.

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1 Introduction

In this paper, we investigate the following parabolic equations

$$u_{it} = \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i), \quad (i = 1, 2, \ldots, k), \ (x, t) \in \Omega \times (0, t^*), \tag{1.1}$$

coupled via nonlinear boundary flux

$$|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} = f_i(u_1, u_2, \ldots, u_k), \quad (i = 1, 2, \ldots, k), \ (x, t) \in \partial\Omega \times (0, t^*), \tag{1.2}$$

with initial data

$$u_i\,(x, 0) = u_{i0}\,(x) \geq 0, \quad (i = 1, 2, \ldots, k), \ x \in \Omega, \tag{1.3}$$

where $p \geq 2$, $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary $\partial\Omega$ assumed sufficiently smooth, Ω is a bounded star-shaped region in $\mathbb{R}^N$ $(N \geq 2)$ and t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$. Moreover the non-negative initial functions $u_{i0}(x), i = 1, 2, \ldots, k$ satisfy the compatibility conditions and $f_i(u_1, u_2, \ldots, u_k) : \mathbb{R}^k \to \mathbb{R}, i = 1, 2, \ldots, k$ are given functions to be specified later. It is well known that the functions $f_i(u_1, u_2, \ldots, u_k), i = 1, 2, \ldots, k$ may greatly affect the behavior of the solution $(u_1, u_2, \ldots, u_k)$ with the development of time.

✉Corresponding author. Email: zhengpan52@sina.com

The blow-up phenomena in nonlinear parabolic equations have been extensively investigated by many authors in the last decades (see [1–5, 8, 16, 24, 26] and the references therein). Nowadays, many methods are known and used in the study of various questions regarding the blow-up phenomena (such as blow-up criterion, blow-up rate and blow-up set, etc.) in nonlinear parabolic problems. In applications, due to the explosive nature of the solutions, it is more important to determine the lower bounds for the blow-up time. Therefore, there exist many interesting results about blow-up time in various problems, such as [11, 12, 14, 17] in parabolic problems, [13, 15, 22, 27] in chemotaxis systems, [23] even in fourth order wave equations, and so on.

In [20], Payne et al. considered the following semilinear heat equation with nonlinear boundary condition

$$\begin{cases} u_t = \Delta u - f(u), & (x,t) \in \Omega \times (0,t^*), \\ \frac{\partial u}{\partial \nu} = g(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

and established sufficient conditions on the nonlinearities to guarantee that the solution $u(x,t)$ exists for all time $t > 0$ or blows up in finite time t^*. Moreover, an upper bound for t^* was derived. Under more restrictive conditions, a lower bound for t^* was also obtained.

Moreover, Payne et al. [21] also studied the following initial-boundary problem

$$\begin{cases} u_t = \nabla(|\nabla u|^{2p} \nabla u), & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{2p} \frac{\partial u}{\partial \nu} = f(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

and obtained upper and lower bounds for the blow-up time under some conditions when blow-up does occur at some finite time.

Recently, for the special case $k = 2$ in (1.1), Liang [7] investigated the following system with nonlinear boundary flux

$$\begin{cases} u_t = \nabla(|\nabla u|^{p-2} \nabla u), v_t = \nabla(|\nabla v|^{p-2} \nabla v), & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = f_1(u,v), |\nabla v|^{p-2} \frac{\partial v}{\partial \nu} = f_2(u,v), & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

and showed that under certain conditions on the nonlinearities and the data, blow-up will occur at some finite time and when blow-up does occur, upper and lower bounds for the blow-up time are obtained.

On the other hand, many authors have studied upper and lower bounds for the blow-up time to nonlinear parabolic equations with local or nonlocal sources (see [6, 9, 10, 18, 19, 25] and the references therein).

