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NORMAL DIRECTION CURVES AND APPLICATIONS

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Abstract. In this study, we define a new type of associated curves in the Euclidean 3-space such as normal-direction curve and normal-donor curve. We obtain characterizations for these curves. Moreover, we give applications of normal-direction curves to some special curves such as helix, slant helix, plane curve or normal-direction (ND)-normal curves in E^3. And, we show that slant helices and rectifying curves can be constructed by using normal-direction curves.

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1. INTRODUCTION

In the curve theory of Euclidean space, the most important subject is to obtain a characterization for a regular curve, since these characterizations allow to classify curves according to some relations. These characterizations can be given for a single curve or for a curve pair. Helix, slant helix, plane curve, spherical curve, etc. are the well-known examples of single special curves [1,10,12,17,20] and these curves, especially the helices, are used in many applications [2,7,9,16]. Moreover, special curves can be defined by considering Frenet planes. If the position vector of a space curve always lies on its rectifying, osculating or normal planes, then the curve is called rectifying curve, osculating curve or normal curve, respectively [4]. In the Euclidean space E^3, rectifying, normal and osculating curves satisfy Cesaro's fixed point condition, i.e., Frenet planes of such curves always contain a particular point [8, 15]. In particular, there exists a simple relationship between rectifying curves and Darboux vectors (centrodes), which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession [4].

Moreover, special curve pairs are characterized by some relationships between their Frenet vectors or curvatures. Involute-evolute curves, Bertrand curves, Mannheim curves are the well-known examples of curve pairs and studied by some mathematicians [3, 11, 14, 19, 20].

Recently, a new curve pair in the Euclidean 3-space E^3 has been defined by Choi and Kim [6]. They have considered an integral curve γ of a unit vector field X defined in the Frenet basis of a Frenet curve α and they have given the definitions and characterizations of principal-directional curve and principal-donor curve in E^3. They also gave some applications of these curves to some special curves.

In the present paper, we consider a new type of associated curves and define a new curve pair such as normal-direction curve and normal-donor curve in E^3. We obtain some characterizations for these curves and show that normal-direction curve is a space evolute of normal-donor curve. Moreover, we give some applications of normal-direction curve to some special curves such as helix, slant helix or plane curve.

2. PRELIMINARIES

This section includes a brief summary of space curves and definitions of general helix and slant helix in the Euclidean 3-space E^3.

A unit speed curve $\alpha : I \to E^3$ is called a general helix if there is a constant vector u, so that $\langle T, u\rangle = \cos\theta$ is constant along the curve, where $\theta \neq \pi/2$ and $T(s) = \alpha'(s)$ is unit tangent vector of α at s. The curvature (or first curvature) of α is defined by $\kappa(s) = \|\alpha''(s)\|$. Then, the curve α is called Frenet curve, if $\kappa(s) \neq 0$, and the unit principal normal vector $N(s)$ of the curve α at s is given by $\alpha''(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector of α at s. Then $\{T, N, B\}$ is called the Frenet frame of α. For the derivatives of the Frenet frame, the following Frenet-Serret formulae hold:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $\tau(s)$ is the torsion (or second curvature) of α at s. It is well-known that the curve α is a general helix if and only if $\frac{\tau}{\kappa}(s) = \text{constant}$ [17,18]. If both $\kappa(s) \neq 0$ and $\tau(s)$ are constants, we call α as a circular helix. A curve α with $\kappa(s) \neq 0$ is called a slant helix if the principal normal lines of α make a constant angle with a fixed direction. Also, a slant helix α in E^3 is characterized by the differential equation of its curvature κ and its torsion τ given by

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = \text{constant}.$$

(See [12]).

Now, we give the definitions of some associated curves defined by Choi and Kim [6]. Let $I \subset \mathbb{R}$ be an open interval. For a Frenet curve $\alpha : I \to E^3$, consider a vector field X given by

$$X(s) = u(s)T(s) + v(s)N(s) + w(s)B(s), \tag{2.2}$$

where u,v and w are arbitrary differentiable functions of s which is the arc length parameter of α. Let

$$u^2(s)+v^2(s)+w^2(s)=1, \tag{2.3}$$

holds. Then the definitions of X-direction curve and X-donor curve in E^3 are given as follows.

