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Compactness of Riemann–Liouville fractional integral operators

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We obtain results on compactness of two linear Hammerstein integral operators with singularities, and apply the results to give new proof that Riemann–Liouville fractional integral operators of order $\alpha \in (0,1)$ map $L^p(0,1)$ to $C[0,1]$ and are compact for each $p \in (\frac{1}{1-\alpha}, \infty]$. We show that the spectral radii of the Riemann–Liouville fractional operators are zero.

Keywords: linear Hammerstein integral operator, Riemann–Liouville fractional integral operator, compactness, spectral radius.

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1 Introduction

Riemann–Liouville left-sided and right-sided fractional integral operators of order $\alpha \in (0,1)$, denoted by $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$, respectively, are two special linear Volterra integral operators with the kernel

$$k(x,y) = \frac{1}{|x-y|^\alpha}. \tag{1.1}$$

The kernel k is singular at each (x,x) and the singularities often make it difficult to study problems such as continuity and compactness of these operators defined in subspaces of $L^1(0,1)$.

It is well known that $I_{0^+}^{1-\alpha}$ is bounded from $L^p(0,1)$ to $L^p(0,1)$ for each $p \in [1,\infty]$, and from $L^p(0,1)$ to $C[0,1]$ for each $p \in (\frac{1}{1-\alpha}, \infty]$, see [6, Theorem 2.6], [11, Theorem 12], [20, Theorem 3.6] and [23, Proposition 3.2 (1) and (3)]. It is implicitly proved in [6, Theorem 6.1] that $I_{0^+}^{1-\alpha}$ is compact from $C[0,1]$ to $C[0,1]$ and in [22, Theorem 4.8] that $I_{0^+}^{1-\alpha}$ is compact from $D \subset C[0,1] \to P$, where D is a subset of $C[0,1]$ and P is the standard positive cone in $C[0,1]$.

In this paper, we prove that both $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ are compact from $L^p(0,1)$ to $C[0,1]$ for each $p \in (\frac{1}{1-\alpha}, \infty]$. This allows one to study the existence of solutions of the initial or

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boundary value problems for nonlinear fractional differential equations with discontinuous nonlinearities by applying the fixed point theorems or fixed point index theories. We refer to [2–6, 8–10, 13, 15, 18, 19, 21–27] for the study of these nonlinear problems.

To study the compactness of $I_{0^+}^{1-\alpha}$, we first study compactness of the following two linear Hammerstein integral operators L and $\mathscr{L}$:

$$Lv(x) = \int_0^1 k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{1.2}$$

and

$$\mathscr{L}v(x) = \int_0^1 \sigma(x,y)k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{1.3}$$

where $k : [0,1] \times [0,1] \setminus \mathscr{D} \to \mathbb{R}$ has singularities in a subset

$$\mathscr{D} = \{(x,y) : x \in [0,1], y \in D(x)\}$$

to be defined in Section 2, and $\sigma(x,y) = \operatorname{sgn}(x-y)$. It is not trivial to prove compactness of these operators due to the singularities of k on $\mathscr{D}$.

Under suitable assumptions on k, we prove that both L and $\mathscr{L}$ map $L^p(0,1)$ to $C[0,1]$ and are compact for $p \in [1,\infty]$. In particular, when $\mathscr{D} = \{(x,x) : x \in [0,1]\}$, we show that $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ are proportional to the sum and substraction of the two operators L_α and $\mathscr{L}_\alpha$, respectively, where L_α and $\mathscr{L}_\alpha$ are the two operators L and $\mathscr{L}$ with the kernel k defined in (1.1). When $p \in \left(\frac{1}{1-\alpha}, \infty\right]$, these relations are used to derive compactness of $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ from the compactness of L_α and $\mathscr{L}_\alpha$. As applications of compactness of $I_{0^+}^{1-\alpha}$, we show that the spectral radius of $I_{0^+}^{1-\alpha}$ is 0, and $I_{0^+}^{1-\alpha}$ has no eigenfunctions.

2 Compactness of linear integral operators

In this section, we study the following two linear Hammerstein integral operators

$$Lv(x) = \int_0^1 k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{2.1}$$

where the kernel k is allowed to have singularities on $[0,1] \times [0,1]$ and

$$\mathscr{L}v(x) = \int_0^1 \sigma(x,y)k(x,y)v(y)\,dy \quad \text{for each } x \in [0,1], \tag{2.2}$$

where $\sigma : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$\sigma(x,y) = \operatorname{sgn}(x-y) = \begin{cases} 1 & \text{if } y < x, \\ 0 & \text{if } y = x, \\ -1 & \text{if } x < y. \end{cases} \tag{2.3}$$

Unless stated otherwise, $p, q \in [1,\infty]$ are the conjugate indices, that is, they satisfy the following condition:

$$1/p + 1/q = 1, \tag{2.4}$$

where if $p = \infty$, then $q = 1$ and if $p = 1$, then $q = \infty$.

We denote by $L^p[0,1]$ and $L^p_+[0,1]$ the Banach space of functions for which the pth power of the absolute values are Lebesgue integrable with the norm $\|\cdot\|_{L^p(0,1)}$, and its positive cone, respectively, and by $C[0,1]$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ with the maximum norm denoted by $\|\cdot\|_{C[0,1]}$ or $\|\cdot\|$.

Let X, Y be Banach spaces. Recall that a linear map $L : X \to Y$ is said to be compact if L is continuous and $\overline{L(S)}$ is compact for each bounded subset $S \subset X$.