Motivated by the above works, we investigate the blow-up condition of the solution and derive upper and lower bounds for the blow-up time t^*. Throughout this paper, we take the functions $f_i(u_1, u_2, \ldots, u_k)$, $i = 1, 2, \ldots, k$ satisfying

$$f_i(u_1, u_2, \ldots, u_k) = a \left| \sum_{j=1}^{k} u_i \right|^{r-1} \left(\sum_{j=1}^{k} u_i \right) + b |u_i|^{\frac{r+1}{k} - 2} u_i |u_1 u_2 \cdots u_{i-1} u_{i+1} \cdots u_k|^{\frac{r+1}{k}}, \tag{1.7}$$

where a, b are positive constants and r satisfies

$$\begin{cases} r > 1, & \text{if } N = 1, 2, \\ 1 < r \le \frac{N+2}{N-2}, & \text{if } N \ge 3. \end{cases} \tag{1.8}$$

Moreover it is easy to see that

$$\sum_{i=1}^{k} u_i f_i(u_1, u_2, \ldots, u_k) = (r+1)F(u_1, u_2, \ldots, u_k) \tag{1.9}$$

and

$$\frac{\partial F(u_1, u_2, \ldots, u_k)}{\partial u_i} = f_i(u_1, u_2, \ldots, u_k), \quad i = 1, 2, \ldots, k, \tag{1.10}$$

where

$$F(u_1, u_2, \ldots, u_k) = \frac{1}{r+1}\left[a|\sum_{i=1}^{k} u_i|^{r+1} + kb|\prod_{i=1}^{k} u_i|^{\frac{r+1}{k}}\right]. \tag{1.11}$$

Our main results of this paper are stated as follows.

Theorem 1.1. *Let $p \leq r+1$. Assume that $(u_1, u_2, \ldots, u_k)$ is the nonnegative solution of problem* (1.1)–(1.3)*. Moreover, suppose that $\Psi(0) > 0$ with*

$$\Psi(t) = p\int_{\partial\Omega} F(u_1, u_2, \ldots, u_k)ds - \sum_{i=1}^{k}\int_{\Omega} |\nabla u_i|^p dx, \tag{1.12}$$

where the function $F(u_1, u_2, \ldots, u_k)$ is defined by (1.11)*. Then for $p > 2$, the solution $(u_1, u_2, \ldots, u_k)$ of problem* (1.1)–(1.3) *blows up in finite time $t^* < T$ with*

$$T = \frac{\Phi(0)}{(p-2)\Psi(0)}, \tag{1.13}$$

where

$$\Phi(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^2 dx. \tag{1.14}$$

When $p = 2$, we have $T = \infty$.

Theorem 1.2. *Assume that $(u_1, u_2, \ldots, u_k)$ is the nonnegative solution of problem* (1.1)–(1.3) *in a bounded star-shaped domain $\Omega \subset \mathbb{R}^3$ assumed to be convex in two orthogonal directions. If the solution $(u_1, u_2, \ldots, u_k)$ does blow up in finite time t^*, then the blow-up time t^* is bounded from below by*

$$t^* \geq \int_{\Theta(0)}^{\infty} \frac{1}{\sum_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi, \tag{1.15}$$

where

$$\Theta(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)} dx \quad \text{with } m \geq \max\left\{4, \frac{2}{r-1}\right\}, \tag{1.16}$$

and l_i, α_i $(i = 1, 2, 3, 4)$ are computable positive constants.

This paper is organized as follows. In Section 2, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time t^*. Moreover, we also give the lower bound for the blow-up time t^* under appropriate assumptions on the data of problem (1.1)–(1.3), and prove Theorem 1.2 in Section 3.

2 Proof of Theorem 1.1

In this section, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time t^*, and prove Theorem 1.1.