Definition 1. (Definition 2.1. in [6]) Let α be a Frenet curve in Euclidean 3-space E^3 and X be a unit vector field satisfying the equations (2.2) and (2.3). The integral curve $\beta : I \to E^3$ of X is called an X-direction curve of α. The curve α whose X-direction curve is β is called the X-donor curve of β in E^3.

Definition 2. (Definition 2.2. in [6]) An integral curve of principal normal vector $N(s)$ (resp. binormal vector $B(s)$) of α in (2.2) is called the principal-direction curve (resp. binormal-direction curve) of α in E^3.

Remark 1. (Remark 2.3. in [6]) A principal-direction (resp. the binormal-direction) curve is an integral curve of $X(s)$ with $u(s)=w(s)=0$, $v(s)=1$ (resp. $u(s)=v(s)=0$, $w(s)=1$) for all s in (2.2).

3. NORMAL-DIRECTION CURVE AND NORMAL-DONOR CURVE IN E^3

In this section, we will give definitions of normal-direction curve and normal donor curve in E^3. We obtain some theorems and results characterizing these curves. First, we give the following definition.

Definition 3. Let α be a Frenet curve in E^3 and X be a unit vector field lying on the normal plane of α and defined by

$$X(s)=v(s)N(s)+w(s)B(s), \qquad v(s)\neq 0, \qquad w(s)\neq 0, \tag{3.1}$$

and satisfying that the vectors $X'(s)$ and $T(s)$ are linearly dependent. The integral curve $\gamma : I \to E^3$ of $X(s)$ is called a normal-direction curve of α. The curve α whose normal -direction curve is γ is called the normal-donor curve in E^3.

The Frenet frame is a rotation-minimizing with respect to the principal normal N [8]. If we consider a new frame given by $\{T,X,M\}$ where $M=T\times X$, we have that this new frame is rotation-minimizing with respect to T, i.e., the unit vector X belongs to a rotation-minimizing frame.

Since, $X(s)$ is a unit vector and $\gamma : I \to E^3$ is an integral curve of $X(s)$, without loss of generality we can take s as the arc length parameter of γ and we can give the following characterizations in the view of these information.

Theorem 1. *Let $\alpha : I \to E^3$ be a Frenet curve and an integral curve of $X(s)=v(s)N(s)+w(s)B(s)$ be the curve $\gamma : I \to E^3$. Then, γ is a normal-direction curve of*

α *if and only if the following equalities hold,*

$$v(s)=\sin\left(\int \tau ds\right)\neq 0, \qquad w(s)=\cos\left(\int \tau ds\right)\neq 0. \tag{3.2}$$

Proof. Since γ is a normal-direction curve of α, from Definition 3, we have

$$X(s)=v(s)N(s)+w(s)B(s), \tag{3.3}$$

and

$$v^2(s)+w^2(s)=1. \tag{3.4}$$

Differentiating (3.3) with respect to s and by using the Frenet formulas, it follows

$$X'(s)=-v\kappa T+(v'-w\tau)N+(w'+v\tau)B. \tag{3.5}$$

Since we have that X' and T are linearly dependent. Then from (3.5) we can write

$$\begin{cases} -v\kappa\neq 0, \\ v'-w\tau=0, \\ w'+v\tau=0. \end{cases} \tag{3.6}$$

The solutions of second and third differential equations are

$$v(s)=\sin\left(\int \tau ds\right)\neq 0, \qquad w(s)=\cos\left(\int \tau ds\right)\neq 0,$$

respectively, which completes the proof. □

Theorem 2. *Let $\alpha : I \to E^3$ be a Frenet curve. If γ is the normal-direction curve of α, then γ is a space evolute of α.*

Proof. Since γ is an integral curve of X, we have $\gamma'=X$. Denote the Frenet frame of γ by $\{\bar{T},\bar{N},\bar{B}\}$. Differentiating $\gamma'=X$ with respect to s and by using Frenet formulas we get

$$X'=\bar{T}'=\bar{\kappa}\bar{N}. \tag{3.7}$$

Furthermore, we know that X' and T are linearly dependent. Then from (3.7) we get $\bar{N}$ and T are linearly dependent, i.e, γ is a space evolute of α. □