Assume that for each $x \in [0,1]$, there exists a subset $D(x)$ of $[0,1]$ satisfying $\mathrm{meas}(D(x)) = 0$. Let

$$\mathscr{D} = \{(x,y) : x \in [0,1],\ y \in D(x)\}.$$

It is easy to verify that $(x,y) \in [0,1] \times [0,1] \setminus \mathscr{D}$ if and only if $x \in [0,1]$ and $y \in [0,1] \setminus D(x)$.

Theorem 2.1. *Let $p, q \in [1,\infty]$ satisfy (2.4). Assume that $k : [0,1] \times [0,1] \setminus \mathscr{D} \to \mathbb{R}$ satisfies the following conditions.*

- (i) *For each $x \in [0,1]$, $k(x,\cdot) : [0,1] \setminus D(x) \to \mathbb{R}$ satisfies $k(x,\cdot) \in L^q(0,1)$.*
- (ii) *For each $\tau \in [0,1]$, $\lim_{x\to\tau} \|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)} = 0$.*

Then the map L defined in (2.1) maps $L^p(0,1)$ to $C[0,1]$ and is compact.

Proof. Let $v \in L^p(0,1)$. By the condition (i) we have

$$|Lv(x)| = \left|\int_0^1 k(x,y)v(y)\,dy\right| \le \|k(x,\cdot)\|_{L^q(0,1)}\|v\|_{L^p(0,1)} < \infty \quad \text{for each } x \in [0,1]$$

and Lv is well defined on $[0,1]$. For $\tau, x \in [0,1]$, we have

$$\begin{aligned} |Lv(x) - Lv(\tau)| &\le \int_0^1 |k(x,y) - k(\tau,y)||v(y)|\,dy \\ &\le \|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)}\|v\|_{L^p(0,1)}. \end{aligned} \tag{2.5}$$

It follows from the condition (ii) that $Lv \in C[0,1]$ for $v \in L^p(0,1)$ and the first part of the result holds.

We define a map $\mathscr{K} : [0,1] \to L^q(0,1)$ by

$$\mathscr{K}(x) = k(x,\cdot).$$

Then the conditions (i) and (ii) are equivalent to the fact that $\mathscr{K} : [0,1] \to L^q(0,1)$ is continuous. Hence, $\|\mathscr{K}(\cdot)\|_{L^q(0,1)} : [0,1] \to \mathbb{R}_+$ is continuous, and thus

$$M_1 := \max\{\|k(x,\cdot)\|_{L^q(0,1)} : x \in [0,1]\} < \infty.$$

Let $S \subset L^p(0,1)$ be a bounded set in $L^p(0,1)$. Then

$$M_2 := \max\{\|v\|_{L^p(0,1)} : v \in S\} < \infty.$$

Hence, for $v \in S$ and $x \in [0,1]$,

$$|Lv(x)| \le \int_0^1 |k(x,y)||v(y)|\,dy \le \|k(x,\cdot)\|_{L^q(0,1)}\|v\|_{L^p(0,1)} \le M_1M_2 < \infty.$$

Hence, $\|Lv\|_{C[0,1]} \le M_1M_2$ and $L(S)$ is bounded in $C[0,1]$. By (2.5), we have for $v \in S$ and $\tau, x \in [0,1]$,

$$|Lv(x) - Lv(\tau)| \le M_2\|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)}.$$

It follows from the condition (ii) that $L(S)$ is equicontinuous. By the Ascoli–Arzelà Theorem, $\overline{L(S)}$ is compact.

Let $\{v_n\} \subset L^p(0,1)$ and $v \in L^p(0,1)$ such that $\|v_n - v\|_{L^p(0,1)} \to 0$. Then we have for $x \in [0,1]$,

$$|Lv_n(x) - Lv(x)| \leq \left| \int_0^1 |k(x,y)||v_n(y) - v(y)|\, dy \right| \leq \|k(x,\cdot)\|_{L^q(0,1)} \|v_n - v\|_{L^p(0,1)}$$

and

$$\|Lv_n - Lv\|_{C[0,1]} \leq M_1 \|v_n - v\|_{L^p(0,1)} \to 0.$$

Hence, $L : L^p(0,1) \to C[0,1]$ is continuous and thus, is compact. □

The compactness result of Theorem 2.1 with $q = 1$ is closely related to [17, Lemma 2.1].

Lemma 2.2. *Assume that* $k : [0,1] \times [0,1] \setminus \{(x,x) : x \in [0,1]\} \to \mathbb{R}$ *satisfies the following condition.*

(H) *There exists* $q \in [1,\infty]$ *such that for each* $x \in [0,1]$, $k(x,\cdot) \in L^q(0,1)$.

Then the following assertions hold.

(1) *If* $q \in [1,\infty)$ *and*

$$\lim_{x\to\tau} \|k(x,\cdot) - k(\tau,\cdot)\|_{L^q(0,1)} = 0 \quad \textit{for some } \tau \in [0,1], \tag{2.6}$$

then

$$\lim_{x\to\tau} \|\sigma(x,\cdot)k(x,\cdot) - \sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^q(0,1)} = 0. \tag{2.7}$$

(2) *If* $q \in [1,\infty]$ *and* $k(x,y) \geq 0$ *for* $x, y \in [0,1]$ *with* $x \neq y$ *and then* (2.7) *implies* (2.6).

