On Linnik's approximation to Goldbach's problem, II

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1 Introduction

Continuing our work [PR] we examine the following problem, initiated by Linnik:

What is the smallest integer K such that every sufficiently large even integer ($N > N_0 = N_0(K)$) could be written as the sum of two primes and K powers of two?

Naturally the binary Goldbach conjecture is equivalent with $K = 0$ and $N_0 = 2$. However, Linnik succeeded about 70 years ago in showing the existence of such a K (without specifying any bound for it) in two subsequent papers [Lin1, Lin2]. The first one assumes the Generalized Riemann Hypothesis (GRH), the second work is unconditional. The first explicit bounds were proven at the end of 1990's.

$K = 54000$ (Liu, Liu, Wang [LLW2]),
$K = 25000$ (Li [Li1]),
$K = 2250$ (Wang [Wan]),
$K = 1906$ (Li [Li2]

Under the assumption of (GRH) these bounds could be reduced to:

(GRH) $\Rightarrow K = 770$ (Liu, Liu, Wang [LLW1]),
(GRH) $\Rightarrow K = 200$ (Liu, Liu, Wang [LLW3]),
(GRH) $\Rightarrow K = 160$ (Wang [Wan]).

In [PR] we showed that $K = 7$ is possible under GRH and announced the result of our present work:

Theorem 1. *Every sufficiently large even number can be written as a sum of two primes and* 8 *powers of two.*

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We mention that independently of us, the results $K = 7$ (on GRH) and $K = 13$ (unconditionally) were proved by D. R. Heath-Brown and J. C. Puchta [HP]. This second bound was improved to $K = 12$ by C. Elsholtz (unpublished) and later independently by Z. Liu and G. Liu [LL].

Finally we remark that all these proofs make use of Gallagher's [Gal] important contribution to this problem who significantly simplified Linnik's work in 1975.

2 Notation. The explicit formula

We will follow closely [PR] in our notation. However, in order to apply the explicit formula of [Pin2] in its original form, we must attach the usual weights $\log p$ to the primes. So we will choose an arbitrary $\varepsilon > 0$ and set $N_1 = N^{1-\varepsilon}$, $N > N_0(\varepsilon, k)$

(2.1)
$$e(\alpha) = e^{2\pi i\alpha}, \quad S(\alpha) = \sum_{N_1 < p \le N} \log p \, e(p\alpha), \quad L = \left[\log_2 N - \sqrt{\log_2 N}\right],$$

where $\log_2 N$ denotes the logarithm to base 2, and p, p', p_i will always denote primes.

Further, let for even N and m

$$r''_k(N) = \sum_{\substack{N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k} \\ 1 \le \nu_i \le L,\ p_i \in (N_1, N)}} \log p_1 \log p_2, \tag{2.2}$$

$$r'_k(N) = \sum_{\substack{N = p + 2^{\nu_1} + \cdots + 2^{\nu_k} \\ 1 \le \nu_i \le L,\ p \in (N_1, N)}} \log p, \tag{2.3}$$

$$r_{k,k}(m) = \#\left\{m = 2^{\nu_1} + \cdots + 2^{\nu_k} - 2^{\mu_1} - \cdots - 2^{\mu_k} : \nu_i, \mu_j \in [1, L]\right\}. \tag{2.4}$$

Similarly to (2.1)–(2.3) of [PR] let

$$2 \le P < Q = \frac{N}{P}, \tag{2.5}$$

and let us define the major ($\mathcal{M}$) and minor ($C(\mathcal{M})$) arcs, respectively by

$$\mathcal{M} = \bigcup_{q \le p} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ},\ \frac{a}{q} + \frac{1}{qQ}\right], \tag{2.6}$$

$$C(\mathcal{M}) = [1/Q,\ 1 + 1/Q] \setminus \mathcal{M}. \tag{2.7}$$

The main difference compared with the results of [Li1], [Li2], [Wan], [LLW1], [LLW2], [LLW3] is a