Theorem 3. *Let $\alpha : I \to E^3$ be a Frenet curve. If γ is the normal direction curve of α, then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of γ are given as follows,*

$$\bar{\kappa}=\kappa\left|\sin\left(\int \tau ds\right)\right|, \qquad \bar{\tau}=\kappa\cos\left(\int \tau ds\right).$$

Proof. From (3.5), (3.6) and (3.7), we have

$$\bar{\kappa}\bar{N}=-v\kappa T. \tag{3.8}$$

By considering (3.8) and (3.2) we obtain

$$\bar{\kappa}\bar{N}=-\kappa\sin\left(\int \tau ds\right)T, \tag{3.9}$$

which gives us

$$\bar{\kappa} = \kappa \left| \sin\left(\int \tau ds \right) \right|. \tag{3.10}$$

Moreover, from (3.9) and (3.10), we can write

$$\bar{N} = T. \tag{3.11}$$

Then, we have

$$\bar{B} = \bar{T} \times \bar{N} = \cos\left(\int \tau ds \right) N - \sin\left(\int \tau ds \right) B. \tag{3.12}$$

Differentiating (3.12) with respect to s gives

$$\bar{B}' = -\kappa \cos\left(\int \tau ds \right) T. \tag{3.13}$$

Since $\bar{\tau} = -\langle \bar{B}', \bar{N} \rangle = -\langle \bar{B}', T \rangle$, from (3.13) it follows

$$\bar{\tau} = \kappa \cos\left(\int \tau ds \right), \tag{3.14}$$

that finishes the proof. □

Corollary 1. *Let γ be a normal-direction curve of the curve α. Then the relationships between the Frenet frames of curves are given as follows,*

$$\begin{aligned} X = \bar{T} &= \sin\left(\int \tau ds \right) N + \cos\left(\int \tau ds \right) B, \\ \bar{N} &= T, \\ \bar{B} &= \cos\left(\int \tau ds \right) N - \sin\left(\int \tau ds \right) B. \end{aligned}$$

Proof. The proof is clear from Theorem 3. □

Theorem 4. *Let γ be a normal-direction curve of α with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$. Then curvature κ and torsion τ of α are given by*

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}, \qquad \tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2 + \bar{\tau}^2} \left(\frac{\bar{\kappa}}{\bar{\tau}} \right)'.$$

Proof. From (3.10) and (3.14), we easily get

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}. \tag{3.15}$$

Substituting (3.15) into (3.10) and (3.14), it follows

$$\left| \sin\left(\int \tau ds \right) \right| = \frac{\bar{\kappa}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \tag{3.16}$$

$$\cos\left(\int \tau ds\right) = \frac{\bar{\tau}}{\sqrt{\bar{\kappa}^2+\bar{\tau}^2}}, \tag{3.17}$$

respectively. Differentiating (3.16) with respect to s, we have

$$\tau\cos\left(\int \tau ds\right) = \frac{\bar{\tau}(\bar{\kappa}'\bar{\tau}-\bar{\kappa}\bar{\tau}')}{(\bar{\kappa}^2+\bar{\tau}^2)^{3/2}}. \tag{3.18}$$

From (3.17) and (3.18), it follows

$$\tau = \frac{\bar{\kappa}'\bar{\tau}-\bar{\kappa}\bar{\tau}'}{\bar{\kappa}^2+\bar{\tau}^2},$$

or equivalently,

$$\tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2+\bar{\tau}^2}\left(\frac{\bar{\kappa}}{\bar{\tau}}\right)'. \tag{3.19}$$

□

Theorem 4 leads us to give the following corollary whose proof is clear.

Corollary 2. *Let γ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ be a normal-direction curve of α. Then*

$$\frac{\tau}{\kappa} = -\frac{\bar{\kappa}^2}{(\bar{\kappa}^2+\bar{\tau}^2)^{3/2}}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)', \tag{3.20}$$

is satisfied, where κ and τ are curvature and torsion of α, respectively.