Proof. (1) Let $q \in [1,\infty)$. Let $x, \tau, y \in [0,1]$ with $x \neq y$ and $\tau \neq y$. If $\tau < x$, then

$$\sigma(x,y) - \sigma(\tau,y) = \begin{cases} 0 & \text{if } y < \tau, \\ 2 & \text{if } \tau < y < x, \\ 0 & \text{if } x < y. \end{cases} \tag{2.8}$$

If $x < \tau$, then

$$\sigma(x,y) - \sigma(\tau,y) = \begin{cases} 0 & \text{if } y < x, \\ -2 & \text{if } x < y < \tau, \\ 0 & \text{if } x < \tau < y. \end{cases} \tag{2.9}$$

For $x, \tau, y \in [0,1]$ with $x \neq y$ and $\tau \neq y$, let

$$\Phi(x,\tau,y) = \sigma(x,y)k(x,y) - \sigma(\tau,y)k(\tau,y).$$

Then

$$\begin{aligned} |\Phi(x,\tau,y)|^q &\leq \Big[|\sigma(x,y)||k(x,y) - k(\tau,y)| + |k(\tau,y)||\sigma(x,y) - \sigma(\tau,y)|\Big]^q \\ &\leq \Big[|k(x,y) - k(\tau,y)| + |k(\tau,y)||\sigma(x,y) - \sigma(\tau,y)|\Big]^q \\ &\leq |k(x,y) - k(\tau,y)|^q + |k(\tau,y)|^q|\sigma(x,y) - \sigma(\tau,y)|^q. \end{aligned} \tag{2.10}$$

Assume that (1) holds. If $x, \tau, y \in [0,1]$ with $x \neq y$, $\tau \neq y$ and $\tau < x$, then by (2.8) and (2.10), we have

$$\begin{aligned}\int_0^1 |\Phi(x,\tau,y)|^q &\leq \int_0^1 |k(x,y)-k(\tau,y)|^q + |k(\tau,y)|^q|\sigma(x,y)-\sigma(\tau,y)|^q\,dy\\ &= \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + \int_0^1 |k(\tau,y)|^q|\sigma(x,y)-\sigma(\tau,y)|^q\,dy\\ &= \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + \int_\tau^x |k(\tau,y)|^q|\sigma(x,y)-\sigma(\tau,y)|^q\,dy\\ &= \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + 2^q\int_\tau^x |k(\tau,y)|^q\,dy.\end{aligned}$$

This, together with the condition (H) implies

$$\lim_{x\to\tau^+}\int_0^1 |\Phi(x,\tau,y)|^q \leq \lim_{x\to\tau^+}\int_0^1 |k(x,y)-k(\tau,y)|^q\,dy + 2^q\lim_{x\to\tau^+}\int_\tau^x |k(\tau,y)|^q\,dy = 0$$

and $\lim_{x\to\tau^+}\int_0^1 |\Phi(x,\tau,y)|^q = 0$. Similarly, if $x < \tau$, by using (2.9) and (2.10), we have $\lim_{x\to\tau^-}\int_0^1 |\Phi(x,\tau,y)|^q = 0$. It follows that (2.7) holds.

(2) Let $q \in [1,\infty]$. Since $k(x,y) \geq 0$ for $x, y \in [0,1]$ with $x \neq y$,

$$\Big|\sigma(x,y)k(x,y)\Big| = k(x,y) \quad \text{for } x,y \in [0,1] \text{ with } x \neq y.$$

Hence, we have for $x, \tau, y \in [0,1]$ with $x \neq y$ and $\tau \neq y$,

$$\begin{aligned}\Big|k(x,y)-k(\tau,y)\Big| &= \Big||\sigma(x,y)k(x,y)| - |\sigma(\tau,y)k(\tau,y)|\Big|\\ &\leq \Big|\sigma(x,y)k(x,y)-\sigma(\tau,y)k(\tau,y)\Big|.\end{aligned} \tag{2.11}$$

If $q = \infty$, then by (2.11), we have

$$\lim_{x\to\tau}\|k(x,\cdot)-k(\tau,\cdot)\|_{L^\infty(0,1)} \leq \lim_{x\to\tau}\|\sigma(x,\cdot)k(x,\cdot)-\sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^\infty(0,1)}. \tag{2.12}$$

If $q \in [1,\infty)$, then by (2.11), we have

$$\begin{aligned}\Big|k(x,y)-k(\tau,y)\Big|^q &= \Big||\sigma(x,y)k(x,y)| - |\sigma(\tau,y)k(\tau,y)|\Big|^q\\ &\leq \Big|\sigma(x,y)k(x,y)-\sigma(\tau,y)k(\tau,y)\Big|^q\end{aligned}$$

and

$$\lim_{x\to\tau}\|k(x,\cdot)-k(\tau,\cdot)\|_{L^q(0,1)} \leq \lim_{x\to\tau}\|\sigma(x,\cdot)k(x,\cdot)-\sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^q(0,1)}. \tag{2.13}$$

By (2.7), (2.12) and (2.13), we see that (2.6) holds. □

By Theorem 2.1 and Lemma 2.2, we obtain the following results.