Proof of Theorem 1.1. Using the Green formula and the hypotheses stated in Theorem 1.1, we have

$$\begin{aligned}
\Phi'(t) &= 2\sum_{i=1}^{k}\int_{\Omega} u_i u_{it} dx \\
&= 2\sum_{i=1}^{k}\int_{\Omega} u_i \text{div}(|\nabla u_i|^{p-2}\nabla u_i) dx \\
&= 2\sum_{i=1}^{k}\int_{\partial\Omega} u_i |\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} ds - 2\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx \\
&= 2\sum_{i=1}^{k}\int_{\partial\Omega} u_i f_i(u_1,u_2,\ldots,u_k) ds - 2\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx \\
&= 2(r+1)\int_{\partial\Omega} F(u_1,u_2,\ldots,u_k) ds - 2\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx \\
&\geq 2\left[p\int_{\partial\Omega} F(u_1,u_2,\ldots,u_k) ds - \sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx\right] \\
&= 2\Psi(t)
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\Psi'(t) &= p\sum_{i=1}^{k}\int_{\partial\Omega} f_i(u_1,u_2,\ldots,u_k) u_{it} ds - p\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^{p-2}\nabla u_i \nabla u_{it} dx \\
&= p\sum_{i=1}^{k}\int_{\partial\Omega} f_i(u_1,u_2,\ldots,u_k) u_{it} ds - p\sum_{i=1}^{k}\int_{\partial\Omega}|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} u_{it} ds \\
&\quad + p\sum_{i=1}^{k}\int_{\Omega} \text{div}(|\nabla u_i|^{p-2}\nabla u_i) u_{it} dx \\
&= p\sum_{i=1}^{k}\int_{\Omega}(u_{it})^2 dx \geq 0.
\end{aligned} \tag{2.2}$$

It follows from $\Psi(0) > 0$ and (2.2) that $\Psi(t)$ is positive for all $t > 0$. By using Hölder's inequality and Cauchy's inequality, we deduce from (2.2) that

$$\begin{aligned}
\left(\sum_{i=1}^{k}\int_{\Omega} u_i u_{it} dx\right)^2 &\leq \left(\sum_{i=1}^{k}\left(\int_{\Omega} u_i^2 dx\right)^{\frac{1}{2}}\left(\int_{\Omega} u_{it}^2 dx\right)^{\frac{1}{2}}\right)^2 \\
&\leq \left(\sum_{i=1}^{k}\int_{\Omega} u_i^2 dx\right)\left(\sum_{i=1}^{k}\int_{\Omega} u_{it}^2 dx\right) \\
&= \frac{1}{p}\Phi(t)\Psi'(t).
\end{aligned} \tag{2.3}$$

Therefore, it follows from (2.1)–(2.3) that

$$\Phi'(t)\Psi(t) \leq \frac{1}{2}(\Phi'(t))^2 = 2\left(\sum_{i=1}^{k}\int_{\Omega} u_i u_{it} dx\right)^2 \leq \frac{2}{p}\Phi(t)\Psi'(t), \tag{2.4}$$

that is,

$$\left(\Psi(t)\Phi^{-\frac{p}{2}}(t)\right)' \geq 0. \tag{2.5}$$

Integrating (2.5) over $(0,t)$, we obtain

$$\Psi(t)\Phi^{-\frac{p}{2}}(t) \geq \Psi(0)\Phi^{-\frac{p}{2}}(0) =: M. \tag{2.6}$$

Combining (2.1) with (2.6), we derive

$$\Phi'(t)\Phi^{-\frac{p}{2}}(t) \geq 2M. \tag{2.7}$$

If $p > 2$, then (2.7) can be written as

$$(\Phi^{1-\frac{p}{2}})'(t) \leq 2M\left(1 - \frac{p}{2}\right). \tag{2.8}$$

Integrating (2.8) over $(0,t)$ again, we have

$$\Phi(t) \geq \left[\Phi^{1-\frac{p}{2}}(0) - M(p-2)t\right]^{-\frac{2}{p-2}}, \tag{2.9}$$

which implies $\Phi(t) \to +\infty$ as $t \to T = \frac{\Phi^{1-\frac{p}{2}}(0)}{M(p-2)} = \frac{\Phi(0)}{(p-2)\Psi(0)}$. Therefore, for $p > 2$, we derive

$$t^* \leq T = \frac{\Phi(0)}{(p-2)\Psi(0)}. \tag{2.10}$$

If $p = 2$, then we infer from (2.7) that

$$\Phi(t) \geq \Phi(0)e^{2Mt}, \tag{2.11}$$

which implies $t^* = \infty$. The proof of Theorem 1.1 is complete. □

3 Proof of Theorem 1.2

In this section, under the assumption that $\Omega \subset \mathbb{R}^3$ is a convex bounded star-shaped domain in two orthogonal directions, we establish a lower bound for the blow-up time t^*. To do this, we need the following lemmas.

