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A coincidence problem for a second-order semi-linear differential equation

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In this paper, we study a class of problems at resonance for a general second-order linear operator $Lu = u'' + p(t)u' + q(t)u$. We impose abstract functional conditions and derive several criteria for the existence of a solution for every resonance scenario.

Keywords: functional condition, semi-linear differential equation, resonance.

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1 Introduction

We consider the semi-linear equation

$$u''(t) + p(t)u'(t) + q(t)u(t) = f(t, u(t), u'(t)), \quad a.e.\ t \in (0,1), \tag{1.1}$$

subject to the linear functional conditions

$$F_1(u) = 0, \qquad F_2(u) = 0, \tag{1.2}$$

where F_1 and F_2 are continuous linear functionals on $C^1[0,1]$.

One of the early works that stimulated interest to applications of the coincidence degree theory to non-local boundary value problems was the paper by Feng and Webb [3]. Our work is motivated by [3] and [2]. In [2], the authors studied the resonant functional problem

$$u''(t) = f(t, u(t), u'(t)), \quad \text{a.e. } t \in (0,1), \tag{1.3}$$

$$B_1(u) = 0, \qquad B_2(u) = 0, \tag{1.4}$$

where f is Carathéodory, B_1 and B_2 are continuous linear functionals on $C^1[0,1]$. Imposing $B_1(t)B_2(1) = B_2(t)B_1(1)$, the problem (1.3), (1.4) is at resonance of dimension one or two. An existence result was obtained for every possible resonance scenario.

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In order to apply the coincidence degree approach of Mawhin and many other methods of functional analysis in ordinary semi-linear differential equations, one relies on the knowledge of a fundamental solution set. In all known to us papers based on these methods, the linear operator L, such as $Lu = (pu')'$ in [7], can be "inverted" by the reduction of order method. The method developed here can be also applied to fractional order problems, that is, when L is an integro-differential operator such as the Riemann–Liouville, Caputo fractional derivatives and their numerous generalizations. Since we deal with a linear differential operator that, in general, does not admit the reduction of order, this work is also a generalization of many results such as [1,6–8]. Moreover, if the boundary conditions, or, for that matter multi-point conditions, or even linear conditions involving Riemann–Stieltjes integrals are chosen, a specific resonance is "fixed". Obviously, in this case, one would only hope to study one or very few resonance conditions per paper. We believe a more productive approach would yield a formalism for solving a class of problems.

In our setting, the problem is abstract since we deal with a large class of general second-order linear differential operators whose fundamental solution set is $\{\phi_1, \phi_2\}$. Not only our work is an abstract generalization of many results in that respect but also due to the functional conditions (1.2) studied here, which certainly include (1.6). In fact, as in [2], we study every "geometric" scenario of resonance. In particular, in [2], the authors considered (1.1) with $p(t) = q(t) = 0$ subject to (1.2). Thus, the present work extends the results of [2], as well.

In [7], the author considered several resonance cases in the framework of the generalized Sturm–Liouville boundary value problem

$$(p(t)u'(t))' - q(t)u(t) = f\left(t, \int_0^t u(s)\,ds, u'(t)\right), \qquad t \in (0,1), \tag{1.5}$$

$$au(0) - bp(0)u'(0) = \mu_1 u(\xi), \qquad cu(1) + dp(1)u'(1) = \mu_2 u(\xi), \tag{1.6}$$

where $a, b, c, d \in \mathbb{R}$, $0 < \xi < 1$, and f is continuous and

$$\mu_1\left(c\int_\xi^1 \frac{1}{p(s)}\,ds + d\right) + \mu_2\left(a\int_0^\xi \frac{1}{p(s)}\,ds + b\right) = ad + bc + ac\int_0^1 \frac{1}{p(s)}\,ds. \tag{1.7}$$

By means of a "shift" operator, a resonant problem can be converted to a non-resonant problem [5] and, thus, need not be studied as a coincidence equation $Lu = Nu$. In [7], the problem is not at resonance if

$$L_0 u(t) = (p(t)u'(t))' - q(t)u(t).$$

Considering

$$Lu(t) = (p(t)u'(t))' = q(t)u(t) + f\left(t, \int_0^t u(s)\,ds, u'(t)\right), \qquad t \in (0,1),$$

the equation (1.7) becomes a resonance condition. The advantage here is that the fundamental solution set of L is easy to obtain while for L_0 we only know that it exists but, in general, there is no hope to obtain it explicitly. It is also worth mentioning that whenever a criterion for the existence of a solution to the coincidence equation $Lu = f(t, u, u')$ is obtained, it can always, with a little effort, be extended to $Lu = f(t, u, T_1(u), u', T_2(u'))$, where T_1 and T_2 are bounded operators such as the primitive of $u(t)$ in (1.5), on a suitable functional space. Indeed, the projection scheme needed to apply the coincidence degree approach to these equations is exactly the same, and the only difference is in the "growth" condition on the function f.