Theorem 2.3. *Let $q \in [1,\infty)$ and $p \in (1,\infty]$ satisfy (2.4). Assume that $k : [0,1]\times[0,1] \setminus \{(x,x) : x \in [0,1]\} \to \mathbb{R}_+$ satisfies the conditions (i) and (ii) of Theorem 2.1. Then the maps L defined in (2.1) and $\mathscr{L}$ defined in (2.2) map $L^p(0,1)$ to $C[0,1]$ and are compact.*

Proof. By Theorem 2.1 with $\mathscr{D} = \{(x,x) : x \in [0,1]\}$, L maps $L^p(0,1)$ to $C[0,1]$ and is compact. By (H), $\sigma(x,\cdot)k(x,\cdot) \in L^q(0,1)$. By Lemma 2.2 (1), Theorem 2.1 (ii) implies (2.7) holds for each $\tau \in [0,1]$. Hence, σk satisfies the conditions (i) and (ii) of Theorem 2.1. It follows from Theorem 2.1 that $\mathscr{L}$ maps $L^p(0,1)$ to $C[0,1]$ and is compact. □

Theorem 2.4. *Assume that* $k : [0,1] \times [0,1] \setminus \{(x,x) : x \in [0,1]\} \to \mathbb{R}_+$ *satisfies the following conditions.*

(i) *For each* $x \in [0,1]$, $k(x,\cdot) \in L^\infty(0,1)$.

(ii) *For each* $\tau \in [0,1]$, $\lim_{x\to\tau} \|\sigma(x,\cdot)k(x,\cdot) - \sigma(\tau,\cdot)k(\tau,\cdot)\|_{L^\infty(0,1)} = 0$.

Then the maps L *defined in* (2.1) *and* $\mathscr{L}$ *defined in* (2.2) *map* $L^1(0,1)$ *to* $C[0,1]$ *and are compact.*

Proof. By the condition (i), $\sigma(x,\cdot)k(x,\cdot) \in L^\infty(0,1)$. This, together with the condition (ii), shows that σk satisfies the conditions (i) and (ii) of Theorem 2.1 with $\mathscr{D} = \{(x,x) : x \in [0,1]\}$. It follows from Theorem 2.1 that $\mathscr{L}$ maps $L^1(0,1)$ to $C[0,1]$ and is compact. By Lemma 2.2 (2), the condition (ii) implies that k satisfies Theorem 2.1 (ii). Hence, k satisfies Theorem 2.1 (i) and (ii) with $q = \infty$. It follows from Theorem 2.1 that L maps $L^1(0,1)$ to $C[0,1]$ and is compact. □

As applications of the above results, we study the following two specific linear Hammerstein integral operators:

$$L_\alpha v(x) = \int_0^1 \frac{1}{|x-y|^\alpha} v(y)\,dy \quad \text{for each } x \in [0,1] \tag{2.14}$$

and

$$\mathscr{L}_\alpha v(x) = \int_0^1 \frac{\sigma(x,y)}{|x-y|^\alpha} v(y)\,dy \quad \text{for each } x \in [0,1], \tag{2.15}$$

where $\alpha \in (0,1)$.

We first prove the following result.

Lemma 2.5.

(1) *If* $\alpha \in (0,1)$, *then*

$$\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \frac{1}{1-\alpha}\left[x^{1-\alpha} + (1-x)^{1-\alpha}\right] \quad \textit{for each } x \in [0,1]$$

and

$$\int_0^1 \frac{\sigma(x,y)}{|x-y|^\alpha}\,dy = \frac{1}{1-\alpha}\left[x^{1-\alpha} - (1-x)^{1-\alpha}\right] \quad \textit{for each } x \in [0,1].$$

(2) *If* $\alpha \in [1,\infty)$, *then* $\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \infty$ *for each* $x \in [0,1]$.

Proof. (1) Let $\alpha \in (0,1)$ and $x \in [0,1]$. Then

$$\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \int_0^x \frac{1}{(x-y)^\alpha}\,dy + \int_x^1 \frac{1}{(y-x)^\alpha}\,dy = \frac{x^{1-\alpha}}{1-\alpha} + \frac{(1-x)^{1-\alpha}}{1-\alpha}.$$

Similarly, the second equality holds.

(2) Let $\alpha \in [1,\infty)$, $x \in \left[0,\frac{1}{2}\right]$ and $z = y - x$ for $y \in [0,1]$. Then $x \le 1 - x$ and

$$\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \int_{-x}^{1-x} \frac{1}{|z|^\alpha}\,dz \ge \int_{-x}^{x} \frac{1}{|z|^\alpha}\,dz = 2\int_0^x \frac{1}{z^\alpha}\,dz = \infty.$$

Let $x \in \left(\frac{1}{2},1\right]$ and let $z = y - x$ for $y \in [0,1]$. Then $-x < -(1-x)$ and

$$\int_0^1 \frac{1}{|x-y|^\alpha}\,dy = \int_{-x}^{1-x} \frac{1}{|z|^\alpha}\,dz \ge \int_{-(1-x)}^{1-x} \frac{1}{|z|^\alpha}\,dz = 2\int_0^{1-x} \frac{1}{z^\alpha}\,dz = \infty. \qquad \square$$

The following result gives an application of Theorem 2.3.

Theorem 2.6. *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha},\infty\right]$. Then the maps L_α defined in (2.14) and $\mathscr{L}_\alpha$ defined in (2.15) map $L^p(0,1)$ to $C[0,1]$ and are compact.*

Proof. Let $p \in \left(\frac{1}{1-\alpha},\infty\right]$ and $q \in \left[1,\frac{1}{\alpha}\right)$ satisfy (2.4). We define $k : [0,1]\times[0,1] \setminus \{(x,x) : x \in [0,1]\} \to \mathbb{R}$ by

$$k(x,y) = \frac{1}{|x-y|^\alpha}. \tag{2.16}$$

Since $\alpha q \in (0,1)$, by Lemma 2.5 (1), we have for each $x \in [0,1]$,

$$\int_0^1 |k(x,y)|^q\,dy = \int_0^1 \frac{1}{|x-y|^{\alpha q}}\,dy < \infty.$$

Hence, $k(x,\cdot) \in L^q(0,1)$ for each $x \in [0,1]$ and Theorem 2.2 (i) holds. Let $\tau \in (0,1)$ and $\delta_1 \in \big(0,\min\{\tau,1-\tau\}\big)$. Let

$$\varepsilon \in \left(0,\frac{2^{4-\alpha q}\delta_1^{1-\alpha q}}{1-\alpha q}\right), \quad \delta_\varepsilon = \left[\frac{\varepsilon(1-\alpha q)}{2^{4-\alpha q}}\right]^{\frac{1}{1-\alpha q}} \quad \text{and } \delta \in (0,\delta_\varepsilon).$$