Lemma 3.1 (see [21, Lemma A.1]). *Let Ω be a bounded star-shaped domain in $\mathbb{R}^N$, $N \geq 2$. Then for any non-negative C^1-function u and $\gamma > 0$, we have*

$$\int_{\partial\Omega} u^\gamma ds \leq \frac{N}{\rho_0}\int_\Omega u^\gamma dx + \frac{\gamma d}{\rho_0}\int_\Omega u^{\gamma-1}|\nabla u| dx, \tag{3.1}$$

where

$$\rho_0 = \min_{x\in\partial\Omega}(x\cdot\nu) > 0 \quad \text{and} \quad d = \max_{x\in\overline{\Omega}}|x|. \tag{3.2}$$

Lemma 3.2 (see [21, Lemma A.2]). *Let Ω be a bounded domain in $\mathbb{R}^3$ assumed to be star-shaped and convex in two orthogonal directions. Then for any non-negative C^1-function u and $n \geq 1$, we have*

$$\int_\Omega u^{\frac{3n}{2}} dx \leq \left[\frac{3}{2\rho_0}\int_\Omega u^n dx + \frac{n}{2}\left(1 + \frac{d}{\rho_0}\right)\int_\Omega u^{n-1}|\nabla u| dx\right]^{\frac{3}{2}}, \tag{3.3}$$

where ρ_0 and d are defined in Lemma 3.1.

Proof of Theorem 1.2. Differentiating $\Theta(t)$ in (1.16), we obtain

$$\begin{aligned}\Theta'(t) &= m(r-1)\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-1}u_{it}dx\\ &= m(r-1)\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-1}\mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i)dx\\ &= m(r-1)\sum_{i=1}^{k}\int_{\partial\Omega} u_i^{m(r-1)-1}f_i(u_1,u_2,\dots,u_k)ds\\ &\quad -m(r-1)[m(r-1)-1]\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-2}|\nabla u_i|^p dx.\end{aligned} \tag{3.4}$$

By the definition of the functions $f_i, i=1,2,\dots,k$ and Lemma 3.1, we have

$$\begin{aligned}&\sum_{i=1}^{k}\int_{\partial\Omega} u_i^{m(r-1)-1}f_i(u_1,u_2,\dots,u_k)ds\\ &\quad\le C\sum_{i=1}^{k}\int_{\partial\Omega} u_i^{(m+1)(r-1)}ds\\ &\quad\le \frac{3C}{\rho_0}\sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)}dx+\frac{C(m+1)(r-1)d}{\rho_0}\sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)-1}|\nabla u_i|dx,\end{aligned} \tag{3.5}$$

where C is a positive constant. Combining (3.4) with (3.5), we derive

$$\begin{aligned}\Theta'(t) \le&\frac{3m(r-1)C}{\rho_0}I_1(t)+\frac{Cm(m+1)(r-1)^2d}{\rho_0}I_2(t)\\ &-m(r-1)[m(r-1)-1]I_3(t),\end{aligned} \tag{3.6}$$

where

$$I_1(t)=\sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)}dx=\sum_{i=1}^{k}I_{1i}(t), \tag{3.7}$$

$$I_2(t)=\sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)-1}|\nabla u_i|dx=\sum_{i=1}^{k}I_{2i}(t), \tag{3.8}$$

and

$$I_3(t)=\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-2}|\nabla u_i|^p dx=\sum_{i=1}^{k}I_{3i}(t). \tag{3.9}$$

