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A time-nonlocal inverse problem for a hyperbolic equation with an integral overdetermination condition

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Abstract. This article is concerned with the study of the unique solvability of a time-nonlocal inverse boundary value problem for second-order hyperbolic equation with an integral overdetermination condition. To study the solvability of the inverse problem, we first reduce the considered problem to an auxiliary system with trivial data and prove its equivalence (in a certain sense) to the original problem. Then using the Banach fixed point principle, the existence and uniqueness of a solution to this system is shown. Further, on the basis of the equivalency of these problems the existence and uniqueness theorem for the classical solution of the inverse coefficient problem is proved for the smaller value of time.

Keywords: inverse problem, hyperbolic equation, overdetermination condition, classical solution, existence, uniqueness.

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1 Introduction

In practice, it is often required to recover the coefficients in an ordinary or partial differential equation from the final overspecified data. Problems of these types are called inverse problems of mathematical physics and are one of the most complicated and practically important problems. The theory of inverse problems is widely used to solve practical problems in almost all fields of science, in particular, in physics, medicine, ecology, and economics. Such problems include the locating groundwater, investigating locations for landfills, acoustics, oil and gas exploration, electromagnetic, X-ray tomography, laser tomography, elasticity, fluid dynamics, and so on.

In the modern mathematical literature, the theory of inverse boundary-value problems for equations of hyperbolic type of the second-order is stated rather satisfactory. In particular, the solvability of the inverse problems in various formulations with different overdetermination

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conditions for partial differential equations is extensively studied in many monographs and papers (see for example, [2,4,5,7,8,10,11,13,14,16,17,20,21,26], and the references therein).

Recently, problems with nonlocal conditions for partial differential equations have been of great interest to applied sciences. In the literature, the term "nonlocal boundary value problems" refers to problems that contain conditions relating the values of the solution and/or its derivatives either at different points of the boundary or at boundary points and some interior points [19]. It is well known that direct nonlocal boundary value problems with integral conditions (with respect to spatial variable) [3,6,9,15] are widely used for thermo-elasticity, chemical engineering, heat conduction, and plasma physics. As well as the direct nonlocal boundary value problems for hyperbolic equations with integral conditions (with respect to time variable) are considered in the papers [12,22] and the references therein. Moreover, In [23–25] the authors present a regularity result for solutions of partial differential equations in the framework of mixed Morrey spaces.

It should also be noted that the statement of the problem and the proof technique used in this paper differ from those of the above articles, and the conditions in the theorems are significantly different from those in them. A distinctive feature of this article is the consideration the inverse boundary value problem for a hyperbolic equation with both spatial and time nonlocal conditions.

2 Mathematical formulation

In the region defined by $D : 0 < x < 1,\ 0 < t < T,\ \ D_T = \overline{D}$, we consider the problem of determining the unknown functions $u(x,t) \in C^1(D_T) \cap C^2(D)$ and $a(t) \in C[0,T]$ such that the pair $\{u(x,t), a(t)\}$ satisfies a one-dimensional hyperbolic equation

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t), \qquad (x,t) \in D, \tag{2.1}$$

with the nonlocal initial conditions

$$u(x,0) + \delta_1 u(x,T) = \varphi(x), \qquad u_t(x,0) + \delta_2 u_t(x,T) = \psi(x), \qquad 0 \le x \le 1, \tag{2.2}$$

the boundary conditions

$$u_x(0,t) = u(1,t) = 0, \qquad 0 \le t \le T, \tag{2.3}$$

and integral overdetermination condition of the first kind

$$\int_0^1 w(x)u(x,t)dx = H(t), \qquad 0 \le t \le T, \tag{2.4}$$

where $\delta_1, \delta_2 \ge 0$, and $0 < T < +\infty$ are given numbers, and $f(x,t), \varphi(x), \psi(x), w(x), H(t)$ are known functions.

To study problem (2.1)–(2.4), we consider the equation

$$y''(t) = \gamma(t)y(t), \qquad 0 < t < T, \tag{2.5}$$

with the boundary conditions

$$y(0) + \delta_1 y(T) = 0, \qquad y'(0) + \delta_2 y'(T) = 0, \tag{2.6}$$

where $\delta_1, \delta_2 \geq 0$ are fixed numbers, $\gamma(t) \in C[0,T]$ is given function, and $y = y(t)$ is desired function.