Then $\delta < \delta_\varepsilon < \delta_1 < \frac{1}{2}$. For $x,y \in [0,1]$ with $x \ne y$ and $y \ne \tau$, let

$$k(x,y) - k(\tau,y) = \frac{1}{|x-y|^\alpha} - \frac{1}{|\tau-y|^\alpha}.$$

Let $D_1 = \{x \in [0,1] : |x-\tau| \le \delta\} \times [0,\tau-\delta_\varepsilon] \cup [\tau+\delta_\varepsilon,1]$. Then D_1 is closed and

$$|x-y| \ge |y-\tau| - |x-\tau| \ge \delta_\varepsilon - \delta > 0 \quad \text{for } (x,y) \in D_1.$$

Hence, $k : D_1 \to \mathbb{R}$ is uniformly continuous on D_1. Let $\sigma \in \left(0,\frac{1}{2(1-2\delta_\varepsilon)}\right)$. It follows that there exists $\delta^* \in (0,\delta)$ such that when $|x-\tau| < \delta^*$,

$$|k(x,y) - k(\tau,y)|^q < \sigma\varepsilon \quad \text{for } y \in [0,\tau-\delta_\varepsilon] \cup [\tau+\delta_\varepsilon,1].$$

Hence, when $|x-\tau| < \delta^*$, we have

$$\begin{aligned}\int_0^{\tau-\delta_\varepsilon} &|k(x,y)-k(\tau,y)|^q\,dy + \int_{\tau+\delta_\varepsilon}^1 |k(x,y)-k(\tau,y)|^q\,dy \\ &\le \sigma\varepsilon(\tau-\delta_\varepsilon) + \sigma\varepsilon(1-\tau-\delta_\varepsilon) = \sigma\varepsilon(1-2\delta_\varepsilon) < \frac{\varepsilon}{2}\end{aligned}$$

and

$$\begin{aligned}
\int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(x,y)|^q\,dy &= \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \frac{1}{|x-y|^{\alpha q}}\,dy = \int_{\tau-x-\delta_\varepsilon}^{\tau-x+\delta_\varepsilon} \frac{1}{|u|^{\alpha q}}\,du \\
&\le \int_{-(\delta^*+\delta_\varepsilon)}^{\delta^*+\delta_\varepsilon} \frac{1}{|u|^{\alpha q}}\,du = 2\int_0^{\delta^*+\delta_\varepsilon} \frac{1}{u^{\alpha q}}\,du = \frac{2(\delta^*+\delta_\varepsilon)^{1-\alpha q}}{1-\alpha q} < \frac{2(2\delta_\varepsilon)^{1-\alpha q}}{1-\alpha q} \\
&= \frac{2^{2-\alpha q}}{1-\alpha q}\,\frac{\varepsilon(1-\alpha q)}{2^{4-\alpha q}} = \frac{\varepsilon}{4}.
\end{aligned}$$

This implies that when $|x-\tau| < \delta^*$,

$$\begin{aligned}
&\int_0^1 |k(x,y)-k(\tau,y)|^q\,dy \\
&= \int_0^{\tau-\delta_\varepsilon} |k(x,y)-k(\tau,y)|^q\,dy + \int_{\tau+\delta_\varepsilon}^1 |k(x,y)-k(\tau,y)|^q\,dy + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(x,y)-k(\tau,y)|^q\,dy \\
&< \frac{\varepsilon}{2} + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \left(|k(x,y)| + |k(\tau,y)|\right)^q\,dy \\
&< \frac{\varepsilon}{2} + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(x,y)|^q\,dy + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} |k(\tau,y)|^q\,dy \\
&= \frac{\varepsilon}{2} + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \frac{1}{|x-y|^{\alpha q}}\,dy + \int_{\tau-\delta_\varepsilon}^{\tau+\delta_\varepsilon} \frac{1}{|\tau-y|^{\alpha q}}\,dy < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Hence,

$$\lim_{x\to\tau} \int_0^1 |k(x,y)-k(\tau,y)|^q\,dy = 0$$

and Theorem 2.2 (ii) holds. The proofs are similar if $\tau = 0$ or $\tau = 1$. The result follows from Theorem 2.3. □

3 Compactness of Riemann–Liouville fractional integral operators

Let $a, b \in \mathbb{R}$ with $a < b$ and $\varphi : [a,b] \to \mathbb{R}$ be a measurable function. The Riemann–Liouville left-sided fractional integral operator of order $\alpha \in (0,\infty)$ is defined by

$$I_{a^+}^{\alpha}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(y)}{(x-y)^{1-\alpha}}\,dy \quad \text{for each } x \in [a,b], \tag{3.1}$$

provided the Lebesgue integral on the right side of (3.1) exists for almost every (a.e.) $x \in [a,b]$, and Γ is the standard Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx,$$

see [6, p. 13], [14, p. 69] and [20, p. 33]. Similarly, the Riemann–Liouville right-sided fractional integral operator of order $\alpha \in (0,\infty)$ is defined by

$$I_{b^-}^{\alpha}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(y)}{(y-x)^{1-\alpha}}\,dy \quad \text{for each } x \in [a,b], \tag{3.2}$$

provided the Lebesgue integral on the right side of (3.2) exists for a.e. $x \in [a,b]$. Hardy and Littlewood [11] called these integrals 'right- handed' integral 'with origin a', and 'left-handed' integral 'with origin b', respectively.

