Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **86**, 1–14; https://doi.org/10.14232/ejqtde.2020.1.86 www.math.u-szeged.hu/ejqtde/

Semi-linear impulsive higher order boundary value problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Feliz Minhós ✉ [1,2] and **Rui Carapinha**[2]

[1]Departamento de Matemática, Escola de Ciências e Tecnologia, Universidade de Évora
[2]Centro de Investigação em Matemática e Aplicações (CIMA),
Instituto de Investigação e Formação Avançada, Universidade de Évora,
Rua Romão Ramalho, 59, 7000-671 Évora, Portugal

Received 18 May 2020, appeared 21 December 2020
Communicated by Alberto Cabada

Abstract. This paper considers two-point higher order impulsive boundary value problems, with a strongly nonlinear fully differential equation with an increasing homeomorphism. It is stressed that the impulsive effects are defined by very general functions, that can depend on the unknown function and its derivatives, till order $n-1$.

The arguments are based on the lower and upper solutions method, together with Leray–Schauder fixed point theorem. An application, to estimate the bending of a one-sided clamped beam under some impulsive forces, is given in the last section.

Keywords: higher order boundary value problems, generalized impulsive conditions, upper and lower solutions, fixed point theory.

2020 Mathematics Subject Classification: 34B37, 34B10, 34B15.

1 Introduction

In this article we study the two point boundary value problem composed by the one-dimensional ϕ-Laplacian equation

$$(\phi(u^{(n-1)}(t)))' + q(t)f(t,u(t),\dots,u^{(n-1)}(t)) = 0, \qquad t\in[a,b]\setminus\{t_1,\dots,t_m\}, \tag{1.1}$$

where ϕ is an increasing homeomorphism such that $\phi(0)=0$ and $\phi(\mathbb{R})=\mathbb{R}$, $q\in L^\infty[a,b]$ is a positive function and $f:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ is a L^1-Carathéodory function, together with the boundary conditions

$$u^{(j)}(a)=A_j,\quad u^{(n-1)}(b)=B, \qquad j=0,1,\dots,n-2,\ \ A_j,B\in\mathbb{R}, \tag{1.2}$$

✉ Corresponding author. Email:fminhos@uevora.pt

and the impulsive conditions

$$\begin{aligned} \Delta u^{(i)}(t_k) &= I_{i,k}(u(t_k), u'(t_k), \ldots, u^{(n-1)}(t_k)), \qquad i = 0,1,\ldots,n-2, \\ \Delta\phi(u^{(n-1)}(t_k)) &= I_{n-1,k}(u(t_k), u'(t_k),, \ldots, u^{(n-1)}(t_k)), \end{aligned} \tag{1.3}$$

being $\Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-)$, $i = 0,1,\ldots,n-1$, $k = 1,2,\ldots,m$, $I_{i,k} \in C(\mathbb{R}^n, \mathbb{R})$, and t_k fixed points such that $a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$.

Impulsive boundary value problems have been studied by many authors where it is highlighted the huge possibilities of applications to phenomena where a sudden variation happens. Indeed, these types of jumps occur in different areas such as population dynamics, engineering, control, and optimization theory, medicine, ecology, biology and biotechnology, economics, pharmacokinetics, and many other fields.

From a large number of items existent in the literature on classical impulsive differential problems, we mention, for instance, [1, 2, 17, 19–21] and the references therein. The most applied arguments are based on critical point theory and variational methods [18, 22, 26], fixed point theory on cones [6, 29], bifurcation results [13, 15], and upper and lower solutions techniques suggested in [4, 5, 12, 14].

In the last years, p-Laplacian and ϕ-Laplacian operators have been applied to semi-linear, quasi-linear, and strongly nonlinear differential equations, in singular and regular cases, increasing the range of theoretical and practical applications, as it can be seen, for example, in [3, 11, 25, 27, 28, 30] and in their references. However, impulsive problems with this type of nonlinear differential equations are scarce.

In [16], the third order differential equation

$$(\phi(u''(t)))' + q(t)f(t, u(t), u'(t), u''(t)) = 0, \qquad t \in [a,b] \backslash \{t_1, \ldots, t_n\},$$

is studied, where ϕ is an increasing homeomorphism, $q \in C([a,b])$ with $q > 0$, $f \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$, the two-point boundary conditions

$$u(a) = A, \quad u'(a) = B, \quad u''(b) = C, \qquad A, B, C \in \mathbb{R},$$

and the impulsive effects are given by

$$\begin{aligned} \Delta u(t_k) &= I_{1k}(t_k, u(t_k), u'(t_k)), \\ \Delta u'(t_k) &= I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)), \\ \Delta\phi(u''(t_k)) &= I_{3k}(t_k, u(t_k), u'(t_k), u''(t_k)), \end{aligned}$$

where $k = 1,2,\ldots,n$, $I_{1k} \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$, and $I_{ik} \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$, $i = 2,3$.

In this work, we found a method that allows generalizing the above results to higher-order boundary value problems with impulsive functions, depending not only on the unknown function but also on its derivatives till order $n-1$. To best of our knowledge, it is the first time, where such nonlinear higher-order problems are considered with this type of generalized impulsive functions.

This paper is organized in the following way: Section 2 contains the functional framework, some definitions and an explicit form for the solution of the associated homogeneous problem. Section 3 presents the main existence and localization theorem obtained via lower and upper solutions technique and a fixed point theorem. The last section gives a technique to estimate the bending of a one-sided clamped beam under some impulsive forces and how it can be obtained some qualitative data about its variation.