(i) much more effective treatment of the exponential sum

$$G(\alpha) = \sum_{\nu=1}^{L} 2^{\nu} \tag{2.8}$$

and

(ii) the possibility of having control of $S(\alpha)$ on $\mathcal{M}$ even if we choose P as large as $N^{\frac{4}{9}-\varepsilon}$. Since the estimate of $S(\alpha)$ on the minor arcs does not improve if P increases from $N^{2/5}$ to $N^{4/9}$ we will choose P suitably with

$$P = \left[N^{0.4}, N^{0.41}\right]. \tag{2.9}$$

While the treatment of $G(\alpha)$, the exponential sum over powers of two was fully worked out in [PR] (we have just to apply Corollary 2 there – our present Lemma 6), the methods yielding (ii) were worked out in [Pin2] in form of the explicit formula. We remark, for comparison, that the choice of P was $P = N^{4/9-\varepsilon}$ in [Pin3] for example. Under the assumption of GRH we could choose $P = \sqrt{N}L^{-8}$ (see (2.5) of [PR]). Our present choice (2.9) comes very close to it. This explains the surprisingly small loss of just one power of two in our present unconditional result compared with the result $K = 7$ of [PR], valid on GRH.

In order to introduce the explicit formula let

$$R(h) := \sum_{\substack{p_1 - p_2 = h \\ p_i \in (N_1, N)}} \log p_1 \log p_2 = R_1(h) + R_2(h) \tag{2.10}$$

where

$$R_1(h) = \int_{\mathcal{M}} |S(\alpha)|^2 e(-h\alpha) d\alpha, \quad R_2(h) = \int_{C(\mathcal{M})} |S(\alpha)|^2 e(-h\alpha) d\alpha. \tag{2.11}$$

The explicit formula evaluates the contribution $R_1(h)$ of the major arcs by the aid of so called primitive 'generalized exceptional characters' χ_i belonging to 'generalized exceptional moduli' $r_i \leq P$. These characters are defined by the property that the corresponding $L(s, \chi)$ functions have 'generalized exceptional zeros'

$$\varrho_i = 1 - \delta_i + \gamma_i, \quad \delta_i \leq \frac{H}{\log N}, \quad |\gamma_i| \leq \sqrt{N}, \tag{2.12}$$

where H is a parameter, which will be chosen as a large constant depending on ε. The formula will contain apart from the main term involving the usual singular series

(2.13)

$$\mathfrak{S}(h) = 2C_0 \prod_{\substack{p\mid h\\ p>2}} \left(1 + \frac{1}{p-2}\right), \quad C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\ldots,$$

a 'generalized singular series' for every pair of χ_i, χ_j generalized exceptional characters, satisfying

$$\left|\mathfrak{S}(\chi_i, \chi_j, h)\right| \le \mathfrak{S}(h). \tag{2.14}$$

An important feature of the explicit formula is that the number of zeros (to be counted with multiplicity) is bounded if H is bounded. Their total number M is by a density theorem of Jutila [Jut]

$$M \le Ce^{3H}. \tag{2.15}$$

Apart from the zeros in (2.12) we will include the pole $\varrho = 1$ of $\zeta(s) = L(s, \chi_0)$ ($\chi_0 \bmod 1$) into the set $\mathcal{E} = \mathcal{E}(N, H)$ of 'generalized exceptional singularities' of $\frac{L'}{L}(s, \chi)$ for primitive characters and will consider χ_0 as a primitive character mod 1, $\mathfrak{S}(m)$ as $\mathfrak{S}(m, \chi_0, \chi_0)$. Further we define

$$I(h, \varrho_i, \varrho_j) = \sum_{\substack{m,\ell\in(N_1,N)\\ m-\ell=h}} m^{\varrho_i - 1} \ell^{\overline{\varrho}_j - 1} \tag{2.16}$$

for $\varrho_i, \varrho_j \in \mathcal{E}(X, H)$. For $\varrho_i = \varrho_j = 1$ we obtain the term

$$I(h) = I(h, 1, 1) = N - |h| + O(N^{1-\varepsilon}). \tag{2.17}$$

We further define

$$A(1) = 1, \quad A(\varrho_i) = -1 \ \text{ if } \ \varrho_i \ne 1. \tag{2.18}$$

After this long preparation we can formulate the result.