By Lemma 3.2 and Hölder's inequality, we obtain

$$\begin{aligned}I_{1i}(t) &= \int_{\Omega} u_i^{(m+1)(r-1)}dx\\ &\le \left[\frac{3}{2\rho_0}\int_{\Omega} u_i^{\frac{2}{3}(m+1)(r-1)}dx+\frac{(m+1)(r-1)}{3}\left(1+\frac{d}{\rho_0}\right)\right.\\ &\quad\left.\times\int_{\Omega} u_i^{\frac{2}{3}(m+1)(r-1)-1}|\nabla u_i|dx\right]^{\frac{3}{2}}\\ &\le \left[\frac{3|\Omega|^{\frac{m-2}{3m}}}{2\rho_0}\left(\int_{\Omega} u_i^{m(r-1)}dx\right)^{\frac{2(m+1)}{3m}}+\frac{(m+1)(r-1)(\rho_0+d)}{3\rho_0}\right.\\ &\quad\left.\times\int_{\Omega} u_i^{\frac{2}{3}(m+1)(r-1)-1}|\nabla u_i|dx\right]^{\frac{3}{2}},\end{aligned} \tag{3.10}$$

where $i = 1, 2, \ldots, k$ and $|\Omega|$ is the measure of Ω. By using Hölder's inequality twice again, we have

$$\begin{aligned}
\int_\Omega u_i^{\frac{2}{3}(m+1)(r-1)-1}|\nabla u_i|dx &\le \left(\int_\Omega u_i^{\frac{2}{3}(m+1)(r-1)(1-\delta_1)}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}} \\
&\le \left(\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+1)(1-\delta_1)}{3m}}|\Omega|^{1-\frac{2(m+1)(1-\delta_1)}{3m}}\right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}},
\end{aligned} \tag{3.11}$$

where $i = 1, 2, \ldots, k$ and $\delta_1 = \frac{(m-2)(r-1)+3p-6}{2(m+1)(r-1)(p-1)} \in (0, 1)$ due to (1.16). Therefore, it follows from (3.10) and (3.11) that

$$\begin{aligned}
I_{1i}(t) &\le \Bigg[\frac{3|\Omega|^{\frac{m-2}{3m}}}{2\rho_0}\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+1)}{3m}} + \frac{(m+1)(r-1)(\rho_0+d)}{3\rho_0} \\
&\quad \times \left(\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+1)(1-\delta_1)}{3m}}|\Omega|^{1-\frac{2(m+1)(1-\delta_1)}{3m}}\right)^{\frac{p-1}{p}}\left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}}\Bigg]^{\frac{3}{2}} \\
&\le c_1\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{m+1}{m}} + c_2\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}\left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{3}{2p}} \\
&\le c_1\Theta^{\frac{m+1}{m}}(t) + c_2\Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}(t)I_3^{\frac{3}{2p}}(t), \quad i = 1, 2, \ldots, k,
\end{aligned} \tag{3.12}$$

where

$$c_1 = \frac{3\sqrt{3}}{2}\rho_0^{-\frac{3}{2}}|\Omega|^{\frac{m-2}{2m}} > 0 \tag{3.13}$$

and

$$c_2 = \frac{\sqrt{6}}{9}\left(\frac{(m+1)(r-1)(\rho_0+d)}{\rho_0}\right)^{\frac{3}{2}}|\Omega|^{\left(1-\frac{2(m+1)(1-\delta_1)}{3m}\right)\frac{3(p-1)}{2p}} > 0. \tag{3.14}$$

Hence, we infer from (3.12) that

$$I_1(t) = \sum_{i=1}^{k} I_{1i} \le kc_1\Theta^{\frac{m+1}{m}}(t) + kc_2\Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}(t)I_3^{\frac{3}{2p}}(t). \tag{3.15}$$