2 Definitions and preliminary results

Let

$$PC^{n-1}[a,b] = \left\{ \begin{array}{l} u : u \in C^{n-1}([a,b];\mathbb{R}) \text{ for } t \neq t_k, u^{(i)}(t_k) = u^{(i)}(t_k^-), u^{(i)}(t_k^+) \\ \quad \text{exists for } k = 1,2,\ldots,m, \text{ and } i = 0,1,\ldots,n-1 \end{array} \right\}.$$

Denote $X := PC^{n-1}[a,b]$. Then X is a Banach space with norm

$$\|u\|_X = \max\{\|u^{(i)}\|_\infty,\ i = 0,1,\ldots,n-1\},$$

where

$$\|w\|_\infty = \sup_{a \le t \le b} |w(t)|.$$

Defining $J := [a,b]$ and $J' = J \backslash \{t_1,\ldots,t_m\}$, for a solution u of problem (1.1)–(1.3) one should consider $u(t) \in E$, where

$$E := PC^{n-1}(J) \cap C^n(J').$$

Next lemma provides a uniqueness result for a linear problem related to (1.1)–(1.3).

Lemma 2.1. *For $v \in PC[a,b]$, the problem composed by the differential equation*

$$(\phi(u^{(n-1)}(t)))' + v(t) = 0 \tag{2.1}$$

together with conditions (1.2), (1.3), *has a unique solution given by*

$$\begin{aligned} u(t) = & \sum_{i=0}^{n-2} \left(\left[A_i + \sum_{k\,:\,t_k<t} I_{i,k}\left(t_k, u(t_k),\ldots,u^{(n-1)}(t_k)\right) \right] \frac{(t-a)^{n-2-i}}{(n-2-i)!} \right) \\ & + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k),\ldots,u^{(n-1)}(t_k)\right) \right) ds. \end{aligned}$$

Proof. Integrating the differential equation (2.1) for $t \in (t_m, b]$ we get, by (1.2),

$$\phi\left(u^{(n-1)}(t)\right) = \phi(B) + \int_{t_m}^b v(s)ds. \tag{2.2}$$

By integration of (2.1) for $t \in (t_{m-1}, t_m]$ one has by (2.2)

$$\begin{aligned} \phi\left(u^{(n-1)}(t)\right) &= \int_t^{t_m} v(s)ds - I_{n-1,m}\left(t_m, u(t_m),\ldots,u^{(n-1)}(t_m)\right) + \phi\left(u^{(n-1)}\left(t_m^+\right)\right) \\ &= \phi\left(B\right) - I_{n-1,m}\left(t_m, u(t_m),\ldots,u^{(n-1)}(t_m)\right) + \int_t^b v(s)ds \end{aligned}$$

and so,

$$u^{(n-1)}(t) = \phi^{-1}\left(\phi\left(B\right) - I_{n-1,m}\left(t_m, u(t_m),\ldots,u^{(n-1)}(t_m)\right) + \int_t^b v(s)ds\right).$$

Therefore, for $t \in [a,b]$, we have

$$u^{(n-1)}(t) = \phi^{-1}\left(\phi(B) + \int_t^b v(s)ds - \sum_{k\,:\,t_k>t} I_{n-1,k}\left(t_k, u(t_k),\ldots,u^{(n-1)}(t_k)\right)\right). \tag{2.3}$$

Integrating (2.3), for $t \in [a, t_1]$,

$$u^{(n-2)}(t) = A_{n-2} + \int_a^t \left(\phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k)\right) \right) \right) ds.$$

By integration of (2.3) on $(t_1, t_2]$ and (1.3)

$$\begin{aligned} u^{(n-2)}(t) = {} & I_{n-2,1}\left(t_1, u(t_1), \ldots, u^{(n-1)}(t_1)\right) \\ & + \int_{t_1}^t \left(\phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k)\right) \right) \right) ds. \end{aligned}$$

Therefore, for $t \in [a, b]$,

$$\begin{aligned} u^{(n-2)}(t) = {} & \sum_{k\,:\,t_k<t} \left(I_{n-2,k}\left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k)\right) \right) + A_{n-2} \\ & + \int_a^t \left(\phi^{-1} \left(\phi(B) + \int_s^b v(r)dr - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k)\right) \right) \right) ds. \end{aligned}$$

Following the same method, by iterate integrations and (1.3), we obtain for $t \in [a, b]$

$$\begin{aligned} u(t) = {} & \sum_{i=0}^{n-2} \left(\left[A_i + \sum_{k\,:\,t_k<t} I_{i,k}\left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k)\right) \right] \frac{(t-a)^{n-2-i}}{(n-2-i)!} \right) \\ & + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1} \left(\phi(B) + \int_s^b v(s)ds - \sum_{k\,:\,t_k>s} I_{n-1,k}\left(t_k, u(t_k), \ldots, u^{(n-1)}(t_k)\right) \right) ds. \end{aligned}$$

□

Lower and upper solutions will play a key role in our method, and they are defined as it follows:

Definition 2.2. A function $\alpha(t) \in E$ with $\phi(\alpha^{(n-1)}(t)) \in PC^1[a, b]$ is a lower solution of problem (1.1), (1.2), (1.3) if

$$\begin{cases} \left(\phi\left(\alpha^{(n-1)}(t)\right)\right)' + q(t) f\left(t, \alpha(t), \alpha'(t), \ldots, \alpha^{(n-1)}(t)\right) \geq 0 \\ \alpha^{(j)}(a) \leq A_j, \ j = 0, 1, \ldots, n-2, \\ \alpha^{(n-1)}(b) \leq B \\ \Delta\alpha^{(i)}(t_k) \leq I_{i,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)), \ i = 0, 1, \ldots, n-3, \\ \Delta\alpha^{(n-2)}(t_k) > I_{n-2,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)) \\ \Delta\phi(\alpha^{(n-1)}(t_k)) > I_{n-1,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)), \end{cases} \tag{2.4}$$

for $k = 1, 2, \ldots, m$.