Theorem 2 (Explicit formula)**.** *For every $P_0 \le N^{\frac{4}{9}-\varepsilon_0}$ we can choose a $P = \left[P_0 N^{-\varepsilon_0}, P_0\right]$ such that for*

(2.19)

$$R_1(h) = \sum_{\varrho_i\in\mathcal{E}} \sum_{\varrho_j\in\mathcal{E}} \mathfrak{S}(\chi_i, \chi_j, h) A(\varrho_i) A(\varrho_j) I(h, \varrho_i, \varrho_j) + O(Ne^{-cH}) + O(N_1)$$

where the generalized singular series satisfy (2.14) *and*

$$\big|\mathfrak{S}(\chi_i, \chi_j, h)\big| \le \varepsilon' \tag{2.20}$$

unless (with a suitable constant $C(\varepsilon')$ *depending on* ε'*)*

$$\text{l.c.m.}[r_i, r_j] \mid C(\varepsilon')h. \tag{2.21}$$

Further we have $R_1(h) \ll \mathfrak{S}(h)N$ *for all* $h \le N$*.*

Remark. In the application we will choose first H as large that

$$\big|O(Ne^{-cH}) + O(N_1)\big| \le \frac{\varepsilon N}{2} \tag{2.22}$$

should hold. Afterwards, let (cf. (2.15))

$$\varepsilon' = \frac{\varepsilon}{6(Ce^{3H}+1)^2}, \quad C(\varepsilon') = C_1(\varepsilon). \tag{2.23}$$

Then by the trivial relation

$$\big|I(h, \varrho_i, \varrho_j)\big| \le I(h) \tag{2.24}$$

we obtain the following

Corollary 1. *For* $h \le \varepsilon N/4$ *we have* $R_1(h) \ll \mathfrak{S}(h)N$*, further*

$$\big|R_1(h) - \mathfrak{S}(h)N\big| \le \varepsilon \mathfrak{S}(h)N \tag{2.25}$$

if for $i = 1, 2, \ldots, M$

$$r_i \nmid C_1(\varepsilon)h, \tag{2.26}$$

where the odd square-free part of r_i*'s satisfies*

$$r_i' = \prod_{p \mid r_i,\ p>2} p \gg L^2 \qquad (i = 1, 2, \ldots, M). \tag{2.27}$$

Proof. The parts (2.25)–(2.26) follow from the explicit formula. In order to see (2.27) we first note that if χ_i is real primitive $(\bmod\ r_i)$ then by Chapter 5 of [Dav] we have

$$r_i = A_i r_i' \quad \text{with} \quad A_i = 1, 4 \text{ or } 8; \tag{2.28}$$

further that the existence of a zero with (2.12) implies by [Pin1] or [GS]

$$\frac{1}{\sqrt{r_i}} \ll \frac{1}{L} \Longleftrightarrow r_i \gg L^2. \tag{2.29}$$

On the other hand, if χ_i is non-real $\bmod r_i > C$, then the zero-free region $\big(\ell_i = \log(r_i(|t|+2))\big)$

$$\sigma > 1 - \frac{1}{4\cdot 10^4\big(\log(2r_i') + (\ell_i \log \ell_i)^{3/4}\big)} \tag{2.30}$$

proved by Iwaniec [Iwa] shows

$$\log r_i \gg L. \tag{2.31}$$

Remark. In our present applications any bound of type

$$r_i \longrightarrow \infty \quad \text{as} \quad N \longrightarrow \infty \tag{2.32}$$

would be sufficient in place of (2.27).

The important point in our Corollary 1 is that although we cannot guarantee the asymptotic formula (2.25) for all relevant values of h but it will be true for almost all h values even if h is restricted to a thin set of numbers like

$$h = 2^{\nu_1} + \dots + 2^{\nu_\ell} - (2^{\mu_1} + \dots + 2^{\mu_\ell}) \qquad \nu_i, \mu_j \in [1, L] \tag{2.33}$$

in our present case, where ℓ will be 1 or k. This is possible since by the explicit formula we know exactly (cf. (2.26)) which values of h might be bad (depending on the finitely many generalized exceptional moduli).