In [7], the author introduces a convenience assumption

$$(c\mu_1 - a\mu_2)\int_0^{\xi} \frac{s}{p(s)}\,ds + c(a-\mu_1)\int_0^1 \frac{s}{p(s)}\,ds + d(a-\mu_1) \neq 0. \tag{1.8}$$

In order to guarantee that the projector Q is well-defined, conditions similar to (1.8) have been imposed in many papers (e.g., see the references in [2] and the remarks therein). In our work, we construct the projection scheme so that Q is well-defined without relying on such "convenience" assumptions that are rather restrictive and simply unnecessary.

In this section, we state the preliminaries and the result due to Mawhin [4] used, in the second section, to obtain a solution of (1.1), (1.2).

In order to develop our method, we need to make several basic assumptions. Of course, we assume that the fundamental solution set $\{\phi_1, \phi_2\}$ is known. We would like to consider a solution of (1.1) in classical spaces and make use of the representation

$$u(t) = \int_0^t k(t,s)Lu(s)\,ds + l_1(u)\phi_1(t) + l_2(u)\phi_2(t), \tag{1.9}$$

where

$$k(t,s) = \frac{\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)}{W(\phi_1,\phi_2)(s)}, \quad l_1(u) = \frac{W(u,\phi_2)(0)}{W(\phi_1,\phi_2)(0)}, \quad l_2(u) = \frac{W(\phi_1,u)(0)}{W(\phi_1,\phi_2)(0)}, \tag{1.10}$$

where

$$\Phi(\phi_1,\phi_2)(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{bmatrix}$$

and $W(\phi_1,\phi_2)(t) = \det\Phi(\phi_1,\phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)$ is the Wronskian of the fundamental solution set on $[0,1]$. Our approach relies on the boundedness of $W(\phi_1,\phi_2)(t)$ and $W(\phi_1,\phi_2)(0) \neq 0$. So, the following would fulfill our wishes:

(L) $p, q \in C[0,1]$, $\gamma_1 = \max_{t,s\in[0,1]} |k(t,s)|$, $\gamma_2 = \sup_{t,s\in[0,1]} \left|\frac{\partial}{\partial t}k(t,s)\right|$, $\gamma = \max\{\gamma_1,\gamma_2\}$.

It should be mentioned that the assumption on p can be weakened, which would force one to use weighted norms.

Introduce $X = C^1[0,1]$, $\|u\|_X = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$. The next standing assumption concerns the linear functions in (1.2):

(F) $F_i : X \to \mathbb{R}$, $|F_i(u)| \leq \rho_i\|u\|_X$, where $\rho_i > 0$, $i = 1,2$, $F_1(\phi_1) = \alpha a$, $F_1(\phi_2) = \alpha b$, $F_2(\phi_1) = a$, $F_2(\phi_2) = b$, $\alpha, a, b \in \mathbb{R}$, $a^2 + b^2 \neq 0$.

Under this assumption the differential operator in (1.1) is not invertible and the functional problem is said to be at resonance. Furthermore, in order to claim that all possible resonance cases have been considered, we also need to study the case $a = b = 0$, which is only briefly discussed in Section 2.

Definition 1.1. Let X and Z be normed spaces. A linear mapping $L\colon \operatorname{dom} L \subset X \to Z$ is called a Fredholm mapping if the following two conditions hold:

(i) $\ker L$ has a finite dimension, and

(ii) $\operatorname{Im} L$ is closed and has a finite co-dimension.

If L is a Fredholm mapping, its (Fredholm) *index* is the integer $\operatorname{Ind} L = \dim\ker L - \operatorname{codim}\operatorname{Im} L$.

Since we work with a Fredholm mapping of index zero, it follows from Definition 1.1 that there exist continuous projectors $P\colon X \to X$ and $Q\colon Z \to Z$ such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} Q \tag{1.11}$$

and that the mapping

$$L|_{\operatorname{dom} L \cap \ker P}\colon \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible. The inverse of $L|_{\operatorname{dom} L \cap \ker P}$ we denote by $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q}\colon Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

Definition 1.2. Let $L\colon \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N\colon E \to Z$ be a mapping. We say that N is L-compact on E if $QN\colon E \to Z$ and $K_{P,Q}N\colon E \to X$ are compact on E. In addition, we say, that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

Let $Z = L^1[0,1]$ with the Lebesgue norm denoted by $\|\cdot\|_1$. Consider the mapping $L\colon \operatorname{dom} L \subset X \to Z$ with

$$\operatorname{dom} L = \{u \in X : u' \in AC[0,1],\ u \text{ satisfies } (1.2)\}$$

defined by

$$Lu(t) = u''(t) + p(t)u'(t) + q(t)u(t).$$

Define the mapping $N\colon X \to Z$ by

$$Nu(t) = f(t, u(t), u'(t)).$$

Thus, (1.1), (1.2) is converted into the coincidence equation $Lu = Nu$ whose solution will be shown to exist by applying the following theorem due to Mawhin [4, Theorem IV.13].