4. APPLICATIONS OF NORMAL-DIRECTION CURVES

In this section, we focus on relations between normal-direction curves and some special curves such as general helix, slant helix, plane curve or rectifying curve in E^3.

4.1. *General helices, slant helices and plane curves*

Considering Corollary 2, we have the following theorems which gives a way to construct the examples of slant helices by using general helices.

Theorem 5. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ be a normal-direction curve of α. Then the followings are equivalent,*

(i) A Frenet curve α is a general helix in E^3.
(ii) α is a normal-donor curve of a slant helix.
(iii) A normal-direction curve of α is a slant helix.

Theorem 6. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ be a normal-direction curve of α. Then the followings are equivalent,*

(i) A Frenet curve α is a plane curve in E^3.
(ii) α is a normal-donor curve of a general helix.

(iii) A normal-direction curve of α *is a general helix.*

Example 1. Let consider the general helix given by the parametrization $\alpha(s)=\left(\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}}\right)$ in E^3 (Fig 1a). The Frenet vectors and curvatures of α are obtained as follows,

$$\begin{aligned}
T(s) &= \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\\
N(s) &= \left(-\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ 0\right),\\
B(s) &= \left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\\
\kappa &= \tau = \frac{1}{2}.
\end{aligned}$$

Then we have $X(s)=(x_1(s),\ x_2(s),\ x_3(s))$ where

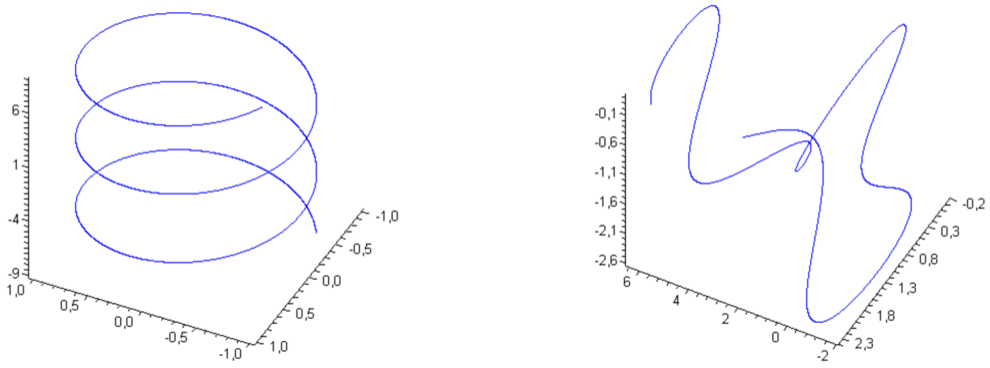
$$\begin{aligned}
x_1(s) &= -\sin\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}},\\
x_2(s) &= \sin\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}},\\
x_3(s) &= \frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right).
\end{aligned}$$

and c is integration constant. Now, we can construct a slant helix γ which is also a normal-direction curve of α (Fig 1b):

$$\gamma=\int_0^s \gamma'(s)ds=\int_0^s X(s)ds=(\gamma_1(s),\ \gamma_2(s),\ \gamma_3(s)),$$

where

$$\begin{aligned}
\gamma_1(s) &= \int_0^s\left[-\sin\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}\right]ds,\\
\gamma_2(s) &= \int_0^s\left[\sin\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}\right]ds,\\
\gamma_3(s) &= \int_0^s\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)ds.
\end{aligned}$$



(A) General helix α.

(B) Slant helix γ.

FIGURE 1. Slant helix γ constructed by α.

4.2. *ND-normal Curves*

In this subsection we define normal-direction (ND)-normal curves in E^3 and give the relationships between normal-direction curves and ND-normal curves.

A space curve whose position vector always lies in its normal plane is called normal curve [5]. Moreover, if the Frenet frame and curvatures of a space curve are given by $\{T, N, B\}$ and κ, τ, respectively, then the vector $\tilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s)$ is called modified Darboux vector of the curve [12, 13].

Let now α be a Frenet curve with Frenet frame $\{T, N, B\}$ and γ a normal-direction curve of α. The curve γ is called normal-direction normal curve (or ND-normal curve) of α, if the position vector of γ always lies on the normal plane of its normal-donor curve α.