A function $\beta(t) \in E$ such that $\phi(\beta^{(n-1)}(t)) \in PC^1[a, b]$ is an upper solution of (1.1)–(1.3) if it satisfies the opposite inequalities.

To control the derivative $u^{(n-1)}(t)$ we will apply the Nagumo condition:

Definition 2.3. An L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition related to a pair of functions $\gamma, \Gamma \in E$, with $\gamma^{(i)}(t) \leq \Gamma^{(i)}(t)$, for $i = 0, 1, \ldots, n-2$, and $t \in [a,b]$, if there exists a function $\psi : C([0,+\infty),]0,+\infty))$ such that

$$|f(t, x_0, x_1, \ldots, x_{n-1})| \leq \psi(|x_{n-1}|), \quad \text{for all } (t, x_0, x_1, \ldots, x_{n-1}) \in S \tag{2.5}$$

with

$$S := \{(t, x_0, x_1, \ldots, x_{n-1}) \in [a,b] \times \mathbb{R}^n : \gamma^{(i)}(t) \leq x_i \leq \Gamma^{(i)}(t), i = 0, 1, \ldots, n-2\},$$

and

$$\int_{\phi(\mu)}^{+\infty} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(s)ds, \tag{2.6}$$

where

$$\mu := \max_{k=0,1,2,\ldots,m} \left\{ \left| \frac{\Gamma^{(n-2)}(t_{k+1}) - \gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\gamma^{(n-2)}(t_{k+1}) - \Gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} \right| \right\}.$$

From the Nagumo condition we deduce an *a priori* estimation for $u^{(n-1)}(t)$:

Lemma 2.4. *If the L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition in the set S, referred to the functions γ and Γ, then there is $N \geq \mu > 0$ such that every solution u of the differential equation (1.1) verifies $\|u^{(n-1)}\|_\infty \leq N$.*

Proof. Let $u(t)$ be a solution of (1.1) such that

$$\gamma^{(i)}(t) \leq u^{(i)}(t) \leq \Gamma^{(i)}(t), \quad \text{for } i = 0, 1, \ldots, n-2 \text{ and } t \in [a,b].$$

By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ with

$$u^{(n-1)}(\eta_0) = \frac{u^{(n-2)}(t_{k+1}) - u^{(n-2)}(t_k)}{t_{k+1} - t_k}, \quad \text{with } k = 0, 1, 2, \ldots, m.$$

Moreover,

$$\begin{aligned} -N \leq -\mu \leq \frac{\gamma^{(n-2)}(t_{k+1}) - \Gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} &\leq u^{(n-1)}(\eta_0) \\ \leq \frac{\Gamma^{(n-2)}(t_{k+1}) - \gamma^{(n-2)}(t_k)}{t_{k+1} - t_k} &\leq \mu \leq N. \end{aligned} \tag{2.7}$$

If $\left|u^{(n-1)}(t)\right| \leq N$ for every $t \in [a,b]$, the proof is complete.

On the contrary, assume that there is $\tau \in [a,b]$ such that $\left|u^{(n-1)}(\tau)\right| > N$. Consider the case where $u^{(n-1)}(\tau) > N$. Therefore there is η_1 such that $u^{(n-1)}(\eta_1) = N$. Suppose, without loss of generality, that $\eta_0 < \eta_1$. So,

$$u^{(n-1)}(t) > 0 \quad \text{and} \quad u^{(n-1)}(\eta_0) \leq u^{(n-1)}(t) \leq N, \quad \text{for } t \in [\eta_0, \eta_1].$$

So

$$|\phi(u^{(n-1)}(t))| = |q(t) f(t, u(t), \ldots, u^{(n-1)}(t))| \leq q(t) |\psi(u^{(n-1)}(t))|, \quad \text{for } t \in [\eta_0, \eta_1],$$

and

$$\int_{\phi(u^{(n-1)}(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} \leq \int_{\eta_0}^{\eta_1} \frac{|(\phi(u^{(n-1)}(t))'|}{\psi(u^{(n-1)}(t))} dt$$
$$= \int_{\eta_0}^{\eta_1} \frac{|q(t)f(t,u(t),\dots,u^{(n-1)}(t))|}{\psi(u^{(n-1)}(t))} dt \leq \int_{\eta_0}^{\eta_1} q(t)dt < \int_a^b q(t)dt.$$

As $u^{(n-1)}(\eta_0) \leq \mu < N$, by the monotony of ϕ,

$$\phi(u^{(n-1)}(\eta_0)) \leq \phi(\mu)$$

and, by (2.6),

$$\int_{\phi(u^{(n-1)}(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} \geq \int_{\phi(\mu)}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s))} > \int_a^b q(t)dt$$

which leads to a contradiction.

The other cases, that is, $u^{(n-1)}(\tau) > N$ with $\eta_1 < \eta_0$, and $u^{(n-1)}(\tau) < -N$ with $\eta_0 < \eta_1$ or $\eta_1 < \eta_0$, follow the same arguments to obtain a contradiction.