Theorem 1.3. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0,1)$;

(ii) $Nu \notin \operatorname{Im} L$ for every $u \in \ker L \cap \partial\Omega$;

(iii) $\deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q\colon Z \to Z$ a continuous projector such that $\ker Q = \operatorname{Im} L$ and $J\colon \operatorname{Im} Q \to \ker L$ is any isomorphism.

Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Lemma 1.4. *The mapping $L\colon \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. By (F), it is clear that $\ker L = \{c(-b\phi_1 + a\phi_2) : c \in \mathbb{R}\} \cong \mathbb{R}$. For convenience, let

$$Tg(t) = \int_0^t k(t,s) g(s)\, ds. \tag{1.12}$$

We claim that $\operatorname{Im} L = \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}$. Now, $g \in \operatorname{Im} L$ if there exists $u \in \operatorname{dom} L$ such that $Lu = g$. Recalling (1.9), that is, $u = Tg + l_1(u)\phi_1 + l_2(u)\phi_2$, we have, by (F),

$$F_1(u) = F_1(Tg) + \alpha(l_1(u)a + l_2(u)b) = 0, \qquad F_2(u) = F_2(Tg) + l_1(u)a + l_2(u)b = 0.$$

It follows, $\operatorname{Im} L \subset \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}$.

Let $g \in \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}$. Define

$$u = Tg - \frac{F_2(Tg)}{a^2+b^2}(a\phi_1 + b\phi_2).$$

Then

$$Lu = LTg - \frac{F_2(Tg)}{a^2+b^2}(aL\phi_1 + bL\phi_2) = g.$$

Also,

$$F_1(u) = F_1(Tg) - \frac{F_2(Tg)}{a^2+b^2}(aF_1(\phi_1) + bF_1(\phi_2)) = F_1(Tg) - \alpha F_2(Tg) = 0$$

and, similarly, $F_2(u) = 0$. That is, $u \in \operatorname{dom} L$, so $g \in \operatorname{Im} L$. We have

$$\{g \in Z : (F_1 - \alpha F_2)Tg = 0\} \subset \operatorname{Im} L.$$

Therefore, $\{g \in Z : (F_1 - \alpha F_2)Tg = 0\} = \operatorname{Im} L$.

We show that there exists $h \in Z$ such that $(F_1 - \alpha F_2)Th \neq 0$. Let $F = F_1 - \alpha F_2$. By (F), $F(\phi_1) = F(\phi_2) = 0$. Since F_1 and F_2 are linearly independent on X, there exists $u_0 \in X$ such that $F(u_0) \neq 0$. Since F is continuous on X, for $\epsilon > 0$, there exists a polynomial p such that $\|p - u_0\|_X < \epsilon$ and $F(p) \neq 0$. Set $h = Lp \in Z$. Again, recall (1.9). Then $F(Th) = F(TLp) = F(p - l_1(p)\phi_1 - l_2(p)\phi_2) = F(p) - l_1(p)F(\phi_1) - l_2(p)F(\phi_2) = F(p) \neq 0$. Since T and F are linear, we may assume, without loss of generality, that $(F_1 - \alpha F_2)Th = 1$. Define $Q : Z \to Z$ by

$$Qg(t) = (F_1 - \alpha F_2)(Tg)h(t) = (F_1 - \alpha F_2)\left(\int_0^t k(t,s)g(s)\,ds\right)h(t).$$

Since $Qh(t) = (F_1 - \alpha F_2)(Th)h(t) = h(t)$, then $Q^2q = Qg$, $g \in Z$. It is obvious that $Q : Z \to Z$ is a continuous map and $Z = \ker Q \oplus \operatorname{Im} Q$, $\operatorname{Im} Q = \{ch : c \in \mathbb{R}\}$ with $\dim \operatorname{Im} Q = 1$, and $\ker Q = \operatorname{Im} L$.

Define $P, \tilde{P}, P_0 : X \to X$ by

$$\begin{aligned} Pu(t) &= \frac{-bW(u,\phi_2)(0) + aW(\phi_1,u)(0)}{(a^2+b^2)W(\phi_1,\phi_2)(0)}(-b\phi_1(t) + a\phi_2(t)) \\ &= \frac{-bl_1(u) + al_2(u)}{a^2+b^2}(-b\phi_1(t) + a\phi_2(t)), \end{aligned} \tag{1.13}$$

$$\begin{aligned} \tilde{P}u(t) &= \frac{aW(u,\phi_2)(0) + bW(\phi_1,u)(0)}{(a^2+b^2)W(\phi_1,\phi_2)(0)}(a\phi_1(t) + b\phi_2(t)) \\ &= \frac{al_1(u) + bl_2(u)}{a^2+b^2}(a\phi_1(t) + b\phi_2(t)), \end{aligned} \tag{1.14}$$