The definition of ND-normal curve allows us to write the following equality,

$$\gamma(s) = m(s)N(s) + n(s)B(s), \tag{4.1}$$

where $m(s)$, $n(s)$ are non-zero differentiable functions of s. Since γ is normal-direction curve of α, from Corollary 1, we have

$$\begin{cases} N = \sin\left(\int \tau ds\right)\bar{T} + \cos\left(\int \tau ds\right)\bar{B}, \\ B = \cos\left(\int \tau ds\right)\bar{T} - \sin\left(\int \tau ds\right)\bar{B}. \end{cases} \tag{4.2}$$

Substituting (4.2) in (4.1) gives

$$\gamma(s) = \left[m\sin\left(\int \tau ds\right) + n\cos\left(\int \tau ds\right)\right]\bar{T} \\ + \left[m\cos\left(\int \tau ds\right) - n\sin\left(\int \tau ds\right)\right]\bar{B}. \tag{4.3}$$

Writing

$$\left\{\begin{array}{l}\rho(s)=m\sin\left(\int\tau ds\right)+n\cos\left(\int\tau ds\right),\\ \sigma(s)=m\cos\left(\int\tau ds\right)-n\sin\left(\int\tau ds\right),\end{array}\right. \tag{4.4}$$

in (4.3) and differentiating the obtained equality we obtain

$$\bar{T}=\rho'\bar{T}+(\rho\bar{\kappa}-\sigma\bar{\tau})\bar{N}+\sigma'\bar{B}. \tag{4.5}$$

Then we have

$$\sigma=a=\text{constant},\qquad\qquad \rho=s+b=\frac{\bar{\tau}}{\bar{\kappa}}a, \tag{4.6}$$

where a, b are non-zero integration constants. From (4.6), it follows that

$$\gamma(s)=a\left(\frac{\bar{\tau}}{\bar{\kappa}}\bar{T}+\bar{B}\right)(s)=a\tilde{\bar{D}}(s), \tag{4.7}$$

where $\tilde{\bar{D}}$ is the modified Darboux vector of γ.

Now we can give the followings which characterize ND-normal curves.

Theorem 7. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ be a normal-direction curve of α. If γ is a ND-normal curve in E^3, then we have the followings,*

(i) γ is a rectifying curve in E^3 whose curvatures satisfy $\frac{\bar{\tau}}{\bar{\kappa}}=\frac{s+b}{a}$ where a, b are non-zero constants .

(ii) The position vector and modified Darboux vector $\tilde{\bar{D}}$ of γ are linearly dependent.

Theorem 7 gives a way to construct a rectifying curve by using normal-donor curve as follows:

Corollary 3. *Let $\alpha : I \to E^3$ be a Frenet curve in $E^3$and γ a ND-normal curve of α in E^3. Then the position vector of γ is obtained as follows,*

$$\begin{aligned}\gamma(s)=&\left[(s+b)\sin\left(\int\tau ds\right)+a\cos\left(\int\tau ds\right)\right]N(s)\\&+\left[(s+b)\cos\left(\int\tau ds\right)-a\sin\left(\int\tau ds\right)\right]B(s)\end{aligned} \tag{4.8}$$

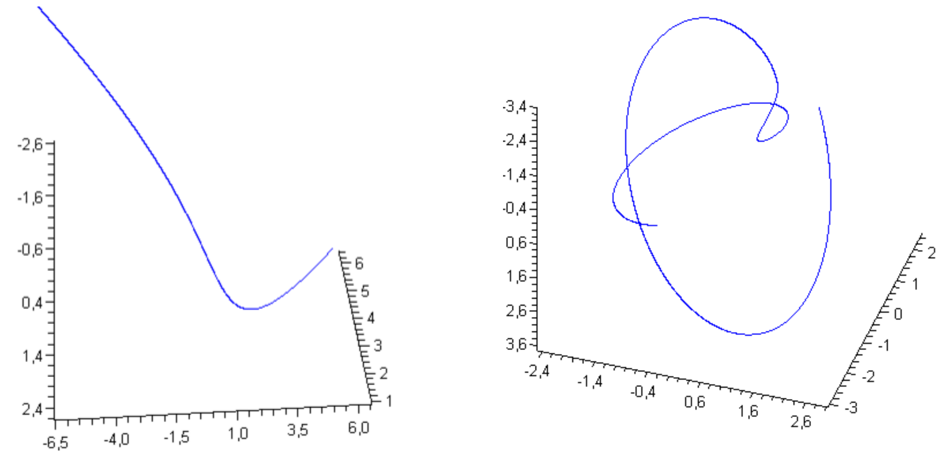
where a, b are non-zero integration constants.