Therefore $\left|u^{(n-1)}(t)\right| \leq N$, for $t \in [a,b]$. □

Forward, in our method, we will use the following lemma, given in [23]:

Lemma 2.5. *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = \max\{v, \min\{u,w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\frac{d}{dx}q(x,u(x))$ exists for a.e. $x \in I$.

(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \quad \text{for a.e. } x \in I.$$

We recall the classical Schauder's fixed point theorem:

Theorem 2.6. *Let M be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $T : M \to M$ is a compact operator. Then T as at least one fixed point in M.*

3 Existence and localization result

The main result is an existence and localization theorem, as it provides not only the existence of solutions but also some of its qualitative properties.

Theorem 3.1. *Suppose that there are α and β lower and upper solutions, respectively, of problem (1.1)–(1.3) such that*

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t), \quad \text{for } t \in [a,b].$$

Assume that the L^1-Carathéodory function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo condition, related to α and β, and verifies

$$f(t,\alpha(t),\dots,\alpha^{(n-3)}(t)),y,z) \leq f(t,x_0,\dots,x_{n-1}) \leq f(t,\beta(t),\dots,\beta^{(n-3)}(t),y,z), \tag{3.1}$$

for $\alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t)$, *for* $i = 0, \ldots, n-3$, *and fixed* $(y, z) \in \mathbb{R}^2$.
Moreover, if the impulsive functions satisfy

$$I_{j,k}(\alpha(t_k), \ldots, \alpha^{(n-1)}(t_k)) \leq I_{j,k}(x_0, \ldots, x_{n-1}) \leq I_{j,k}(\beta(t_k), \ldots, \beta^{(n-1)}(t_k)), \tag{3.2}$$

for $j = 0, \ldots, n-3$, $\alpha^{(i)}(t_k) \leq x_i \leq \beta^{(i)}(t_k)$, *for* $i = 0, 1, \ldots, n-2$, $k = 1, 2, \ldots, m$, *and*

$$\begin{aligned} I_{n-2,k}(\alpha(t_k), \ldots, \alpha^{(n-3)}(t_k), y, z) &\geq I_{n-2,k}(x_0, \ldots, x_{n-1}) \\ &\geq I_{n-2,k}(\beta(t_k), \ldots, \beta^{(n-3)}(t_k), y, z) \end{aligned} \tag{3.3}$$

for $\alpha^{(i)}(t) \leq x_i \leq \beta^{(i)}(t)$, *for* $i = 0, \ldots, n-3$, *and fixed* $(y, z) \in \mathbb{R}^2$, *then problem* (1.1)–(1.3) *has at least one solution* $u \in E$, *such that*

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \quad \text{for } i = 0, 1, \ldots, n-2 \text{ and } -N \leq u^{(n-1)}(t) \leq N,$$

for $t \in [a, b]$ *and* N *given by* (2.7).

Proof. Define the continuous functions δ_i, for $i = 0, 1, \ldots, n-2$,

$$\delta_i(t, u^{(i)}(t)) = \begin{cases} \beta^{(i)}(t), & u^{(i)}(t) \geq \beta^{(i)}(t) \\ u^{(i)}(t), & \alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & u^{(i)}(t) \leq \alpha^{(i)}(t) \end{cases}$$

and consider the following modified and perturbed equation

$$\begin{aligned} (\phi(u^{(n-1)}(t)))' + q(t) f\left(t, \delta_0(t, u(t)), \ldots, \delta_{n-2}(t, u^{(n-2)}(t)), \frac{d}{dt}\left(\delta_{n-2}(t, u^{(n-2)}(t))\right)\right) \\ + \frac{\delta_{n-2}(t, u^{(n-2)}(t)) - u^{(n-2)}(t)}{1 + |u^{(n-2)}(t) - \delta_{n-2}(t, u^{(n-2)}(t))|} = 0, \end{aligned} \tag{3.4}$$

coupled with boundary conditions (1.2) and the truncated impulsive conditions, for $i = 0, 1, \ldots, n-2$,

$$\begin{aligned} \Delta u^{(i)}(t_k) &= I_{i,k}\left(\begin{array}{c} \delta_0(t_k, u(t_k)), \ldots, \delta_{n-2}(t_k, u^{(n-2)}(t_k), \\ \frac{d}{dt}(\delta_{n-2}(t_k, u^{(n-2)}(t_k))) \end{array} \right) := I^*_{i,k}(t_k), \\ \Delta \phi(u^{(n-1)}(t)) &= I_{n-1,k}\left(\begin{array}{c} \delta_0(t_k, u(t_k)), \ldots, \delta_{n-2}(t_k, u^{(n-2)}(t_k)), \\ \frac{d}{dt}(\delta_{n-2}(t_k, u^{(n-2)}(t_k))) \end{array} \right) := I^*_{n-1,k}(t_k). \end{aligned} \tag{3.5}$$

Define the operator $T : E \to E$ by

$$\begin{aligned} T(u)(t) := &\sum_{i=0}^{n-2} \left(\left[A_i + \sum_{k\,:\,t_k < t} I^*_{i,k} \right] \frac{(t-a)^{n-2-i}}{(n-2-i)!} \right) \\ &+ \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \phi^{-1}\left(\phi(B) + \int_s^b v(s) ds - \sum_{k\,:\,t_k > s} I^*_{n-1,k} \right) ds. \end{aligned}$$

By Lemma 2.1, it is clear that the fixed points of T, u_*, are solutions of the initial problem (1.1)-(3.5), if they verify

$$\alpha^{(i)}(t) \leq u_*^{(i)}(t) \leq \beta^{(i)}(t), \quad \text{for } t \in [a, b] \text{ and } i = 0, 1, \ldots, n-2.$$