Next, we estimate $I_2(t)$. By using Hölder's inequality, we have

$$\begin{aligned}
I_{2i}(t) &= \int_\Omega u_i^{(m+1)(r-1)-1}|\nabla u_i|dx \\
&\le \left(\int_\Omega u_i^{(m+2)(r-1)(1-\delta_2)}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}} \\
&\le \left(\left(\int_\Omega u_i^{(m+2)(r-1)}dx\right)^{1-\delta_2}|\Omega|^{\delta_2}\right)^{\frac{p-1}{p}} I_{3i}^{\frac{1}{p}}(t) \\
&= |\Omega|^{\frac{(p-1)\delta_2}{p}}\left(\int_\Omega u_i^{(m+2)(r-1)}dx\right)^{\frac{(p-1)(1-\delta_2)}{p}} I_{3i}^{\frac{1}{p}}(t), \quad i = 1, 2, \ldots, k,
\end{aligned} \tag{3.16}$$

where

$$\delta_2 = \frac{r(p-2)}{(m+2)(r-1)(p-1)} \in (0,1). \tag{3.17}$$

It follows from Lemma 3.2 and Hölder's inequality that

$$\begin{aligned}\int_\Omega u_i^{(m+2)(r-1)}dx &\le \left[\frac{3}{2\rho_0}\int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)}dx + \frac{(m+2)(r-1)}{3}\left(1+\frac{d}{\rho_0}\right)\right.\\ &\qquad \left.\times \int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)-1}|\nabla u_i|dx\right]^{\frac{3}{2}}\\ &\le \left[\frac{3|\Omega|^{\frac{m-4}{3m}}}{2\rho_0}\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+2)}{3m}} + \frac{(m+2)(r-1)(\rho_0+d)}{3\rho_0}\right.\\ &\qquad \left.\times \int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)-1}|\nabla u_i|dx\right]^{\frac{3}{2}}, \quad i=1,2,\ldots,k.\end{aligned} \tag{3.18}$$

By using Hölder's inequality twice again, we have

$$\begin{aligned}\int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)-1}|\nabla u_i|dx &\le \left(\int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)(1-\delta_3)}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}}\\ &\le \left(\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+2)(1-\delta_3)}{3m}}|\Omega|^{1-\frac{2(m+2)(1-\delta_3)}{3m}}\right)^{\frac{p-1}{p}}\\ &\qquad \times \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}}\\ &= |\Omega|^{(1-\frac{2(m+2)(1-\delta_3)}{3m})\frac{p-1}{p}}\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+2)(p-1)(1-\delta_3)}{3mp}}(t) I_{3i}^{\frac{1}{p}}(t),\end{aligned} \tag{3.19}$$

where $i=1,2,\ldots,k$ and

$$\delta_3 = \frac{(m-4)(r-1)+3p-6}{2(m+2)(r-1)(p-1)} < \delta_1 < 1. \tag{3.20}$$

Combining (3.18) with (3.19), we obtain

$$\begin{aligned}&\int_\Omega u_i^{(m+2)(r-1)}dx\\ &\quad \le \frac{3\sqrt{3}}{2}|\Omega|^{\frac{m-4}{2m}}\rho_0^{-\frac{3}{2}}\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{m+2}{m}}(t) + \frac{\sqrt{6}}{9}\left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0}\right)^{\frac{3}{2}}\\ &\qquad \times|\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3(p-1)}{2p}}\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{(m+2)(p-1)(1-\delta_3)}{mp}}(t) I_{3i}^{\frac{3}{2p}}(t), \quad i=1,2,\ldots,k.\end{aligned} \tag{3.21}$$

Substituting (3.21) into (3.16) and applying the following inequality

$$(a_1+a_2)^s \le 2^s(a_1^s+a_2^s), \quad a_1,a_2>0 \quad \text{and} \quad s>0,$$

we derive

$$\begin{aligned} I_{2i}(t) \le{}& |\Omega|^{\frac{(p-1)\delta_2}{p}} \left(\frac{3\sqrt{3}}{2} |\Omega|^{\frac{m-4}{2m}} \rho_0^{-\frac{3}{2}} \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{m+2}{m}} (t) + \frac{\sqrt{6}}{9} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3}{2}} \right. \\ & \left. \times |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3(p-1)}{2p}} \cdot \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{(m+2)(p-1)(1-\delta_3)}{mp}} (t) I_{3i}^{\frac{3}{2p}}(t) \right)^{\frac{(p-1)(1-\delta_2)}{p}} I_{3i}^{\frac{1}{p}}(t) \\ \le{}& c_3 \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_{3i}^{\frac{1}{p}}(t) + c_4 \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_{3i}^{\beta}(t) \\ \le{}& c_3 \Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_3^{\frac{1}{p}}(t) + c_4 \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_3^{\beta}(t), \quad i = 1, 2, \dots, k, \end{aligned} \tag{3.22}$$