Clearly, the problem

$$y''(t) = 0, \qquad y(0) + \delta_1 y(T) = 0, \qquad y'(0) + \delta_2 y'(T) = 0 \tag{2.7}$$

has unique trivial solution, for all nonnegative values of δ_1 and δ_2.

It is known [18] that boundary value problem (2.7) has a Green's function of the form

$$G(t,\tau) = \begin{cases} -\frac{\delta_2 t + \delta_1 (T-\tau) + \delta_1 \delta_2 (t-\tau)}{(1+\delta_1)(1+\delta_2)}, & \in [0,\tau], \\ -\frac{\delta_2 t + \delta_1 (T-\tau) - (1+\delta_1+\delta_2)(t-\tau)}{(1+\delta_1)(1+\delta_2)}, & t \in [\tau, T]. \end{cases} \tag{2.8}$$

Lemma 2.1. *Suppose that the function $\gamma(t)$ is continuous on the interval $[0,T]$. If $\delta_1, \delta_2 \geq 0$ and*

$$\frac{1 + 2\delta_1 + 3\delta_2 + \delta_1 \delta_2}{2(1+\delta_1)(1+\delta_2)} \|\gamma(t)\|_{C[0,T]} T^2 < 1, \tag{2.9}$$

then problem (2.5), (2.6) has only a trivial solution.

Proof. Since problem (2.7) has a unique Green function defined by formula (2.8), then it could be argued [18] that boundary-value problem (2.5), (2.6) is equivalent to the integral equation

$$y(t) = \int_0^T G(t,\tau)\gamma(\tau)y(\tau)d\tau. \tag{2.10}$$

Let us introduce the notation

$$A(y(t)) = \int_0^T G(t,\tau)\gamma(\tau)y(\tau)d\tau. \tag{2.11}$$

Then the equation (2.10) can be rewritten as

$$y(t) = A(y(t)). \tag{2.12}$$

Obviously, the operator A is continuous in the space $C[0,T]$.

Now we prove that A is a contraction operator in the space $C[0,T]$. It is easy to see that the inequality

$$\|A(y_1(t)) - A(y_2(t))\|_{C[0,T]} \leq \frac{1 + 2\delta_1 + 3\delta_2 + \delta_1\delta_2}{2(1+\delta_1)(1+\delta_2)} T^2 \|\gamma(t)\|_{C[0,T]} \|y_1(t) - y_2(t)\|_{C[0,T]} \tag{2.13}$$

holds for any functions $y_1(t), y_2(t) \in C[0,T]$.

In view of (2.9) and (2.13) it is clear that the operator A is contractive in $C[0,T]$. Therefore, the operator A has a unique fixed point $y(t)$ in the space $C[0,T]$ which is a solution of equation (2.12). Thus, the integral equation (2.10) has a unique solution in $C[0,T]$. Consequently, problem (2.5), (2.6) also has a unique solution in the indicated space. Since $y(t) = 0$ is a solution to problem (2.5), (2.6), it follows that this problem has a unique trivial solution. $\square$

Now, to study problem (2.1)–(2.4), we consider the following auxiliary inverse boundary value problem: it is required to find a pair of functions $u(x,t) \in C^1(D_T) \cap C^2(D)$, $a(t) \in C[0,T]$ from (2.1)–(2.3) and

$$H''(t) - \int_0^1 w(x)u_{xx}(x,t)dx = H(t)a(t) + \int_0^1 w(x)f(x,t)dx, \qquad 0 < t < T. \tag{2.14}$$

Theorem 2.2. *Assume that* $\varphi(x),\ \psi(x) \in C[0,1],\ H(t) \in C^1[0,T] \cap C^2(0,T),\ H(t) \neq 0,\ 0 \leq t \leq T,\ f(x,t) \in C(D_T)$, *and that the following compatibility conditions are fulfilled*

$$\int_0^1 w(x)\varphi(x)dx = H(0) + \delta_1 H(T), \qquad \int_0^1 w(x)\psi(x)dx = H'(0) + \delta_2 H'(T). \tag{2.15}$$

Then the following statements are true:

(i) each classical solution $\{u(x,t), a(t)\}$ *of problem* (2.1)–(2.4) *is a solution of problem* (2.1)–(2.3), (2.14), *as well;*

(ii) each solution $\{u(x,t), a(t)\}$ *of problem* (2.1)–(2.3), (2.14) *under the circumstance*

$$\frac{(1+2\delta_1+3\delta_2+\delta_1\delta_2)T^2}{2(1+\delta_1)(1+\delta_2)} \|a(t)\|_{C[0,T]} < 1 \tag{2.16}$$

is a classical solution of problem (2.1)–(2.4).