Note that in (3.1), we still use the symbol $I^{\alpha}_{a+}\varphi(x)$ to denote the Lebesgue integral on the right side of (3.2) at x even when the integral does not exist at x. Hence, we treat (3.1) to hold for each $x \in [a,b]$. Similarly, we treat (3.2) to hold for each $x \in [a,b]$.

It is well known that both I^{α}_{a+} and I^{α}_{b-} map $L^1(0,1)$ to $L^1(0,1)$, see [6, Theorem 2.1], [14, Lemma 2.1], [20, Theorem 2.6], [14, Lemma 2.1] and [20, Theorem 2.6].

We only use I^{α}_{0+} and I^{α}_{1-} to denote the following operators:

$$I^{\alpha}_{0+}\varphi(x) := \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(y)}{(x-y)^{1-\alpha}}\,dy \quad \text{for each } x \in [0,1], \tag{3.3}$$

and

$$I^{\alpha}_{1-}\varphi(x) := \frac{1}{\Gamma(\alpha)}\int_x^1 \frac{\varphi(y)}{(y-x)^{1-\alpha}}\,dy \quad \text{for each } x \in [0,1]. \tag{3.4}$$

We give the following relationships among the above operators given in (3.1), (3.2), (3.3) and (3.4). They are well known to experts, but we give the proofs for completeness because we have not found them anywhere else.

Proposition 3.1. *Let $\varphi \in L^1(a,b)$ and let*

$$t(x) = (1-x)a + xb \quad \textit{for each } x \in [0,1]. \tag{3.5}$$

and

$$(\varphi(t))(x) = \varphi(t(x)) \quad \textit{for a.e. } x \in [0,1]. \tag{3.6}$$

Then the following assertions hold.

(1) *If $(I^{\alpha}_{a+}\varphi)(t(x))$ exists for some $x \in [0,1]$, then*

$$(I^{\alpha}_{a+}\varphi)(t(x)) = (b-a)^{\alpha}(I^{\alpha}_{0+}(\varphi(t))(x). \tag{3.7}$$

(2) *If $(I^{\alpha}_{b-}\varphi)(t(x))$ exists for some $x \in [0,1]$, then*

$$(I^{\alpha}_{b-}\varphi)(t(x)) = (b-a)^{\alpha}(I^{\alpha}_{1-}(\varphi(t))(x). \tag{3.8}$$

Proof. Let $\varphi \in L^1(a,b)$ and let $y = t(x)$ for $x \in [0,1]$. Then

$$\int_a^b |\varphi(y)|\,dy = (b-a)\int_0^1 |\varphi(t(x))|\,dx = (b-a)\int_0^1 |(\varphi(t))(x)|\,dx$$

This implies $\varphi(t) \in L^1(0,1)$.

(1) Assume that $(I^{\alpha}_{a+}\varphi)(t(x))$ exists for some $x \in [0,1]$. Then by (3.1), we have

$$(I^{\alpha}_{a+}\varphi)(t(x)) = \frac{1}{\Gamma(\alpha)}\int_a^{t(x)} \frac{\varphi(s)}{(t(x)-s)^{1-\alpha}}\,ds. \tag{3.9}$$

Let $s = t(y) = (1-y)a + yb$ for $y \in [0,x]$. Then

$$\begin{aligned}\int_a^{t(x)} \frac{\varphi(s)}{(t(x)-s)^{1-\alpha}}\,ds &= \int_0^x \frac{\varphi(t(y))}{[(b-a)(x-y)]^{1-\alpha}}(b-a)dy \\ &= (b-a)^{\alpha}\int_0^x \frac{(\varphi(t))(y)}{(x-y)^{1-\alpha}}\,dy.\end{aligned}$$

This, together with (3.9), implies (3.7).

(2) Assume that $(I^{\alpha}_{b-}\varphi)(t(x))$ exists for some $x \in [0,1]$. Then by (3.2), we have

$$(I^{\alpha}_{b-}\varphi)(t(x)) = \frac{1}{\Gamma(\alpha)} \int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds. \tag{3.10}$$

Let $s = t(y) = (1-y)a + yb$ for $y \in [x,b]$. Then

$$\begin{aligned}\int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds &= \int_{x}^{b} \frac{\varphi(t(y))}{[(b-a)(y-x)]^{1-\alpha}} (b-a)dy \\ &= (b-a)^{\alpha} \int_{x}^{b} \frac{(\varphi(t))(y)}{(y-x)^{1-\alpha}}\, dy.\end{aligned}$$

This, together with (3.10), implies (3.8). □

Proposition 3.2. *Let $\varphi \in L^1(a,b)$ and let*

$$t(x) = a + b - x \quad \textit{for each } x \in [a,b] \tag{3.11}$$

and

$$(\varphi(t))(x) = \varphi(t(x)) \quad \textit{for a.e. } x \in [a,b]. \tag{3.12}$$