As T is compact, by Schauder's fixed point theorem, T has a fixed point $u \in E$, which is a solution of (3.4), (1.2), (3.5). To prove that this solution verifies

$$\alpha^{(i)}(t) \le u^{(i)}(t) \le \beta^{(i)}(t), \quad \text{for } t \in [a,b], \text{ and } i = 0,1,\ldots,n-2,$$

suppose, by contradiction, that, for $i = n-2$, there is $t \in [a,b]$ such that

$$u^{(n-2)}(t) > \beta^{(n-2)}(t).$$

Define $\zeta \in [a,b]$ as

$$\sup_{t\in[a,b]} (u^{(n-2)}(t) - \beta^{(n-2)}(t)) := u^{(n-2)}(\zeta) - \beta^{(n-2)}(\zeta)) > 0. \tag{3.6}$$

By (1.2) and Definition 2.2, $u^{(n-2)}(a) - \beta^{(n-2)}(a) \le 0$, then $\zeta \neq a$. On the other hand $u^{(n-1)}(b) - \beta^{(n-1)}(b) < 0$ and then $\zeta \neq b$, by (3.6).

Therefore $\zeta \in]a,b[$.

Case 1: Assume that there is $p \in \{1,2,\ldots,m\}$ such that $\zeta \in (t_p, t_{p+1})$.

Consider $\epsilon > 0$ small enough such that

$$u^{(n-2)}(t) - \beta^{(n-2)}(t) > 0 \quad \text{and} \quad u^{(n-1)}(t) - \beta^{(n-1)}(t) \le 0, \quad \text{for } t \in (\zeta, \zeta+\epsilon). \tag{3.7}$$

Therefore, by (3.1) and (3.7), for all $t \in (\zeta, \zeta+\epsilon)$, we have the following contradiction

$$\begin{aligned}
0 &\ge \phi\left(u^{(n-1)}(t)\right)' - \phi\left(\beta^{(n-1)}(t)\right)' \\
&\ge -q(t)f\left(t, \delta_0(t,u(t)),\ldots,\delta_{n-2}(t,u^{(n-2)}(t)), \frac{d}{dt}\left(\delta_{n-2}(t,u^{(n-2)}(t))\right)\right) \\
&\quad - \frac{\delta_{n-2}\left(t,u^{(n-2)}(t)\right) - u^{(n-2)}(t)}{1+\left|u^{(n-2)}(t) - \delta_{n-2}\left(t,u^{(n-2)}(t)\right)\right|} + q(t)f\left(t,\beta(t),\ldots,\beta^{(n-1)}(t)\right) \\
&= -q(t)f\left(t,\delta_0(t,u(t)),\ldots,\delta_{n-3}\left(t,u(t)\right),\beta^{(n-2)}(t),\beta^{(n-1)}(t)\right) \\
&\quad - \frac{\beta^{(n-2)}(t) - u^{(n-2)}(t)}{1+\left|u^{(n-2)}(t)-\beta^{(n-2)}(t)\right|} + q(t)f\left(t,\beta(t),\ldots,\beta^{(n-1)}(t)\right) \\
&\ge -q(t)f\left(t,\beta(t),\ldots,\beta^{(n-1)}(t)\right) - \frac{\beta^{(n-2)}(t) - u^{(n-2)}(t)}{1+\left|u^{(n-2)}(t)-\beta^{(n-2)}(t)\right|} \\
&\quad + q(t)f\left(t,\beta(t),\ldots,\beta^{(n-1)}(t)\right) = \frac{u^{(n-2)}(t) - \beta^{(n-2)}(t)}{1+\left|u^{(n-2)}(t)-\beta^{(n-2)}(t)\right|} > 0.
\end{aligned}$$

Case 2: Consider that there exists $k \in \{1,2,\ldots,m\}$ such that, or

$$\max_{t\in[a,b]}\left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) := u^{(n-2)}(t_k^-) - \beta^{(n-2)}(t_k^-) > 0 \tag{3.8}$$

or

$$\sup_{t\in[a,b]}\left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) := u^{(n-2)}(t_k^+) - \beta^{(n-2)}(t_k^+) > 0. \tag{3.9}$$

If (3.8) holds, then

$$\Delta\left(u^{(n-2)}(t) - \beta^{(n-2)}(t)\right) \le 0$$

and, by (3.3) and Definition 2.2, we have the contradiction

$$\begin{aligned} 0 &\geq \Delta u^{(n-2)}(t_k) - \Delta\beta^{(n-2)}(t_k) = I^*_{n-2,k} - \Delta\beta^{(n-2)}(t_k) \\ &= I_{n-2,k}\left(\delta_0(t,u(t)),\ldots,\delta_{n-3}\left(t,u(t)\right),\beta^{(n-2)}(t),\beta^{(n-1)}(t)\right) - \Delta\beta^{(n-2)}(t_k) \\ &\geq I_{n-2,k}\left(t_k,\beta(t_k),\ldots,\beta^{(n-1)}(t_k)\right) - \Delta\beta^{(n-2)}(t_k) > 0. \end{aligned}$$