Proof. Let $\{u(x,t), a(t)\}$ be a classical solution of problem (2.1)–(2.4). Multiplying the both sides of Eq.(2.1) by a special function $w(x)$ and integrating from 0 to 1 with respect to x gives

$$\frac{d^2}{dt^2}\int_0^1 w(x)u(x,t)dx - \int_0^1 w(x)u_{xx}(x,t)dx$$
$$= a(t)\int_0^1 w(x)u(x,t)dx + \int_0^1 w(x)f(x,t)dx, \qquad 0<t<T. \tag{2.17}$$

Taking into account the condition $H(t) \in C^1[0,T] \cap C^2(0,T)$, and differentiating (2.4) twice, we have

$$\int_0^1 w(x)u_{tt}(x,t)dx = H''(t), \qquad 0<t<T. \tag{2.18}$$

From (2.17), taking into account (2.4) and (2.18) we arrive at (2.14).

Now, suppose that $\{u(x,t), a(t)\}$ is a solution to problem (2.1)–(2.3), (2.14). Then from (2.17), by allowing for (2.14), we find:

$$\frac{d^2}{dt^2}\left(\int_0^1 w(x)u(x,t)dx - H(t)\right) = a(t)\left(\int_0^1 w(x)u(x,t)dx - H(t)\right), \tag{2.19}$$

for $0<t<T$.

By using the initial conditions (2.2) and the compatibility conditions (2.15), we may write

$$\int_0^1 w(x)u(x,0)dx - H(0) + \delta_1\left(\int_0^1 w(x)u(x,T)dx - H(T)\right)$$
$$= \int_0^1 w(x)(u(x,0)+\delta_1 u(x,T))dx - (H(0)+\delta_1 H(T))$$
$$= \int_0^1 w(x)\varphi(x)dx - (H(0)+\delta_1 H(T)) = 0,$$

$$\int_0^1 w(x)u_t(x,0)dx - H'(0) + \delta_2\left(\int_0^1 w(x)u_t(x,T)dx - H'(T)\right)$$
$$= \int_0^1 w(x)(u_t(x,0)+\delta_2 u_t(x,T))dx - (H'(0)+\delta_2 H'(T))$$
$$= \int_0^1 w(x)\psi(x)dx - (H'(0)+\delta_2 H'(T)) = 0. \tag{2.20}$$

Lemma 2.1 enables us to conclude that the problem (2.19), (2.20) has only a trivial solution. Then, $\int_0^1 w(x)u(x,t)dx - H(t) = 0,\ 0 \leq t \leq T$, i.e., the condition (2.4) is satisfied. □

3 Existence and uniqueness of the solution of the inverse problem

We impose the following conditions on the numbers δ_1, δ_2, and the functions φ, ψ, f, w, and H:

$H_1)$ $\delta_1 \geq 0,\ \delta_2 \geq 0,\ 1 + \delta_1\delta_2 > \delta_1 + \delta_2;$

$H_2)$ $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1),\ \varphi'(0) = \varphi(1) = \varphi''(1) = 0;$

$H_3)$ $\psi(x) \in C^1[0,1],\ \psi''(x) \in L_2(0,1),\ \psi'(0) = \psi(1) = 0;$

$H_4)$ $f(x,t), f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T),\ f_x(0,t) = f(1,t) = 0,\ \ 0 \leq t \leq T;$

$H_5)$ $w(x) \in L_2(0,1),\ H(t) \in C^2[0,T],\ H(t) \neq 0,\ 0 \leq t \leq T.$

We seek the first component of solution $\{u(x,t), a(t)\}$ of the problem (2.1)–(2.3), (2.14) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \cos \lambda_k x, \qquad \lambda_k = \frac{\pi}{2}(2k-1), \tag{3.1}$$

where

$$u_k(t) = 2\int_0^1 u(x,t) \cos \lambda_k x dx, \qquad k = 1,2,\ldots,$$

are twice-differentiable functions on an interval $[0,T]$.