Proof. The proof is clear from (4.1), (4.4) and (4.6). □

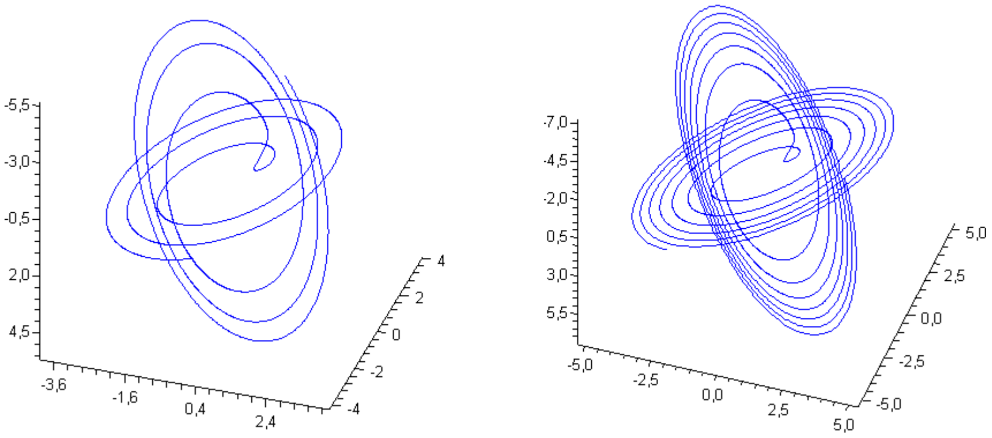
Example 2. Let consider the general helix given by the parametrization

$$\alpha(s)=\left(\sqrt{1+s^2},s,\ln(s+\sqrt{1+s^2})\right),$$

and drawn in Fig 2a.



(A) General helix α. (B) ND-normal curve γ for $-\pi \leq s \leq \pi$.



(C) ND-normal curve γ for $\frac{-3\pi}{2} \leq s \leq \frac{3\pi}{2}$. (D) ND-normal curve γ for $-2\pi \leq s \leq 2\pi$.

FIGURE 2. ND-normal curve γ constructed by α.

Frenet vectors and curvatures of the curve are

$$\begin{aligned}
T(s) &= \frac{1}{\sqrt{2}\sqrt{1+s^2}}\left(s, \sqrt{1+s^2}, 1\right), \\
N(s) &= \frac{1}{\sqrt{1+s^2}}(1, 0, -s), \\
B(s) &= \frac{1}{\sqrt{2}\sqrt{1+s^2}}\left(-s, \sqrt{1+s^2}, -1\right), \\
\kappa = \tau &= \frac{1+s^2}{2},
\end{aligned}$$

respectively. Then from Corollary 3, a ND-normal curve γ is obtained as follows,

$$\begin{aligned}\gamma(s)=\Bigg(&\frac{1}{\sqrt{1+s^2}}\left[(s+b)\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)+a\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right]\\ &-\frac{s}{\sqrt{2(1+s^2)}}\left[(s+b)\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)-a\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right],\\ &-\frac{1}{\sqrt{2}}\left[(s+b)\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)-a\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right],\\ &-\frac{s}{\sqrt{1+s^2}}\left[(s+b)\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)+a\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right]\\ &-\frac{1}{\sqrt{2(1+s^2)}}\left[(s+b)\cos\left(\frac{s}{2}+\frac{s^3}{6}+c\right)-a\sin\left(\frac{s}{2}+\frac{s^3}{6}+c\right)\right]\Bigg)\end{aligned}$$

which is also a rectifying curve in the view of Theroem 7 and drawn in Figures 2b, 2c, 2d by choosing $a=b=1,\ c=0$.

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