where

$$\alpha = 1 - \frac{1}{p}, \quad \beta = \frac{1}{p} + \frac{3\alpha(1-\delta_2)}{2p} < 1, \tag{3.23}$$

$$c_3 = \left(3\sqrt{3} \rho_0^{-\frac{3}{2}} \right)^{\alpha(1-\delta_2)} |\Omega|^{\frac{\alpha}{2m}(m-4+(m+4)\delta_2)}, \tag{3.24}$$

and

$$c_4 = \left(\frac{2\sqrt{6}}{9} \right)^{\alpha(1-\delta_2)} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3\alpha(1-\delta_2)}{2}} |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3\alpha^2(1-\delta_2)}{2} + \alpha\delta_2}. \tag{3.25}$$

Hence, we deduce from (3.22) that

$$I_2(t) = \sum_{i=1}^{k} I_{2i}(t) \le k c_3 \Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_3^{\frac{1}{p}}(t) + k c_4 \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_3^{\beta}(t). \tag{3.26}$$

Therefore, it follows from (3.6), (3.15) and (3.26) that

$$\begin{aligned} \Theta'(t) \le{}& l_1 \Theta^{\frac{m+1}{m}}(t) + \widetilde{l_2} \Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}} (t) I_3^{\frac{3}{2p}}(t) + \widetilde{l_3} \Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_3^{\frac{1}{p}}(t) \\ & + \widetilde{l_4} \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_3^{\beta}(t) - m(r-1)[m(r-1)-1] I_3(t), \end{aligned} \tag{3.27}$$

where

$$l_1 = \frac{9\sqrt{3} mk(r-1)C}{2\rho_0} \rho_0^{-\frac{3}{2}} |\Omega|^{\frac{m-2}{2m}} > 0, \tag{3.28}$$

$$\widetilde{l_2} = \frac{\sqrt{6} mk(r-1)C}{3\rho_0} \left(\frac{(m+1)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3}{2}} |\Omega|^{\left(1-\frac{2(m+1)(1-\delta_1)}{3m}\right)\frac{3(p-1)}{2p}} > 0, \tag{3.29}$$

$$\widetilde{l_3} = \frac{mk(m+1)(r-1)^2 Cd}{\rho_0} \left(3\sqrt{3} \rho_0^{-\frac{3}{2}} \right)^{\alpha(1-\delta_2)} |\Omega|^{\frac{\alpha}{2m}(m-2+(m+2)\delta_2)} > 0, \tag{3.30}$$

and

$$\begin{aligned} \widetilde{l_4} ={}& \left(\frac{2\sqrt{6}}{9} \right)^{\alpha(1-\delta_2)} \frac{mk(m+1)(r-1)^2 Cd}{\rho_0} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3\alpha(1-\delta_2)}{2}} \\ & \times |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3\alpha^2(1-\delta_2)}{2} + \alpha\delta_2} > 0. \end{aligned} \tag{3.31}$$

Next, by using the fundamental inequality

$$a_1^{r_1}a_2^{r_2} \le r_1a_1 + r_2a_2, \quad a_1, a_2 > 0, r_1, r_2 > 0 \quad \text{and} \quad r_1 + r_2 = 1, \tag{3.32}$$

we have

$$\begin{aligned}\Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}(t)I_3^{\frac{3}{2p}}(t) &= (\varepsilon_1 I_3(t))^{\frac{3}{2p}}\left[\frac{\Theta^{\frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}}(t)}{\varepsilon_1^{\frac{3}{2p-3}}}\right]^{1-\frac{3}{2p}} \\ &\le \frac{3}{2p}\varepsilon_1 I_3(t) + \left(1-\frac{3}{2p}\right)\varepsilon_1^{\frac{3}{3-2p}}\Theta^{\frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}}(t),\end{aligned} \tag{3.33}$$

where ε_1 is an arbitrary positive constant.