Applying formal scheme of the Fourier method and using (2.1) and (2.2), we get

$$u_k''(t) + \lambda_k^2 u_k(t) = F_k(t;a,u), \qquad k = 1,2,\ldots; \qquad 0 < t < T, \tag{3.2}$$

$$u_k(0) + \delta_1 u_k(T) = \varphi_k, \qquad u_k'(0) + \delta_2 u_k'(T) = \psi_k, \qquad k = 1,2,\ldots, \tag{3.3}$$

where

$$F_k(t;u,a) = f_k(t) + a(t)u_k(t), \qquad f_k(t) = 2\int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$\varphi_k = 2\int_0^1 \varphi(x) \cos \lambda_k x dx, \qquad \psi_k = 2\int_0^1 \psi(x) \cos \lambda_k x dx, \qquad k = 1,2,\ldots$$

Solving the problem (3.2),(3.3) gives

$$\begin{aligned} u_k(t) = &\frac{1}{\rho_k(T)} \left[\varphi_k(\cos \lambda_k t + \delta_2 \cos \lambda_k(T-t)) + \frac{\psi_k}{\lambda_k}(\sin \lambda_k t - \delta_1 \sin \lambda_k(T-t)) \right] \\ &+ \int_0^T G_k(t,\tau) F_k(\tau;u,a) d\tau, \end{aligned} \tag{3.4}$$

where

$$\rho_k(T) = 1 + (\delta_1 + \delta_2) \cos \lambda_k T + \delta_1\delta_2, \tag{3.5}$$

$$G_k(t,\tau) = \begin{cases} -\frac{1}{\lambda_k \rho_k(T)}[\delta_1 \sin \lambda_k(T-\tau) \cos \lambda_k t \\ \qquad + \delta_2 \cos \lambda_k(T-\tau) \sin \lambda_k t + \delta_1\delta_2 \sin \lambda_k(t-\tau)], & t \in [0,\tau], \\ -\frac{1}{\lambda_k \rho_k(T)}[\delta_1 \sin \lambda_k(T-\tau) \cos \lambda_k t + \delta_2 \cos \lambda_k(T-\tau) \sin \lambda_k t \\ \qquad + \delta_1\delta_2 \sin \lambda_k(t-\tau)] + \frac{1}{\lambda_k} \sin \lambda_k(t-\tau), & t \in [\tau,T]. \end{cases} \tag{3.6}$$

Substituting the expression of (3.4) into (3.1), we find the component $u(x,t)$ of the classical solution to problem (2.1)–(2.3), (2.14) to be

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\frac{1}{\rho_k(T)}\left[\varphi_k(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))\right.\right.$$
$$\left.\left.+\frac{\psi_k}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right\}\cos\lambda_k x. \quad (3.7)$$

Thus the problem (2.7), taking into account (2.14), yields

$$a(t)=[H(t)]^{-1}\left\{H''(t)-\int_0^1 w(x)f(x,t)dx+\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)w_k\right\}, \quad (3.8)$$

where

$$w_k=2\int_0^1 w(x)\cos\lambda_k x dx, \qquad k=1,2,\ldots$$

After substituting (3.4) into (3.8), we find the second component $a(t)$ of the solution to problem (2.1)–(2.3), (2.14) in the form

$$a(t)=[H(t)]^{-1}\left\{H''(t)-\int_0^1 w(x)f(x,t)dx+\frac{1}{2}\sum_{k=1}^{\infty}w_k\lambda_k^2\left(\frac{1}{\rho_k(T)}\left[\varphi_k(\cos\lambda_k t+\delta_2\cos\lambda_k(T-t))\right.\right.\right.$$
$$\left.\left.\left.+\frac{\psi_k}{\lambda_k}(\sin\lambda_k t-\delta_1\sin\lambda_k(T-t))\right]+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right)\right\}. \quad (3.9)$$

Thus the solution of problem (2.1)–(2.3), (2.14) was reduced to the solution of systems (3.7), (3.9) with respect to unknown functions $u(x,t)$ and $a(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (2.1)–(2.3), (2.14):

Lemma 3.1. *If $\{u(x,t),a(t)\}$ is any solution to problem (2.1)–(2.3), (2.14), then the functions*

$$u_k(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx, \qquad k=1,2,\ldots$$

satisfy the system (3.4) on an interval $[0,T]$.