and

$$P_0(t) = \frac{W(u,\phi_2)(0)}{W(\phi_1,\phi_2)(0)}\phi_1(t) + \frac{W(\phi_1,u)(0)}{W(\phi_1,\phi_2)(0)}\phi_2(t) = l_1(u)\phi_1(t) + l_2(u)\phi_2(t), \tag{1.15}$$

where the second expression of each map is obtained using (1.10). Since

$$P\phi_1 = -\frac{b}{a^2+b^2}(-b\phi_1 + a\phi_2), \quad P\phi_2 = \frac{a}{a^2+b^2}(-b\phi_1 + a\phi_2),$$

then $P(-b\phi_1 + a\phi_2) = -b\phi_1 + a\phi_2$. Therefore, $P^2 = P$, $X = \ker P \oplus \operatorname{Im} P$, where $\operatorname{Im} P = \{c(-b\phi_1 + a\phi_2) : c \in \mathbb{R}\} = \ker L$. Similarly, $\tilde{P}^2 = \tilde{P}$, $X = \ker \tilde{P} \oplus \operatorname{Im} \tilde{P}$, where $\operatorname{Im} \tilde{P} =$

$\{c(a\phi_1 + b\phi_2) : c \in \mathbb{R}\}$. Moreover, $P_0^2 = P_0$, $X = \ker P_0 \oplus \operatorname{Im} P_0$, where $\operatorname{Im} P_0 = \{c_1\phi_1 + c_2\phi_2 : c_1, c_2 \in \mathbb{R}\}$. Finally,

$$P + \tilde{P} = P_0 \tag{1.16}$$

and $P\tilde{P} = \tilde{P}P = 0$ on X.

Since the relationships (1.11) hold, the projectors P and Q are exact. In summary, L is a Fredholm mapping of index zero. □

The next two results provide the generalized inverse of L and its norm-estimates. Recall (1.12).

Lemma 1.5. *If the map $K_P : Z \to X$ is defined by*

$$K_p g = -\frac{1}{a^2 + b^2} F_2(Tg)(a\phi_1 + b\phi_2) + Tg, \tag{1.17}$$

then $LK_P g = g$, $g \in Z$, and $K_p Lu = u$, $u \in \operatorname{dom} L \cap \ker P$.

Proof. It is easy to see that $LK_P g = g$, $g \in Z$. Let $u \in \operatorname{dom} L \cap \ker P$ and $g = Lu$. Using (1.9) and (1.15),

$$Tg = u - l_1(u)\phi_1 - l_2(u)\phi_2 = u - P_0 u.$$

Then $F_2(Tg) = F_2(u) - l_1(u)F_2(\phi_1) - l_2(u)F_2(\phi_2) = -al_1(u) - bl_2(u)$ since $u \in \operatorname{dom} L$. As a result,

$$K_P Lu = \frac{al_1(u) + bl_2(u)}{a^2 + b^2}(a\phi_1 + b\phi_2) + u - P_0 u = \tilde{P}u + u - P_0 u = u - Pu = u$$

by (1.16) and since $u \in \ker P$. □

Obviously,

$$\|Tg\|_0 \leq \gamma_1 \|g\|_1, \quad \|(Tg)'\|_0 \leq \gamma_2 \|g\|_1, \quad \|Tg\|_X \leq \gamma \|g\|_1.$$

Also, $|F_2(Tg)| \leq \rho_2 \|Tg\|_X \leq \gamma\rho_2 \|g\|_1$. Hence,

$$\|K_P g\|_0 \leq \frac{\rho_2 \|a\phi_1 + b\phi_2\|_0}{(a^2 + b^2)} \|Tg\|_X + \|Tg\|_0 \leq \left(\frac{\rho_2 \gamma \|a\phi_1 + b\phi_2\|_0}{a^2 + b^2} + \gamma_1 \right) \|g\|_1,$$

$$\|(K_P g)'\|_0 \leq \frac{\rho_2 \|a\phi_1' + b\phi_2'\|_0}{(a^2 + b^2)} \|Tg\|_X + \|(Tg)'\|_0 \leq \left(\frac{\rho_2 \gamma \|a\phi_1' + b\phi_2'\|_0}{a^2 + b^2} + \gamma_2 \right) \|g\|_1.$$

The estimates on the generalized inverse are summarized in the next result.

Lemma 1.6. *The map $K_P : Z \to X$ satisfies*

(a) $\|K_P g\|_0 \leq A \|g\|_1$, *where*

$$A = \frac{\rho_2 \gamma \|a\phi_1 + b\phi_2\|_0}{a^2 + b^2} + \gamma_1,$$

(b) $\|(K_P g)'\|_0 \leq B \|g\|_1$, *where*

$$B = \frac{\rho_2 \gamma \|a\phi_1' + b\phi_2'\|_0}{a^2 + b^2} + \gamma_2,$$

(c) $\|K_P g\|_X \leq \|K_P\| \|g\|_1$, *where* $\|K_P\| = \max\{A, B\}$.