According to this, the contribution of the generalized exceptional moduli might be estimated by the aid of the following

Lemma 1. *Let $m \le N$ be arbitrary, q be an odd squarefree number. Then for any $\eta > 0$*

$$A(m,q) := \sum_{\substack{\nu \le L \\ 2^\nu < m \\ q \mid m - 2^\nu}} \mathfrak{S}(m - 2^\nu) \le \eta L \tag{2.34}$$

if $\min(q, N) > C_0(\eta)$.

Proof. Let, as in the following always, $\sum'$ mean summation over odd square-free integers. Let, further, for any odd d with $[a,b] = \mathrm{l.c.m.}[a,b]$

$$k(d) = \prod_{p|d,\ p>2} \frac{1}{p-2}, \quad \xi(d) = \min\{\nu; 2^\nu \equiv 1(\bmod d)\}. \tag{2.35}$$

Then

$$\begin{aligned} \frac{A(m,q)}{2C_0} &\le {\sum_{d<m}}' k(d) \sum_{\substack{\nu \le L \\ 2^\nu \equiv m \bmod ([d,q])}} 1 \\ &\le {\sum_{d<m}}' k(d) \frac{L}{\xi([d,q])} + {\sum_{d<m}}' k(d) S(m,d) = \sum\nolimits_1 + \sum\nolimits_2 \end{aligned} \tag{2.36}$$

where $S(m,d) = 1$ if there exists a $\nu \le L$ with $d \,|\, m - 2^\nu$ and $S(m,d) = 0$ otherwise. Let us choose $D = D(\eta)$ in such a way that

$$\sum_{d>D}{}' \frac{k(d)}{\xi(d)} < \frac{\eta}{8}. \tag{2.37}$$

This is possible, since the infinite series (2.37) is convergent according to Romanov's basic result (the complete series is, in fact, less than 1.94 – cf. (8.14) of [PR]). Since we have trivially $\xi(m) \ge \log_2 m$ we obtain from (2.37) by $\sum_{d \le x} k(d) \le C \log x$:

$$L^{-1} \sum\nolimits_1 \le \sum_{d \le D} \frac{k(d)}{\xi(q)} + \sum_{d>D} \frac{k(d)}{\xi(d)} \le \frac{C \log D}{\log q} + \frac{\eta}{8} \le \frac{\eta}{4} \tag{2.38}$$

if $C_0(\eta)$ was chosen large enough. Further we have

$$P(m) := \prod_{2^\nu < m,\ \nu \le L} (m - 2^\nu) \le N^L \le e^{L^2}. \tag{2.39}$$

Consequently we have by $\sum_{p|P(m)} \log p \ll L^2$:

$$\begin{aligned} \sum\nolimits_2 &\le \prod_{2<p|P(m)} \left(1 + \frac{1}{p-2}\right) \ll \exp\left(\sum_{p|P(m)} \frac{1}{p}\right) \\ &\le \exp\left(\sum_{\substack{p|P(m) \\ p>L^3}} \frac{\log p}{p} + \sum_{p \le L^3} \frac{1}{p}\right) \le \exp(\log\log L + O(1)) \\ &\ll \log L = o(L). \end{aligned} \tag{2.40}$$

Hence, (2.36), (2.38) and (2.40) prove our lemma. □

3 Two basic results about primes

In this and later sections we will closely follow the structure of proof of [PR] with the appropriate changes adapted to our present situation when we work without any unproved hypothesis.

The estimate on the minor arcs is the celebrated result of Vinogradov [Vin] which can be proved more easily by the method of Vaughan [Vau].