Consider now (3.9). So, there is $\epsilon > 0$ such that, for $t \in (t_k, t_k + \epsilon)$,

$$u^{(n-1)}(t) - \beta^{(n-1)}(t) \leq 0,$$

and the arguments follow by the same technique as in Case 1, to have

$$u^{(n-2)}(t) \leq \beta^{(n-2)}(t), \qquad \forall t \in [a,b].$$

To prove that $u^{(n-2)}(t) \geq \alpha^{(n-2)}(t),\ \forall t \in [a,b]$, the method is similar. Therefore

$$\alpha^{(n-2)}(t) \leq u^{(n-2)}(t) \leq \beta^{(n-2)}, \quad \text{for } t \in [a,b].$$

Integrating the first inequality in $[a,t_1]$, we have

$$\begin{aligned} \alpha^{(n-3)}(t) &\leq u^{(n-3)}(t) - u^{(n-3)}(a) + \alpha^{(n-3)}(a) \\ &= u^{(n-3)}(t) - A_{n-3} + \alpha^{(n-3)}(a) \leq u^{(n-3)}(t). \end{aligned} \tag{3.10}$$

For $t \in (t_1,t_2]$, by (3.2) and (3.10),

$$\begin{aligned} \alpha^{(n-3)}(t) &\leq u^{(n-3)}(t) - u^{(n-3)}(t_1^+) + \alpha^{(n-3)}(t_1^+) \\ &\leq u^{(n-3)}(t) - I^*_{n-3,1}(t_1) - u^{(n-3)}(t_1) \\ &\quad + I_{n-3,1}\left(t_1,\alpha(t_1),\ldots,\alpha^{n-1}(t_1)\right) + \alpha^{(n-3)}(t_1) \\ &\leq u^{(n-3)}(t) - I^*_{n-3,1}(t_1) + I_{n-3,1}\left(t_1,\alpha(t_1),\ldots,\alpha^{(n-1)}(t_1)\right) \\ &\leq u^{(n-3)}(t). \end{aligned}$$

Applying this method for each interval $(t_k,t_{k+1}]$, $k = 2,\ldots,m$, we obtain

$$\alpha^{(n-3)}(t) \leq u^{(n-3)}(t), \qquad \forall t \in [a,b],$$

and, by the same technique,

$$\beta^{(n-3)}(t) \geq u^{(n-3)}(t), \qquad \forall t \in [a,b].$$

By iteration of these arguments, we conclude

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \quad \text{for } i = 0,1,\ldots,n-2, \text{ and } t \in [a,b].$$

The estimation $\left|u^{(n-1)}(t)\right| \leq N$ is a trivial consequence of Lemma 2.4. □

4 Estimation for the bending of one-sided clamped beam under impulsive effects

Problems related to beam structures and especially beams that support some forces as impulses, are part of a vast field of investigation in boundary value problems theory, see, for example, [7–10, 24].

In this application we consider a model to describe the bending of a beam with length $L > 1$, given by the fourth-order equation

$$\frac{EI}{A}u^{(4)}(x) + \frac{3}{2}\sqrt[3]{u'(x)}|u''(x)| - ku(x) - \gamma u'''(x) = 0, \quad \text{for } x \in]0, L[, \tag{4.1}$$

where $E > 0$ is the Young modulus, $I > 0$ the mass moment of inertia, $A > 0$ the cross section area, $k > 0$ the tension of a spring force vertically applied on the beam, and $\gamma > 0$ the shear force coefficient.

At the end points the behavior of the beam is given by the following boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0, \quad u'''(L) = 0, \tag{4.2}$$

meaning that the beam is clamped on the left end side.

For clearance, we consider only one moment of impulse which occurs at $t_1 = 1$. The impulsive effects are given by generalized functions with dependence on the unknown function itself, and on several derivatives till order three,

$$\begin{aligned} \Delta u(1) &= u(1) + u'(1) - 2u''(1) - u'''(1) \\ \Delta u'(1) &= u(1) + u'(1) - 2u''(1) - u'''(1) \\ \Delta u''(1) &= -u(1) - u'(1) + u''(1) + 5u'''(1) - 1 \\ \Delta u'''(1) &= u(1) - u'(1) + u''(1) + u'''(1) - 1. \end{aligned} \tag{4.3}$$

This problem (4.1)-(4.3) is a particular case of (1.1)-(1.3) with $[a, b] = [0, L]$, $n = 4$,

$$f(x, y_0, y_1, y_2, y_3) = \frac{A}{EI}\left(\frac{3}{2}\sqrt[3]{y_1}|y_2| - ky_0 - \gamma y_3\right), \tag{4.4}$$

$\phi(w) = w, q(t) \equiv 1, m = 1$, $t_1 = 1$, and the impulsive functions given by

$$\begin{aligned} I_{0,1}(w_0, w_1, w_2, w_3) &= w_0 + w_1 - 2w_2 - w_3 \\ I_{1,1}(w_0, w_1, w_2, w_3) &= w_0 + w_1 - 2w_2 - w_3 \\ I_{2,1}(w_0, w_1, w_2, w_3) &= -w_0 - w_1 + w_2 + 5w_3 - 1 \\ I_{3,1}(w_0, w_1, w_2, w_3) &= w_0 - w_1 + w_2 + w_3 - 1. \end{aligned}$$

As a numeric example we can consider $A = 1$, $EI = 1$, $k = 1$, $\gamma = 6$, $L = 2$. In this case, the continuous functions

$$\alpha(x) = 0, \quad \beta(x) = \frac{x^3}{6} + x^2 + x, \quad \text{for } x \in [0, 2],$$

are, respectively, lower and upper solutions of problem (4.1)–(4.3), according to Definition 3.6.