Proof. Let $\{u(x,t),a(t)\}$ be any solution of the problem (2.1)–(2.3), (2.14). Multiplying both sides of the Eq. (2.1) by the special functions $2\cos\lambda_k x$ ($k=1,2,\ldots$), integrating from 0 to 1 with respect to x, and using the relations

$$2\int_0^1 u_{tt}(x,t)\cos\lambda_k x dx=\frac{d^2}{dt^2}\left(2\int_0^1 u(x,t)\cos\lambda_k x dx\right)=u_k''(t), \qquad k=1,2,\ldots,$$

$$2\int_0^1 u_{xx}(x,t)\cos\lambda_k x dx=-\lambda_k^2\left(2\int_0^1 u(x,t)\cos\lambda_k x dx\right)=-\lambda_k^2 u_k(t), \qquad k=1,2,\ldots,$$

we obtain that Eq. (3.2) is satisfied.

In like manner, it follows from (2.2) that condition (3.3) is also satisfied.

Thus, the system of functions $u_k(t)$ ($k=1,2,\ldots$) is a solution of problem (3.2), (3.3). From this fact it follows directly that the functions $u_k(t)$ ($k=1,2,\ldots$) also satisfy the system (3.4) on $[0,T]$. □

Obviously, if $u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx,\ k = 1,2,\dots,$ is a solution to system (3.4), then the pair $\{u(x,t), a(t)\}$ of functions $u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\lambda_k x$ and $a(t)$ is also a solution to system (3.7), (3.9).

The next statement follows from Lemma 3.1.

Corollary 3.2. *Assume that the system (3.7), (3.9) has a unique solution. Then the problem (2.1)–(2.3), (2.14) has at most one solution, i.e., if the problem (2.1)–(2.3), (2.14) has a solution, then it is unique.*

Let us consider the functional space that is introduced in [1]. Denote by $B_{2,T}^3$ a set of all functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x, \qquad \lambda_k = \frac{\pi}{2}(2k-1), \qquad k = 1,2,\dots,$$

considered in D_T with the norm $\|u(x,t)\|_{B_{2,T}^3} = J_T(u)$, where $u_k(t) \in C[0,T]$ and

$$J_T(u) \equiv \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]}\right)^2 \right\}^{\frac{1}{2}} < +\infty.$$

Henceforth we shall denote by E_T^3 the topological product of $B_{2,T}^3 \times C[0,T]$, where the norm of an element $z = \{u, a\}$ is determined by the formula

$$\|z\|_{E_T^3} = \|u(x,t)\|_{B_{2,T}^3} + \|a(t)\|_{C[0,T]}.$$

It is known that the spaces $B_{2,T}^3$ and E_T^3 are Banach spaces [27].

Let us now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

in the space E_T^3, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t)\cos\lambda_k x, \qquad \Phi_2(u,a) = \tilde{a}(t),$$

and the functions $\tilde{u}_k(t)$ $(k = 1,2,\dots)$ and $\tilde{a}(t)$ are equal to the right-hand sides of (3.4) and (3.9), respectively.

It is easy to see that under conditions $\delta_1 \geq 0,\ \delta_2 \geq 0,\ 1 + \delta_1\delta_2 > \delta_1 + \delta_2$, we have

$$\frac{1}{\rho_k(T)} \leq \frac{1}{1 - (\delta_1 + \delta_2) + \delta_1\delta_2} \equiv \rho > 0.$$

Taking into account this relation, we obtain

$$\begin{aligned}
&\left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]}\right)^2 \right\}^{\frac{1}{2}} \\
&\quad \leq 2\rho(1+\delta_2)\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 |\varphi_k|\right)^2\right)^{\frac{1}{2}} + 2\rho(1+\delta_1)\left(\sum_{k=1}^{\infty} \left(\lambda_k^2 |\psi_k|\right)^2\right)^{\frac{1}{2}} \\
&\quad + 2(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^2 |f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} \\
&\quad + 2(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},
\end{aligned} \tag{3.10}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[H(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| H''(t) - \int_0^1 w(x) f(x,t) dx \right\|_{C[0,T]} \right.$$
$$+ \frac{1}{2} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\rho(1+\delta_2) \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2 \right)^{\frac{1}{2}} + \rho(1+\delta_1) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_k|)^2 \right)^{\frac{1}{2}} \right.$$
$$+ (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}}$$
$$\left.\left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2)) T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \right\}. \tag{3.11}$$