If $(I^{\alpha}_{b-}\varphi)(t(x))$ exists for some $x \in [a,b]$, then

$$(I^{\alpha}_{b-}\varphi)(t(x)) = (I^{\alpha}_{a+}\varphi(t))(x). \tag{3.13}$$

Proof. Assume that $(I^{\alpha}_{b-}\varphi)(t(x))$ exists for some $x \in [a,b]$. Then by (3.2), we have

$$(I^{\alpha}_{b-}\varphi)(t(x)) = \frac{1}{\Gamma(\alpha)} \int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds. \tag{3.14}$$

Let $s = t(y) = a + b - y$ for $y \in [a,x]$. Then

$$\int_{t(x)}^{b} \frac{\varphi(s)}{(s-t(x))^{1-\alpha}}\, ds = \int_{a}^{x} \frac{\varphi(t(y))}{(x-y)^{1-\alpha}}\, dy = \int_{a}^{x} \frac{(\varphi(t))(y)}{(x-y)^{1-\alpha}}\, dy.$$

This, together with (3.14), implies (3.13). □

Proposition 3.3. *Let $\varphi \in L^1(a,b)$ and let*

$$t^*(x) = xa + (1-x)b \quad \textit{for each } x \in [0,1] \tag{3.15}$$

and

$$(\varphi(t^*))(x) = \varphi(t^*(x)) \quad \textit{for a.e. } x \in [0,1]. \tag{3.16}$$

If $(I^{\alpha}_{b-}\varphi)(t^*(x))$ exists for some $x \in [0,1]$, then

$$(I^{\alpha}_{b-}\varphi)(t^*(x)) = (b-a)^{\alpha}(I^{\alpha}_{0+}\varphi(t^*))(x) \quad \textit{for each } x \in [a,b].$$

Proof. Note that $t^*(x) = t(1-x)$ for each $x \in [0,1]$, where t is the same as in (3.5). By (3.8), we have for each $x \in [0,1]$,

$$\begin{aligned}(I^{\alpha}_{b-}\varphi)(t^*(x)) &= (I^{\alpha}_{b-}\varphi)(t(1-x)) = (b-a)^{\alpha}(I^{\alpha}_{1-}(\varphi(t)))(1-x) \\ &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_{1-x}^{1} \frac{\varphi(t(y))}{(y-(1-x))^{1-\alpha}}\, dy = \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_{0}^{x} \frac{\varphi(t(1-z))}{((1-z)-(1-x))^{1-\alpha}}\, dz \\ &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha)} \int_{0}^{x} \frac{\varphi(t^*(z))}{(x-z)^{1-\alpha}}\, dz = (b-a)^{\alpha}(I^{\alpha}_{0+}\varphi(t^*))(x)\end{aligned}$$

and the result holds. □

By Proposition 3.3, we obtain the following result.

Corollary 3.4. *Assume that $v \in L^1(0,1)$ satisfies that $(I_{1^-}^{1-\alpha}v)(x)$ exists for some $x \in [0,1]$. Then*

$$(I_{1^-}^{1-\alpha}v)(x) = (I_{0^+}^{1-\alpha}v^*)(1-x),$$

where $v^*(s) = v(1-s)$ for a.e. $a \in [0,1]$.

To apply the results in Section 2, in the following we always assume $\alpha \in (0,1)$ and consider the following Riemann–Liouville fractional integral operators:

$$I_{0^+}^{1-\alpha}v(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{v(y)}{(x-y)^\alpha}\, dy \quad \text{for each } x \in [0,1] \tag{3.17}$$

and

$$I_{1^-}^{1-\alpha}v(x) := \frac{1}{\Gamma(1-\alpha)} \int_x^1 \frac{v(y)}{(y-x)^\alpha}\, dy \quad \text{for each } x \in [0,1], \tag{3.18}$$

where $v \in L^1(0,1)$.

Now, we prove that if $p \in \left(\frac{1}{1-\alpha}, \infty\right)$, then both $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ map $L^p(0,1)$ to $C[0,1]$ and are compact.

Theorem 3.5. *Let $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. Then the following assertions hold.*

(1) *The maps $I_{0^+}^{1-\alpha}$ and $I_{1^-}^{1-\alpha}$ map $L^p(0,1)$ to $C[0,1]$ and are compact.*

(2) *For each $v \in L^p(0,1)$, $I_{0^+}^{1-\alpha}v(0) = I_{1^-}^{1-\alpha}v(1) = 0$.*

(3) *For each $x \in [0,1]$,*

$$I_{0^+}^{1-\alpha}\hat{1}(x) = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} \quad \text{and } I_{1^-}^{1-\alpha}\hat{1}(x) = \frac{(1-x)^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{3.19}$$

where $\hat{1}(x) \equiv 1$ for each $x \in [0,1]$.

Proof. (1) Let $v \in L^p(0,1)$ and $x \in [0,1]$. Then

$$\begin{aligned}\mathscr{L}_\alpha v(x) &= \int_0^x \frac{1}{(x-y)^\alpha} v(y)\, dy - \int_x^1 \frac{1}{(y-x)^\alpha} v(y)\, dy \\ &= \Gamma(1-\alpha)\left[(I_{0^+}^{1-\alpha}v)(x) - (I_{1^-}^{1-\alpha}v)(x)\right]\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}L_\alpha v(x) &= \int_0^x \frac{1}{(x-y)^\alpha} v(y)\, dy + \int_x^1 \frac{1}{(y-x)^\alpha} v(y)\, dy \\ &= \Gamma(1-\alpha)\left[(I_{0^+}^{1-\alpha}v)(x) + (I_{1^-}^{1-\alpha}v)(x)\right].\end{aligned} \tag{3.21}$$

By (3.20) and (3.21), we have for $x \in [0,1]$,

$$I_{0^+}^{1-\alpha}v(x) = \frac{1}{2\Gamma(1-\alpha)}\left[L_\alpha v(x) + \mathscr{L}_\alpha v(x)\right] \tag{3.22}$$

and

$$I_{1^-}^{1-\alpha}v(x) = \frac{1}{2\Gamma(1-\alpha)}\left[L_\alpha v(x) - \mathscr{L}_\alpha v(x)\right]. \tag{3.23}$$

The results follow from Theorem 2.6.