In fact, for $\alpha(x) \equiv 0$ the inequalities are trivially satisfied and for β, we have,

$$\beta(0) = 0, \quad \beta'(0) = 1, \quad \beta''(0) = 2 > 0, \quad \beta'''(2) = 1 > 0,$$
$$\Delta\beta(1) = 0 \geq \beta(0) + \beta'(0) - 2\beta''(0) - \beta'''(0) = -\frac{4}{3},$$
$$\Delta\beta'(1) = 0 \geq \beta(0) + \beta'(0) - 2\beta''(0) - \beta'''(0) = -\frac{4}{3},$$
$$\Delta\beta''(1) = 0 < -\beta(0) - \beta'(0) + \beta''(0) + 5\beta'''(0) - 1 = \frac{4}{3}$$
$$\Delta\beta'''(1) = 0 < \beta(0) - \beta'(0) + \beta''(0) + \beta'''(0) - 1 = \frac{5}{3}.$$

The nonlinear part $f(x, y_0, y_1, y_2, y_3)$, given by (4.4), verifies a Nagumo condition on the set

$$S_* = \left\{ \begin{array}{l} (t, y_0, y_1, y_2, y_3) \in [0,2] \times \mathbb{R}^n : 0 \leq y_0 \leq \frac{x^3}{6} + x^2 + x, \\ \qquad\qquad 0 \leq y_1 \leq \frac{x^2}{2} + 2x + 1, 0 \leq y_2 \leq x + 2 \end{array} \right\}$$

with

$$\mu = \max\left\{ |\beta''(2)|, |\beta''(0)|, |\beta''(1)| \right\} = 4,$$
$$\psi(|y_3|) := |y_3| + \frac{22}{3},$$

and

$$\int_\mu^{+\infty} \frac{ds}{s + \frac{22}{3}} = +\infty > \int_0^L 1\, ds = L.$$

Moreover, f is nondecreasing on y_0 and, by Theorem 3.1, there exists a solution $u(x)$ of problem (4.1)–(4.3) such that

$$\alpha^{(i)}(x) \leqslant u^{(i)}(x) \leqslant \beta^{(i)}(x), \qquad i = 0, 1, 2, \text{ for } x \in [0,2],$$

that is

$$0 \leq u(x) \leq \frac{x^3}{6} + x^2 + x,$$
$$0 \leq u'(x) \leq \frac{x^2}{2} + 2x + 1,$$
$$0 \leq u''(x) \leq x + 2, \quad \text{for } x \in [0,2].$$

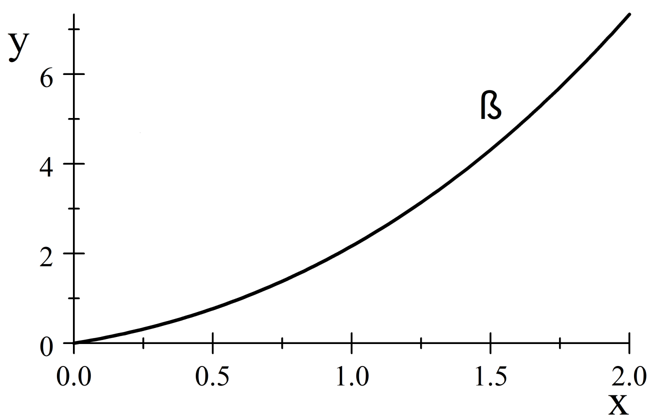

Figure 4.1: Strip of u localization.

References

[1] M. Benchohra, J. Henderson, S. Ntouyas, Impulsive differential equations and inclusions, *Contemporary Mathematics and Its Applications*, Vol. 2, Hindawi Publishing Corporation, 2006. https://doi.org/10.1155/9789775945501; MR2322133

[2] P. Chen, X. Tang, Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems, *Appl. Math. Comput.* **218**(2012), 11775–11789. https://doi.org/10.1016/j.amc.2012.05.027; MR2945181

[3] J. R. Graef, L. Kong, F. Minhós, Higher order boundary value problems with ϕ-Laplacian and functional boundary conditions, *Comput. Math. Appl.* **61**(2011), 236–249. https://doi.org/10.1016/j.camwa.2010.10.044; MR2754132

[4] J. Graef, L. Kong, F. Minhós, J. Fialho, On the lower and upper solutions method for higher order functional boundary value problems, *Appl. Anal. Discrete Math.* **5**(2011), No. 1, 133–146. https://doi.org/10.2298/AADM110221010G; MR2809041

[5] M. R. Grossinho, F. Minhós, A. I. Santos, A note on a class of problems for a higher order fully nonlinear equation under one sided Nagumo type condition, *Nonlinear Anal.* **70**(2009), 4027–4038. https://doi.org/10.1016/j.na.2008.08.011; MR2515319

[6] X. Hao, L. Liu, Y. Wu, Positive solutions for nth-order singular nonlocal boundary value problems, *Bound. Value Probl.* **2007**, Art. ID 74517, 10 pp. https://doi.org/10.1155/2007/74517; MR2304521

[7] T. S. Jang, A new semi-analytical approach to large deflections of Bernoulli–Euler–v. Karman beams on a linear elastic foundation: Nonlinear analysis of infinite beams, *Int. J. Mech. Sci.* **66**(2013), 22–32. https://doi.org/10.1016/j.ijmecsci.2012.10.005

[8] P. Kang, Z. Wei, Existence of positive solutions for systems of bending elastic beam equations, *Electron. J. Differential Equations* **2012**, No. 19, 1–9. MR2889625