2 Main results

Assume that the following conditions on the function $f(t,x_1,x_2)$ are satisfied:

(H_1) there exists a constant $M_0 > 0$ such that, for each $u \in \operatorname{dom} L \setminus \ker L$ with $|u(t)| + |u'(t)| > M_0$, $t \in [0,1]$, we have $QNu(t) \neq 0$,

(H2) there exist functions $\delta_0, \delta_1, \delta_2 \in L^1[0,1]$ such that, for all $(x_1,x_2) \in \mathbb{R}^2$ and *a.e.* $t \in [0,1]$,

$$|f(t,x_1,x_2)| \le \delta(t) + \delta_1(t)|x_1| + \delta_2(t)|x_2|.$$

(H3) there exists a constant $M_1 > 0$ such that if $|c| > M_1$, then $c(F_1 - \alpha F_2)(TNu_c) > 0$, where $u_c = c(-b\phi_1 + a\phi_2)$.

In the next result, $\|\Phi^{-1}(\phi_1,\phi_2)(t)\|$ is the matrix norm compatible with the norm $\max\{|a_1|,|a_2|\}$ of a vector $[a_1,a_2]^T \in \mathbb{R}^2$.

Theorem 2.1. *If (L), (F), $(H1)$–$(H3)$ hold, then the functional problem (1.1), (1.2) has at least one solution provided*

$$D_1(\|\delta_1\|_1 + \|\delta_2\|_1) < 1, \tag{2.1}$$

where

$$\begin{aligned} D_1 = \max\Big\{&\gamma_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1\|_0 + \|\phi_2\|_0),\\ &\gamma_2 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1'\|_0 + \|\phi_2'\|_0)\Big\}. \end{aligned}$$

Proof. Let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L \colon Lu = \lambda Nu,\ \lambda \in (0,1)\}$. If $u \in \Omega_1$, it follows, from (H_1), that there exists $t_0 \in [0,1]$ such that $|u(t_0)|, |u'(t_0)| \le M_0$. Now,

$$u = \lambda TNu + l_1(u)\phi_1 + l_2(u)\phi_2, \quad u' = \lambda (TNu)' + l_1(u)\phi_1' + l_2(u)\phi_2'. \tag{2.2}$$

Thus,

$$\begin{bmatrix} l_1(u) \\ l_2(u) \end{bmatrix} = \Phi^{-1}(\phi_1,\phi_2)(t_0) \begin{bmatrix} u(t_0) - \lambda TNu(t_0) \\ u'(t_0) - \lambda(TNu)'(t_0) \end{bmatrix}.$$

In what follows, C_i, $i = 1,\dots,5$, are positive constants whose exact values are ignored. Hence,

$$\begin{aligned} |l_1(u)|, |l_2(u)| &= \max\{|l_1(u)|, |l_2(u)|\}\\ &= \|\Phi^{-1}(\phi_1,\phi_2)(t_0)\| \max\{|u(t_0) - \lambda TNu(t_0)|, |u'(t_0) - \lambda(TNu)'(t_0)|\}\\ &\le \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\| \max\{|u(t_0)| + \lambda|TNu(t_0)|, |u'(t_0)| + \lambda|(TNu)'(t_0)|\}\\ &\le \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\| \max\{M_0 + \lambda|TNu(t_0)|, M_0 + \lambda|(TNu)'(t_0)|\}\\ &< \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\| \max\{M_0 + \gamma_1\|Nu\|_1, M_0 + \gamma_2\|Nu\|_1\}\\ &= C_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\| \|Nu\|_1. \end{aligned}$$

We have

$$\begin{aligned} \|u\|_0 &\le \gamma_1\|Nu\|_1 + |l_1(u)|\|\phi_1\|_0 + |l_2(u)|\|\phi_2\|_0\\ &< C_2 + \left(\gamma_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1\|_0 + \|\phi_2\|_0)\right)\|Nu\|_1 \end{aligned}$$

and, similarly,

$$\|u'\|_0 < C_3 + \left(\gamma_2 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1'\|_0 + \|\phi_2'\|_0)\right)\|Nu\|_1.$$

Hence,

$$\begin{aligned}\|u\|_X < C_4 + \max\{\gamma_1 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1\|_0 + \|\phi_2\|_0),\\ \gamma_2 + \gamma \max_{t\in[0,1]} \|\Phi^{-1}(\phi_1,\phi_2)(t)\|(\|\phi_1'\|_0 + \|\phi_2'\|_0)\}\|Nu\|_1.\end{aligned}$$

By (H_2), $\|Nu\|_1 \le \|\delta_0\|_1 + \|\delta_1\|_1\|u\|_0 + \|\delta_2\|_1\|u'\|_0 \le \|\delta_0\|_1 + (\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X$, so

$$\|u\|_X < C_5 + D_1(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X$$

for all $u \in \Omega_1$. In view of the inequality (2.1), Ω_1 is bounded.