Lemma 2. *For* $\alpha \in C(\mathcal{M})$ *we have*

$$S(\alpha) \ll \left(\frac{N}{\sqrt{P}} + N^{4/5} + \sqrt{NP}\right) L^4 \ll L^4 N^{4/5}. \tag{3.1}$$

It follows by sieve methods that $R(h)$, the actual number of solution of $p - p' = h$ (cf. (2.10)) is at most constant times more than the expected one. The classical result of this type is the following one of Chen Jing Run [Che].

Lemma 3. *For* $N > N_0$ *we have with* $C^* = 3.9171$ *and* $h < N$

$$R(h) \le C^* \mathfrak{S}(h) N. \tag{3.2}$$

4 Numbers of the form $p + 2^\nu$

Similarly to (8.2)–(8.3) of [PR], using the notation (2.1), (2.3), (2.8) we introduce

(4.1)

$$S(N) := \sum_{\substack{p_1 - p_2 = 2^{m_1} - 2^{m_2} \\ p_i \in (N_1, N],\ m_i \in [1, L]}} \log p_1 \cdot \log p_2 = \sum_n \big(r_1'(n)\big)^2 = \int_0^1 |S(\alpha)G(\alpha)|^2 d\alpha.$$

The following result is Lemma 10 of [PR]. Here and later we omit the condition $N > N_0$, which we assumed at any rate from the beginning.

Lemma 4. $S(N) \le 2C_2 N L^2$ *with* $C_2 = C_0 R_0 C^* + \frac{\log 2}{2}$ *where* $R_0 \in (1.936, 1.94)$.

Actually we will need only an estimation of the integral of $(SG)^2$ on the minor arcs. Lemma 4 serves just as an auxiliary result to show

Lemma 5. *With* $C_2' < 4.0826$ *we have*

$$S_2(N) := \int_{C(\mathcal{M})} |S(\alpha)G(\alpha)|^2 d\alpha \le 2C_2' N L^2. \tag{4.2}$$

Remark. This is slightly weaker than the corresponding Lemma 11 of [PR], valid under GRH, where we had the estimate $C_2' < 3.9095$. However, its proof is much more difficult since we cannot use GRH. Here is where the explicit formula and Lemma 1 comes into play. Here and later we need the definition of the exceptional set $\mathcal{H}$ from Corollary 1:

$$\mathcal{H} = \bigcup_{i=1}^{M} \mathcal{H}_i, \qquad \mathcal{H}_i = \left\{ h \le \frac{\varepsilon N}{4};\ r_i \mid C(\varepsilon)h \right\}. \tag{4.3}$$

We remark that $\mathcal{H}$ may be empty if there are no generalized exceptional zeros.

Proof of Lemma 5. Analogously to (8.17)–(8.22) of [PR] we have (cf. (2.11))

$$S_2(N) = \int_0^1 - \int_{\mathcal{M}} = S(N) - 2 \sum_{1 \le \nu_1 < \nu_2 \le L} R_1(2^{\nu_2} - 2^{\nu_1}) - LR_1(0). \tag{4.4}$$

Now, from Corollary 1 and Lemma 1 we have with the notation $\sum^*$ for the condition $1 \le \nu_1 < \nu_2 \le L$

$$\begin{aligned} 2 \sum\nolimits^* R_1(2^{\nu_2} - 2^{\nu_1}) &= (1 + O(\varepsilon)) 2N \sum\nolimits^* \mathfrak{S}(2^{\nu_2} - 2^{\nu_1}) \\ &\quad + O\left(N \sum_{i=1}^{M} \sum_{2^{\nu_2} - 2^{\nu_1} \in \mathcal{H}_i}^{*} \mathfrak{S}\big(2^{\nu_2} - 2^{\nu_1}\big) \right). \end{aligned} \tag{4.5}$$

Now the error term is here for any fixed class $\mathcal{H}_i$ and for any fixed ν_2 at most $\frac{\varepsilon L}{M}$ by Lemma 1 if $N > \widetilde{C}(\varepsilon)$ with a suitable constant ε. On the other hand, we have by (8.8)–(8.14) of [PR]

$$2 \sum\nolimits^* \mathfrak{S}(2^{\nu_2} - 2^{\nu_1}) \sim 2C_0 R_0 L^2 \quad \text{as} \quad L \to \infty. \tag{4.6}$$