(2) For $v \in L^p(0,1)$, by (2.14) and (2.15), it is easy to see that

$$L_\alpha v(0) = -\mathscr{L}_\alpha v(0) \quad \text{and} \quad L_\alpha v(1) = \mathscr{L}_\alpha v(1).$$

This, together with (3.22) and (3.23), implies

$$(I_{0+}^{1-\alpha})v(0) = I_{1-}^{1-\alpha}v(1) = 0.$$

(3) By Lemma 2.5 (1) and (3.20) and (3.21) with $v = \hat{1}$, we see that (3.19) holds. □

Remark 3.6. In a personal communication, Professor J. R. L. Webb informed me that there is another known proof of compactness of $I_{0+}^{1-\alpha}$ which I reproduce below. By [11, Theorem 12] (or [6, Theorem 2.6], [20, Theorem 3.6] and [23, Proposition 3.2 (3)]), $I_{0+}^{1-\alpha}$ maps $L^p(0,1)$ to the Hölder space $C^{0,\beta}$, and the Hölder space $C^{0,\beta}$ with the norm

$$\|u\|_{0,\beta} := \max_{x\in[0,1]} |u(x)| + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\beta}$$

is compactly imbedded in $C[0,1]$, where $\beta = 1-\alpha-\frac{1}{p}$. Indeed, if $\{u_n\}$ is a bounded sequence in $C^{0,\beta}[0,1]$, say $\|u_n\|_{0,\beta} \le M < \infty$, then we have for $x \neq y$,

$$|u_n(x)-u_n(y)| = \frac{|u_n(x)-u_n(y)|}{|x-y|^\beta}|x-y|^\beta \le M|x-y|^\beta,$$

so $\{u_n\}$ is bounded and equicontinuous, and hence relatively compact in $C[0,1]$ by the Ascoli–Arzelà theorem.

Remark 3.7. By Proposition 3.4, $I_{1-}^{1-\alpha} : L^p(0,1) \to C[0,1]$ is compact for each $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. By (3.22), (3.23) and compactness of $I_{0+}^{1-\alpha}$ and $I_{1-}^{1-\alpha}$ obtained in Remark 3.6, we see that Theorem 3.5 (1) is equivalent to Theorem 2.6.

By Propositions 3.1, 3.2 and 3.3, we see that all the theorems which are proved for one operator of the operators: I_{a+}^α, I_{b-}^α, I_{0+}^α and I_{1-}^α, will apply, with the obvious changes, to the others. Therefore, in the following, we only consider the operator $I_{0+}^{1-\alpha}$.

As an application of continuity and compactness of $I_{0+}^{1-\alpha}$, we prove the following result on the eigenfunctions and spectral radius of $I_{0+}^{1-\alpha}$.

Theorem 3.8. *Let $p \in \left(\frac{1}{1-\alpha}, \infty\right]$. Then the following assertions hold.*

(1) *If there exist $\varphi \in L_+^p(0,1)$ and $\mu \in (0,\infty)$ such that*

$$\varphi(x) = \mu I_{0+}^{1-\alpha}\varphi(x) \quad \text{for a.e. } x \in [0,1]. \tag{3.24}$$

Then $\varphi(x) = 0$ for each $x \in [0,1]$.

(2) *$r(I_{0+}^{1-\alpha}) = 0$, where $r(I_{0+}^{1-\alpha})$ is the spectral radius of $I_{0+}^{1-\alpha}$.*

Proof. Let P be the standard positive cone in $C[0,1]$, that is,

$$P = \{u \in C[0,1] : u(x) \ge 0 \quad \text{for } x \in [0,1]\}. \tag{3.25}$$

Then P is a total and normal cone in $C[0,1]$.

(1) By Theorem 3.5 (1), $I_{0^+}^{1-\alpha}\varphi \in P$. By (3.24), $\varphi \in P$. By (3.24) and weakly singular Gronwall inequality [12, Lemma 7.1.1] or ([6, Lemma 6.19], [7, Lemma 4.3] and [22, Theorem 3.2]), we have $\varphi(x) = 0$ for a.e. $x \in [0,1]$. Since $\varphi \in C[0,1]$, $\varphi(x) = 0$ for each $x \in [0,1]$.

(2) By Theorem 3.5 (1), for $p \in \left(\frac{1}{1-\alpha}, \infty\right]$, the operator $I_{0^+}^{1-\alpha}$ maps P to P and is compact. If $r(I_{0^+}^{1-\alpha}) > 0$, it would follow from Krein–Rutman theorem (see [1, Theorem 3.1] or [16]) that there exists an eigenvector $\varphi \in P \setminus \{0\}$ such that

$$I_{0^+}^{1-\alpha}\varphi(x) = r(I_{0^+}^{1-\alpha})\varphi(x) \quad \text{for each } x \in [0,1].$$

By the result (i), we obtain $\varphi(x) = 0$ for each $x \in [0,1]$, which contradicts the fact $\varphi \in P \setminus \{0\}$. $\square$

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