Define $\Omega_2 = \{u \in \ker L\colon Nu \in \operatorname{Im} L\}$. Then $u = c(-b\phi_1 + a\phi_2)$ for some $c \in \mathbb{R}$. Since $Nu \in \operatorname{Im} L = \ker Q$, $(F_1 - \alpha F_2)TNu = 0$. By (H_3), $|c| \le M_1$, that is, Ω_2 is bounded.

Define $J : Z \to X$ by

$$Jg(t) = (F_1 - \alpha F_2)(Tg)(-b\phi_1(t) + a\phi_2(t)).$$

Recall the characterization of $\operatorname{Im} Q$ in the proof of Lemma 1.4. Since $J(ch)(t) = c(F_1 - \alpha F_2)(Th)(-b\phi_1 + a\phi_2) = c(-b\phi_1 + a\phi_2)$, $J\colon \operatorname{Im} Q \to \ker L$ is an isomorphism.

Let $\Omega_3 = \{u \in \ker L\colon \lambda u + (1-\lambda)JQNu = 0, \lambda \in [0,1]\}$. Let $u \in \Omega_3$ be denoted by $u_c = c(-b\phi_1 + a\phi_2)$. Then $\lambda u + (1-\lambda)JQNu = 0$ implies $\lambda c + (1-\lambda)(F_1 - \alpha F_2)TNu_c = 0$. If $\lambda = 0$, then $JQNu_c = 0$, that is, $u \in \Omega_2$, which is bounded. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0,1)$, then, by (H_2),

$$0 < \lambda c^2 = -(1-\lambda)c(F_1 - \alpha F_2)TNu_c < 0,$$

which is a contradiction. Thus, Ω_3 is bounded.

Let Ω be open and bounded such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. Then the assumptions (i) and (ii) of Theorem 1.3 are fulfilled. It is a routine exercise to show that the mapping N is L-compact on $\overline{\Omega}$. Lemma 1.4 states that L if Fredholm of index zero. We now demonstrate that the third assumption of Theorem 1.3 is verified.

We apply the degree property of invariance under a homotopy to

$$H(u,\lambda) = \lambda I u + (1-\lambda)JQNu, \quad (u,\lambda) \in X \times [0,1].$$

If $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}\ker(JQN|_{\ker L\cap\partial\Omega}, \Omega \cap \ker L, 0) &= \ker(H(\cdot,0), \Omega\cap\ker L, 0)\\ &= \ker(H(\cdot,1), \Omega\cap\ker L, 0)\\ &= \ker(I, \Omega\cap\ker L, 0)\\ &\ne 0,\end{aligned}$$

that is, the assumption (iii) of Theorem 1.3 is checked and the proof is completed. □

It is worth mentioning that the inequality in (H_3) may be reversed since the proof will carry over with a slight modification.

We will replace (H_1) of Theorem 2.1 with

(H_4) there exists a constant $M_0 > 0$ such that, for each $u \in \operatorname{dom} L \setminus \ker L$ with $|u(t)| > M_0$, $t \in [0,1]$, we have $QNu(t) \neq 0$.

Theorem 2.2. *If (L), (F), (H_2)–(H_4) hold, then the boundary value problem (1.1), (1.2) has at least one solution provided $-b\phi_1(t) + a\phi_2(t) \neq 0$ on $[0,1]$, and*

$$D_2(\|\delta_1\|_1 + \|\delta_2\|_1) < 1, \tag{2.3}$$

where

$$D_2 = \frac{A\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]}|-b\phi_1(t) + a\phi_2(t)|} + \|K_P\|.$$

Proof. As in the proof of Theorem 2.1, let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L \colon Lu = \lambda Nu,\ \lambda \in (0,1)\}$. For $u \in \Omega_1$, it follows from (H_4) that there exists $t_0 \in [0,1]$ such that $|u(t_0)| \leq M_0$.
Remark: Note that it does not follow from (H_4) that $|u'(t_0)| \leq M_0$, so we cannot apply the approach taken in the proof of Theorem 2.1 to the present case. Likewise, the inequality $|u(t_0)| \leq M_0$ can not be obtained from (H_5) of Theorem 2.3, which will not allows us to apply the argument of Theorem 2.1. For this reason, here and in the proof of Theorem 2.3 we rely on $u = Pu + (I-P)u$.