Now (4.4)–(4.6) together imply by $R_1(0) > 0$

$$S_2(N) \le 2NL^2 \left(C_0 R_0 (C^* - 1) + \frac{\log 2}{2} + O(\varepsilon) \right) \qquad \text{Q.E.D.} \tag{4.7}$$

Remark. Evaluating $R_1(0)$ the same way as in (8.20) of [PR] we can show the relation

$$R_1(0) = (1 + o(1)) N \log P \ge \frac{2 \log 2 (1 + o(1))}{5} NL, \tag{4.8}$$

which improves (4.7). This leads still to $K = 8$ but enables to apply Lemma 3 with $C^* = 4 + o(1)$ obtainable by Selberg's sieve.

5 Sums of powers of 2

In this section we quote from [PR] two basic results for sums of powers of two. The first one is exactly Corollary 2 of [PR].

Lemma 6. *We have* ($\mu(S)$ *is the Lebesgue measure of* S)

$$(5.1) \qquad |G(\alpha)| = \left|\sum_{j=1}^{L} e(2^j\alpha)\right| < 0.789401L =: c_1 L$$

if $\alpha \in [0,1] \setminus \mathcal{E}^*$ *where* $\mu(\mathcal{E}^*) \ll N^{-3/5}L^{-100}$.

Lemma 7 is a consequence of Theorems 1 and 2 of [KP] (for this form see Theorem 4 of [PR]). Lemma 8 is the nearly trivial Lemma 12 of [PR] (originally Lemma 5 of [Gal]).

Lemma 7. *We have for fixed* $k \geq 1$ *and* $L \to \infty$

$$(5.2) \qquad S(k,L) := \sum_{m=-\infty}^{\infty} r_{k,k}(m)\mathfrak{S}(m) \sim 2L^{2k}(1 + A(k))$$

where $A(k)$ *is a positive constant depending on* k *and*

$$(5.3) \qquad A(4) \in (0.003, 0.004).$$

Lemma 8. $r_{k,k}(0) \leq 2L^{2k-2}$.

6 Proof of Theorem 1

Our crucial estimate, the following Lemma 9 is an exact analogue of Lemma 13 of [PR]. However, since we are not allowed to use GRH, its proof will again use Corollary 1 of the explicit formula and Lemma 1. We will use also the unconditional Lemma 6 of [PR]. $\sum\limits_{m}$ will mean that m runs through all integers.

Lemma 9. *Let* $c_1 = 0.789401$, $C_2' = 4.0826$. *For* $N > N_0(k,\varepsilon)$ *we have*

$$(6.1) \qquad \sum_{m \leq N} \big(r_k'(m)\big)^2 \leq 2NL^{2k}\big(1 + A(k) + C_2' c_1^{2k-2} + \varepsilon\big).$$

Proof. Parseval's identity implies

$$\sum_{1\le m\le N}\big(r'_k(m)\big)^2 \le \int_0^1 \big|S(\alpha)G^k(\alpha)\big|^2 d\alpha = \int_{\mathcal{M}} + \int_{C(\mathcal{M})\cap\mathcal{E}^*} + \int_{C(\mathcal{M})\cap C(\mathcal{E}^*)} . \tag{6.2}$$

Using again Corollary 1, Lemma 1, further Lemma 8, we obtain similarly to Lemma 5

$$\begin{aligned}
\int_{\mathcal{M}} |SG^k|^2 &= \sum_m r_{k,k}(m)\int_{\mathcal{M}} |S(\alpha)|^2 e(m\alpha)d\alpha \\
&\le r_{k,k}(0)\int_0^1 |S(\alpha)|^2 d\alpha + \sum_{m\ne 0} r_{k,k}(m)R_1(m) \\
&\le 2L^{2k-2}\cdot 2N\log N + N(1+O(\varepsilon))\sum_{m\ne 0} r_{k,k}(m)\mathfrak{S}(m) \\
&\quad + O\Bigg(N\sum_{i=1}^{M}\sum_{\substack{m=2^{\nu_1}+\dots+2^{\nu_k}-2^{\mu_1}-\dots-2^{\mu_{k-1}}\\ m-2^{\mu_k}\in\mathcal{H}_i}} \mathfrak{S}(m-2^{\mu_k})\Bigg) \\
&\le N(1+O(\varepsilon))S(k,L) + O(\varepsilon N L^{2k}) \\
&\le 2NL^{2k}(1+A(k)+O(\varepsilon)).
\end{aligned} \tag{6.3}$$