Then from (3.10) and (3.11), respectively, we find that

$$\left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \leq 4\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)}$$
$$+ 4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$+ 2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2)) T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3},$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[H(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| H''(t) - \int_0^1 w(x) f(x,t) dx \right\|_{C[0,T]} \right.$$
$$+ \frac{1}{2} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[2\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} \right.$$
$$+ 2\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)} + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2)) 2\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$\left.\left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2)) T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3} \right] \right\},$$

or

$$\left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \leq A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \tag{3.12}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \tag{3.13}$$

where

$$A_1(T) = 4\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)}$$
$$+ 4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)},$$
$$B_1(T) = 2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T,$$
$$A_2(T) = \left\|[H(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| H''(t) - \int_0^1 w(x) f(x,t) dx \right\|_{C[0,T]} \right.$$
$$+ \frac{1}{2} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[2\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} + 2\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)} \right.$$
$$\left.\left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2)) 2\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)} \right] \right\},$$

$$B_2(T) = \frac{1}{2}\left\|[H(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T.$$

Finally, from (3.12) and (3.13) we conclude:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{3.14}$$

where

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem.

Theorem 3.3. *Let the assumptions $H_1)$–$H_5)$ and the condition*

$$(A(T)+2)^2 B(T) < 1 \tag{3.15}$$

be satisfied. Then problem (2.1)–(2.3), (2.14) has a unique classical solution in the ball $K = K_R(\|z\|_{E^3_T} \le R = A(T)+2)$ of the space E^3_T.

Remark 3.4. Inequality (3.15) is satisfied for sufficiently small values of T.

Proof. We consider the operator equation

$$z = \Phi z \tag{3.16}$$

in the space E^3_T, where $z = \{u, a\}$, and the components $\Phi_i(u,a),\ i = 1,2$ are defined by the right sides of equations (3.7) and (3.9), respectively.

Similar to (3.14) we obtain that for any $z, z_1, z_2 \in K_R$ the following inequalities hold

$$\|\Phi z\|_{E^3_T} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} \le A(T) + B(T)(A(T)+2)^2, \tag{3.17}$$

$$\|\Phi z_1 - \Phi z_s\|_{E^3_T} \le B(T)TR(\|a_1(t)-a_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B^3_{2,T}}). \tag{3.18}$$

Then by virtue of (3.15) from (3.17) and (3.18) it follows that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore, the operator Φ has a unique fixed point $\{u, a\}$ in the ball $K = K_R$, which is a solution of equation (3.16).

In this way we conclude that the function $u(x,t)$ as an element of space $B^3_{2,T}$ is continuous and has continuous derivatives $u(x,t)$ and $u_{xx}(x,t)$ in D_T.

From (3.2) it is easy to see that

$$\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u''_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$
$$\le \sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \|\,\|f_x(x,t)+a(t)u_x(x,t)\|\,\|_{L_2(0,1)}\right].$$

Thus $u_{tt}(x,t)$ is continuous in the region D_T.

Further, it is possible to verify that Eq. (2.1) and conditions (2.2), (2.3), and (2.14) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is a solution of (2.1)–(2.3), (2.14), and by Lemma 3.1 it is unique. □

On the basis of Theorem 2.2 it is easy to prove the following theorem.

Theorem 3.5. *Suppose that all assumptions of Theorem 3.3, and the conditions*

$$\frac{(1+2\delta_1+3\delta_2+\delta_1\delta_2)T^2(A(T)+2)}{2(1+\delta_1)(1+\delta_2)}<1,$$

$$\int_0^1 w(x)\varphi(x)dx = H(0)+\delta_1 H(T), \qquad \int_0^1 w(x)\psi(x)dx = H'(0)+\delta_2 H'(T)$$

hold. Then problem (2.1)–(2.4) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^3} \le A(T)+2)$ of the space E_T^3.

4 Conclusion

The unique solvability of a time-nonlocal inverse boundary value problem for a second-order hyperbolic equation with an integral overdetermination condition is investigated. Considered problem was reduced to an auxiliary problem in a certain sense and using the contraction mappings principle a unique existence conditions for a solution of equivalent problem are established. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is proved for the smaller value of time.

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