Consider $u \in \Omega_1$ and $u = u_1 + u_2$, $u_1 = Pu \in \operatorname{Im} P = \ker L$, $u_2 = (I-P)u = K_P L u = \lambda K_P Nu$. We have, by Lemma 1.6,

$$\|u_2\|_0 < A\|Nu\|_1, \quad \|u_2\|_X < \|K_P\|\|Nu\|_1. \tag{2.4}$$

Now, $u_1 = u - u_2$, so that $|Pu(t_0)| = |u_1(t_0)| \leq |u(t_0)| + |u_2(t_0)| < M_0 + A\|Nu\|_1$. We have

$$|u_1(t_0)| = \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2}|-b\phi_1(t_0) + a\phi_2(t_0)| < M_0 + A\|Nu\|_1.$$

In particular,

$$\frac{|-bl_1(u) + al_2(u)|}{a^2+b^2} \leq \frac{M_0 + A\|Nu\|_1}{\min_{t\in[0,1]}|-b\phi_1(t) + a\phi_2(t)|}.$$

Hence,

$$\begin{aligned}
\|u_1\|_X = \|Pu\|_X &\leq \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2}\|-b\phi_1 + a\phi_2\|_X \\
&\leq \frac{\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]}|-b\phi_1(t) + a\phi_2(t)|}(M_0 + A\|Nu\|_1).
\end{aligned} \tag{2.5}$$

Combining (2.5) and (2.4), we conclude

$$\begin{aligned}
\|u\|_X &\leq \|u_1\|_X + \|u_2\|_X \\
&< C_1 + \left(\frac{A\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]}|-b\phi_1(t) + a\phi_2(t)|} + \|K_P\|\right)\|Nu\|_1 \\
&< C_2 + \left(\frac{A\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]}|-b\phi_1(t) + a\phi_2(t)|} + \|K_P\|\right)(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X \\
&< C_2 + D_2(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X.
\end{aligned}$$

Therefore, by (2.3), Ω_1 is bounded. The rest of the proof is identical to that of Theorem 2.1. □

The next result relies on the assumption

(H_5) there exists a constant $M_0 > 0$ such that, for each $u \in \operatorname{dom} L \setminus \ker L$ with $|u'(t)| > M_0$, $t \in [0,1]$, we have $QNu(t) \neq 0$.

Theorem 2.3. *If (L), (F), (H_2), (H_3), and (H_5) hold, then the boundary value problem (1.1), (1.2) has at least one solution provided $-b\phi_1'(t) + a\phi_2'(t) \neq 0$ on $[0,1]$, and*

$$D_3(\|\delta_1\|_1 + \|\delta_2\|_1) < 1, \tag{2.6}$$

where

$$D_3 = \frac{B\|-b\phi_1(t) + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|} + \|K_P\|.$$

Proof. Again, let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L \colon Lu = \lambda Nu,\ \lambda \in (0,1)\}$ and $u \in \Omega_1$. By (H_5), there exists $t_0 \in [0,1]$ such that $|u'(t_0)| \leq M_0$.

As in the proof of Theorem 2.2, choose $u \in \Omega_1$, where $u = u_1 + u_2$, $u_1 = Pu \in \operatorname{Im} P = \ker L$, $u_2 = (I-P)u = K_P Lu = \lambda K_P Nu$. We have, by Lemma 1.6,

$$\|u_2'\|_0 < B\|Nu\|_1, \qquad \|u_2\|_X < \|K_P\|\|Nu\|_1. \tag{2.7}$$

Since $u_1 = u - u_2$, then $|(Pu)'(t_0)| = |u_1'(t_0)| \leq |u'(t_0)| + |u_2'(t_0)| < M_0 + B\|Nu\|_1$. We have

$$|u_1'(t_0)| = \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2}|-b\phi_1'(t_0) + a\phi_2'(t_0)| < M_0 + A\|Nu\|_1.$$

For $u \in \Omega_1$, we have

$$\frac{|-bl_1(u) + al_2(u)|}{a^2+b^2} \leq \frac{M_0 + B\|Nu\|_1}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|}.$$

We infer

$$\begin{aligned}\|u_1\|_X = \|Pu\|_X &\leq \frac{|-bl_1(u) + al_2(u)|}{a^2+b^2}\|-b\phi_1 + a\phi_2\|_X \\ &\leq \frac{\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|}(M_0 + B\|Nu\|_1).\end{aligned} \tag{2.8}$$

Applying (2.7) and (2.8), we deduce

$$\begin{aligned}\|u\|_X &\leq \|u_1\|_X + \|u_2\|_X \\ &< C_1 + \left(\frac{B\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|} + \|K_P\|\right)\|Nu\|_1 \\ &< C_2 + \left(\frac{B\|-b\phi_1 + a\phi_2\|_X}{\min_{t\in[0,1]} |-b\phi_1'(t) + a\phi_2'(t)|} + \|K_P\|\right)(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X \\ &< C_2 + D_3(\|\delta_1\|_1 + \|\delta_2\|_1)\|u\|_X.\end{aligned}$$

Therefore, Ω_1 is bounded in view of (2.6). The rest of the proof replicates those of the previous theorems. □

Note that the preceding results depend on $a^2 + b^2 \neq 0$ and deal with such resonance conditions that $\dim\ker L = 1$. If $a = b = 0$, then $\dim\ker L = 2$ and the projector P is simply P_0. We can find linearly independent $h_1, h_2 \in Z$ such that $\operatorname{Im} Q = \{c_1h_1 + c_2h_2 : c_1, c_2 \in \mathbb{R}\}$. Moreover, the generalized inverse has a simple form, namely, $K_P g = Tg$. Finally, we observe that the method of proof of Theorem 2.1 applies directly to this case.