Using Lemmas 5 and 6 we have

$$\int_{C(\mathcal{M})\cap C(\mathcal{E}^*)} |SG^k|^2 \le (c_1L)^{2k-2}\int_{C(\mathcal{M})} |S(\alpha)G(\alpha)|^2 d\alpha \le 2NL^{2k}C_2'c_1^{2k-2}. \tag{6.4}$$

Finally, using $|\mathcal{E}^*| \ll N^{-3/5}L^{-100}$ from Lemma 6 we conclude by Lemma 2

$$\int_{C(\mathcal{M})\cap\mathcal{E}} |SG^k|^2 \ll |\mathcal{E}^*|N^{8/5}L^8 \ll L^{-92}N. \tag{6.5}$$

The three estimates (6.3)–(6.5) prove our lemma.

Now, the last step of the proof is apart from the different numerical data the same as in (10.7)–(10.16) of [PR], so we will be brief.

Using the almost trivial consequence of the prime number theorem we have

$$\sum_{n\le N} r'_k(n) \sim NL^k. \tag{6.6}$$

Thus the average value of $r'_k(n)$ is $2L^k$ for odd n's. So denoting for an even $K = 2k$ (in our case $K = 8$)

$$s_k(n) = r'_k(n) - 2L^k \quad \text{for} \quad 2 \nmid n, \tag{6.7}$$

by (6.6) we have ($\sum^\dagger$ will denote summation over odd numbers)

$$\sum_{m \le N}{}^{\dagger} s_k(m) = o(NL^k). \tag{6.8}$$

Our final goal is to show the positivity of

$$\begin{aligned} r''_K(N) &= \sum_{m+n=N}{}^{\dagger} r'_k(m) r'_k(n) \\ &= 4L^{2k} \sum_{m+n=N}{}^{\dagger} 1 + 4L^k \sum_{n \le N}{}^{\dagger} s_k(n) + \sum_{m+n=N}{}^{\dagger} s_k(m) s_k(n) \\ &= 2L^{2k} N + o(NL^{2k}) + \sum_{m+n=N}{}^{\dagger} s_k(m) s_k(n). \end{aligned} \tag{6.9}$$

However, the last term here is by Cauchy's inequality, (6.6) and Lemma 9

(6.10)
$$\begin{aligned} \left| \sum_{m+n=N}{}^{\dagger} \right| s_k(m) s_k(n) &\le \sum_{n \le N}{}^{\dagger} s_k^2(n) \\ &= \sum_{n \le N}{}^{\dagger} \left(r'_k(n) - 2L^k \right)^2 \\ &= \sum_{n \le N}{}^{\dagger} (r'_k(n))^2 - 4L^k \sum_{n \le N}{}^{\dagger} r_k(n) + 4L^{2k} \cdot \frac{N}{2} \\ &\le 2NL^{2k} \left(1 + A(k) + C'_2 c_1^{2k-2} + \frac{\varepsilon}{2} \right) - 2NL^{2k} \left(1 - \frac{\varepsilon}{2} \right) \\ &\le 2NL^{2k} \cdot \left(A(k) + C'_2 c_1^{2k-2} + \varepsilon \right) := 2NL^{2k} C_3(k). \end{aligned}$$

Now in our case $K = 8$, $k = 4$ our constant is by Lemmas 5–7

$$C_3(4) = A(4) + C'_2 c_1^6 + \varepsilon < 0.992, \tag{6.11}$$

which proves our Theorem 1 in view of (6.9)–(6.10).

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