[9] Y. Kuo, S. Lee, Deflection of non-uniform beams resting on a nonlinear elastic foundation, *Comput. Struct.* **51**(1994), 513–519. https://doi.org/10.1016/0045-7949(94)90058-2

[10] Y. Li, Existence of positive solutions for the cantilever beam equations with fully nonlinear terms, *Nonlinear Anal. Real World Appl.* **27**(2016), 221–237. https://doi.org/10.1016/j.nonrwa.2015.07.016; MR3400525

[11] B. Liu, J. Yu, Existence of solution for m-point boundary value problems of second-order differential systems with impulses, *Appl. Math. Comput.* **125**(2002), 155–175. https://doi.org/10.1016/S0096-3003(00)00110-7

[12] X. Liu, D. Guo, Method of upper and lower solutions for second-order impulsive integro-differential equations in a Banach space, *Comput. Math. Appl.* **38**(1999), 213–223. https://doi.org/10.1016/S0898-1221(99)00196-0; MR1703418

[13] Y. Liu, D. O'Regan, Multiplicity results using bifurcation techniques for a class of boundary value problems of impulsive differential equations, *Commun. Nonlinear Sci. Numer. Simul.* **16**(2011), 1769–1775. https://doi.org/10.1016/j.cnsns.2010.09.001; MR2736044

[14] F. Minhós, Location results: an under used tool in higher order boundary value problems, in: *Mathematical models in engineering, biology and medicine*, AIP Conf. Proc., Vol. 1124, Amer. Inst. Phys., Melville, NY, 2009, pp. 244–253. MR2657151

[15] R. Ma, B. Yang, Z. Wang, Positive periodic solutions of first-order delay differential equations with impulses, *Appl. Math. Comput.* **219**(2013), 6074–6083. https://doi.org/10.1016/j.amc.2012.12.020; MR3018451

[16] F. Minhós, R. Carapinha, On higher order nonlinear impulsive boundary value problems, *Discrete Contin. Dyn. Syst.* 2015, *Dynamical Systems, Differential Equations and Applications 10th AIMS Conference. Suppl.*, 851–860. https://doi.org/10.3934/proc.2015.0851; MR3462521

[17] F. Minhós, H. Carrasco, *High order boundary value problems on unbounded domains. Types of solutions, functional problems and applications*, Trends in Abstract and Applied Analysis, Vol. 5, World Scientific, 2017. https://doi.org/10.1142/10448

[18] J. Nieto, J. Uzal, Nonlinear second-order impulsive differential problems with dependence on the derivative via variational structure, *J. Fixed Point Theory Appl.* **22**(2020), Paper No. 19, 13 pp. https://doi.org/10.1007/s11784-019-0754-3; MR4050175

[19] J. Nieto, R. López, Boundary value problems for a class of impulsive functional equations, *Comput. Math. Appl.* **55**(2008) 2715–2731. https://doi.org/10.1016/j.camwa.2007.10.019; MR2401424

[20] J. Nieto, D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.* **10**(2009), 680–690. https://doi.org/10.1016/j.nonrwa.2007.10.022; MR2474254

[21] L. Skóra, Boundary value problem for the second orderimpulsive delay differential equations, *Ann. Univ. Paedagog. Crac. Stud. Math.* **14**(2015), 27–35. https://doi.org/10.1515/aupcsm-2015-0002

[22] Y. Tian, W. Ge, Variational methods to Sturm–Liouville boundary value problem for impulsive differential equations, *Nonlinear Anal.* **72**(2010), 277–287. https://doi.org/10.1016/j.na.2009.06.051; MR2574937

[23] M. X. Wang, A. Cabada, J. Nieto, Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions, *Ann. Polon. Math.* **58**(1993), 221–235. https://doi.org/10.4064/ap-58-3-221-235; MR1244394

[24] Y. Wang, *Some fourth order equations modeling suspension bridges*, Ph.D. dissertation, Politecnico di Milano, 2015.

[25] X. Wang, Q. Liu, D. Qian, Existence and multiplicity results for some nonlinear problems with singular φ-Laplacian via a geometric approach, *Bound. Value Probl.* **2016**, Paper No. 47, 27 pp. https://doi.org/10.1186/s13661-016-0554-2; MR3460037

[26] J. Xiao, J. Nieto, Z. Luo, Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods, *Commun. Nonlinear Sci. Numer. Simul.* **17**(2012), 426–432. https://doi.org/10.1016/j.cnsns.2011.05.015; MR2826020

[27] Y. Xin, Z. Cheng, Positive periodic solution for φ-Laplacian Rayleigh equation with strong singularity, *Bound. Value Probl.* **2016**, Paper No. 139, 9 pp. https://doi.org/10.1186/s13661-016-0648-x; MR3532368

[28] G. C. Yang, H. B. Feng, New results of positive solutions for the Sturm–Liouville problem, *Bound. Value Probl.* **2016**, Paper No. 64, 17 pp. https://doi.org/10.1186/s13661-016-0571-1; MR3479374

[29] X. Zhang, M. Feng, W. Ge, Existence of solutions of boundary value problems with integral boundary conditions for second-order impulsive integro-differential equations in Banach spaces, *J. Comput. Appl. Math.* **233**(2010), 1915–1926. https://doi.org/10.1016/j.cam.2009.07.060; MR2564027

[30] J. Zhao, B. Sun, Yu Wang, Existence and iterative solutions of a new kind of Sturm–Liouville-type boundary value problem with one-dimensional p-Laplacian, *Bound. Value Probl.* **2016**, Paper No. 86, 11 pp. https://doi.org/10.1186/s13661-016-0590-y; MR3489882