Note that (1.3), (1.4) is a special case of (1.1), (1.2), that is, the former serves as an example of the latter. In conclusion, we present an example that cannot be so cheaply obtained.

Consider

$$Lu(t) = u''(t) - u(t) = \kappa(1 + 2\sin u'(t) + u(t)), \quad \text{a.e. } t \in (0,1), \tag{2.9}$$

where $\kappa \neq 0$, and

$$F_1(u) = u(0) - u(1) = 0, \qquad F_2(u) = u'(0) + u'(1) = 0. \tag{2.10}$$

In this case, $\phi_1(t) = e^t$ and $\phi_2(t) = e^{-t}$ with $W(\phi_1, \phi_2)(t) = -2$, $k(t,s) = \sinh(t-s)$. The equation (1.9) becomes

$$u(t) = \int_0^t \sinh(t-s)Lu(s)\,ds + u'(0)\sinh t + u(0)\cosh t.$$

Then $F_1(\phi_1) = 1 - e$, $F_1(\phi_2) = 1 - e^{-1}$, $F_2(\phi_1) = 1 + e$, $F_2(\phi_2) = -1 - e^{-1}$, that is, we have (F) with $a = 1 + e$, $b = -1 - e^{-1}$, and $\alpha = \frac{1-e}{1+e}$. Hence,

$$\ker L = \{c(-b\phi_1(t) + a\phi_2(t)) : c \in \mathbb{R}\} = \{c(e^t + e^{1-t}) : c \in \mathbb{R}\}.$$

Note that $-b\phi_1(t) + a\phi_2(t) \neq 0$ on $[0,1]$.

We also derive

$$\begin{aligned}(F_1 - \alpha F_2)Tg &= -\int_0^1 \sinh(1-s)g(s)\,ds + \frac{1-e}{1+e}\int_0^1 \cosh(1-s)g(s)\,ds\\ &= -\int_0^1 \left(\sinh(1-s) + \frac{e-1}{e+1}\cosh(1-s)\right)g(s)\,ds.\end{aligned}$$

In particular,

$$\begin{aligned}\operatorname{Im} L &= \{g \in Z : (F_1 - \alpha F_2)Tg = 0\}\\ &= \left\{g \in Z : \int_0^1 \left(\sinh(1-s) + \frac{e-1}{e+1}\cosh(1-s)\right)g(s)\,ds = 0\right\}.\end{aligned}$$

Introduce, for convenience,

$$\mathcal{K}(s) = -\sinh(1-s) - \frac{e-1}{e+1}\cosh(1-s) < 0$$

on $[0,1]$. As a result, if $|u(t)| > M_0 = 4$, we have

$$(F_1 - \alpha F_2)TNu = \kappa\int_0^1 \mathcal{K}(s)(1 + 2\sin u'(s) + u(s))\,ds \neq 0.$$

Hence (H_4) holds. It is also easy to find $M_1 > 0$ such that $|c| > M_1$ implies $c(F_1 - \alpha F_2)TNu_c \neq 0$. Indeed,

$$c(F_1 - \alpha F_2)TNu_c = c\kappa\int_0^1 \mathcal{K}(s)(1 + 2\sin u_c'(s))\,ds + c^2\kappa\int_0^1 \mathcal{K}(s)(-b\phi_1(s) + a\phi_2(s))\,ds,$$

where the first integral is bounded in c and the second integral is a constant. Thus, if $|c|$ is large enough, the assumption (H_3) is fulfilled.

Obviously, if $|\kappa|$ is small enough, then also (2.1) holds. Indeed,

$$|\kappa(1 + 2\sin u'(t) + u(t))| \leq |\kappa| + 2|\kappa||u'(t)| + |\kappa||u(t)|,$$

that is, $\|\delta_1\|_1 = 2|\kappa|$ and $\|\delta_2\|_1 = |\kappa|$ can be made small enough to fulfill (H_3) by choosing a sufficiently small $|\kappa|$. By Theorem 2.2, the problem (2.9), (2.10) has a solution. Finally, since $-b\phi_1'(1/2) + a\phi_2'(1/2) = 0$, Theorem 2.3 cannot be applied to this particular problem at resonance.

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