Similarly, we obtain

$$\Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}}(t)I_3^{\frac{1}{p}}(t) \le \frac{1}{p}\varepsilon_2 I_3(t) + \left(1-\frac{1}{p}\right)\varepsilon_2^{\frac{1}{1-p}}\Theta^{\frac{\alpha p(m+2)(1-\delta_2)}{m(p-1)}}(t) \tag{3.34}$$

and

$$\Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}}(t)I_3^{\beta}(t) \le \beta\varepsilon_3 I_3(t) + (1-\beta)\varepsilon_3^{\frac{\beta}{\beta-1}}\Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m(1-\beta)}}(t), \tag{3.35}$$

where $\varepsilon_i, i = 2,3$ are arbitrary positive constants.

Choosing the arbitrary positive constants ε_i $(i = 1,2,3)$ such that

$$\frac{3}{2p}\varepsilon_1\widetilde{l_2} + \frac{1}{p}\varepsilon_2\widetilde{l_3} + \beta\varepsilon_3\widetilde{l_4} - m(r-1)[m(r-1)-1] = 0, \tag{3.36}$$

it follows from (3.27),(3.33)–(3.35) that

$$\begin{aligned}\Theta'(t) &\le l_1\Theta^{\frac{m+1}{m}}(t) + l_2\Theta^{\frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}}(t) + l_3\Theta^{\frac{\alpha p(m+2)(1-\delta_2)}{m(p-1)}}(t) + l_4\Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m(1-\beta)}}(t) \\ &= l_1\Theta^{\alpha_1}(t) + l_2\Theta^{\alpha_2}(t) + l_3\Theta^{\alpha_3}(t) + l_4\Theta^{\alpha_4}(t),\end{aligned} \tag{3.37}$$

where

$$l_2 = \left(1-\frac{3}{2p}\right)\varepsilon_1^{\frac{3}{3-2p}}\widetilde{l_2}, \tag{3.38}$$

$$l_3 = \left(1-\frac{1}{p}\right)\varepsilon_2^{\frac{1}{1-p}}\widetilde{l_3}, \tag{3.39}$$

$$l_4 = (1-\beta)\varepsilon_3^{\frac{\beta}{\beta-1}}\widetilde{l_4}, \tag{3.40}$$

$$\alpha_1 = \frac{m+1}{m}, \tag{3.41}$$

$$\alpha_2 = \frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}, \tag{3.42}$$

$$\alpha_3 = \frac{\alpha p(m+2)(1-\delta_2)}{m(p-1)}, \tag{3.43}$$

and

$$\alpha_4 = \frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m(1-\beta)}. \tag{3.44}$$

Integrating (3.37) over $(0,t)$, we derive

$$t \geq \int_{\Theta(0)}^{\Theta(t)} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi. \tag{3.45}$$

As $(u_1, u_2, \ldots, u_k)$ blows up, we obtain the lower bound for the blow-up time t^* as follows

$$t \geq \int_{\Theta(0)}^{\infty} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi. \tag{3.46}$$

Clearly, it is unlikely that the quantity $\int_{\Theta(0)}^{\infty} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi$ can be evaluated exactly. However a lower bound for the integral may be obtained as follows. Let

$$g(\Theta) = \begin{cases} L\Theta^{\alpha_m}, & \text{if } \Theta(t) < 1, \\ L\Theta^{\alpha_M}, & \text{if } \Theta(t) > 1, \end{cases} \tag{3.47}$$

where $\alpha_m = \min_i\{\alpha_i\}$, $\alpha_M = \max_i\{\alpha_i\}$, $(i = 1, 2, 3, 4)$ and $L = \sum_{i=1}^{4} l_i$. Then we have

$$t \geq \int_{\Theta(0)}^{\infty} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi \geq \int_{\Theta(0)}^{\infty} \frac{1}{g(\xi)} d\xi. \tag{3.48}$$

The proof of Theorem 1.2 is complete. □

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