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ON CENTRAL FUBINI-LIKE NUMBERS AND POLYNOMIALS

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Abstract. We introduce *the central Fubini-like* numbers and polynomials using Rota approach. Several identities and properties are established as generating functions, recurrences, explicit formulas, parity, asymptotics and determinantal representation.

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1. INTRODUCTION

We start by giving some definitions that will be used throughout this paper.
For $n \geq 1$, the falling factorial denoted $x^{\underline{n}}$ is defined by

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1),$$

and the central factorial $x^{[n]}$, see [4, 9], is defined by

$$x^{[n]} = x(x+n/2-1)(x+n/2-2)\cdots(x-n/2+1).$$

We use the convention, $x^{\underline{0}} = x^{[0]} = 1$.

It is well-known that, for all non-negative integers n and k ($k \leq n$), Stirling numbers of the second kind are defined as the coefficients $S(n,k)$ in the expansion

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}. \tag{1.1}$$

Riordan, in his book [15], shows that, for all non-negative integers n and k ($k \leq n$), the central factorial numbers of the second kind are the coefficients $T(n,k)$ in the expansion

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}. \tag{1.2}$$

In combinatorics, the number of ways to partition a set of n elements into k nonempty subsets are counted by Stirling numbers $S(n,k)$, and the central factorial numbers $T(2n, 2n-2k)$ count the number of ways to place k rooks on a 3D-triangle board of size $(n-1)$, see [11].

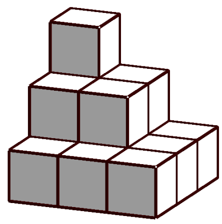

FIGURE 1. 3D-triangle board of size 3.

The coefficients $S(n,k)$ and $T(n,k)$ satisfy, respectively, the triangular recurrences

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) \quad (1 \le k \le n) \tag{1.3}$$

and

$$T(n,k) = \left(\frac{k}{2}\right)^2 T(n-2,k) + T(n-2,k-2) \quad (2 \le k \le n), \tag{1.4}$$

where

$$S(n,k) = T(n,k) = 0 \text{ for } k > n,\ S(0,0) = T(0,0) = T(1,1) = 1 \text{ and } T(1,0) = 0.$$

$S(n,k)$ and $T(n,k)$ admit also the explicit expressions

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n, \tag{1.5}$$

$$T(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{k}{2}-j\right)^n. \tag{1.6}$$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

TABLE 1. The first few values of $S(n,k)$.

	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	0	1				
3	0	$\frac{1}{4}$	0	1			
4	0	0	1	0	1		
5	0	$\frac{1}{16}$	0	$\frac{5}{2}$	0	1	
6	0	0	1	0	5	0	1

TABLE 2. The first few values of $T(n,k)$.

The usual difference operator Δ, the shift operator E^a and the central difference operator δ are given respectively by

$$\Delta f(x) = f(x+1) - f(x),$$
$$\mathsf{E}^a f(x) = f(x+a)$$

and

$$\delta f(x) = f(x+1/2) - f(x-1/2).$$

Riordan, [15], mentioned that the central factorial operator δ satisfies the following property

$$\delta f_n(x) = n f_{n-1}(x), \tag{1.7}$$

where $(f_n(x))_{n\geq 0}$ is a sequence of polynomials with $f_0(x) = 1$.

We can also express δ by means of both Δ and E^a, see [9, 15], as follows:

$$\delta f(x) = \Delta \mathsf{E}^{-1/2} f(x). \tag{1.8}$$

For more details about difference operators, we refer the reader to [9].

2. Central Fubini-like numbers and polynomials

In 1975, Tanny [17], introduced the Fubini polynomials (or ordered Bell polynomials) $F_n(x)$ by applying a linear transformation $\mathcal{L}$ defined as

$$\mathcal{L}(x^{\underline{n}}) := n! x^n.$$

The polynomials $F_n(x)$ are given by

$$F_n(x) := \sum_{k\geq 0}^{n} k! S(n,k) x^k, \tag{2.1}$$

according to,

$$F_n(x) := \mathcal{L}(x^n) = \mathcal{L}\left(\sum_{k=0}^{n} S(n,k) x^{\underline{k}}\right) = \sum_{k=0}^{n} S(n,k) \mathcal{L}(x^{\underline{k}}) = \sum_{k=0}^{n} k! S(n,k) x^k.$$

Putting $x = 1$ in (2) we get

$$F_n := F_n(1) = \sum_{k=0}^{n} k! S(n,k), \tag{2.2}$$

which is the n-th Fubini number.

The Fubini polynomial $F_n(x)$ has the exponential generating function given by, see [17],

$$\sum_{n=0} F_n(x) \frac{t^n}{n!} = \frac{1}{1 - x(e^t - 1)}. \tag{2.3}$$

For more details concerning Fubini numbers and polynomials, see [3, 6, 8, 12, 17, 18, 20] and papers cited therein.

Now, we introduce the linear transformation $\mathcal{Z}$ as follows.

Definition 1. For $n \geq 0$, we define the transformation

$$\mathcal{Z}(x^{[n]}) = n!x^n. \tag{2.4}$$

Then, we have

$$\mathcal{Z}(x^n) = \mathcal{Z}\left(\sum_{k=0}^{n} T(n,k)x^{[k]}\right) = \sum_{k=0}^{n} T(n,k)\mathcal{Z}(x^{[k]}) = \sum_{k=0}^{n} k!T(n,k)x^k. \tag{2.5}$$

And due to Formula (1.6), we are now able to introduce the main notion of the present paper.

Definition 2. The *n-th central Fubini-like* polynomial is given by

$$\mathfrak{C}_n(x) := \sum_{k=0}^{n} k!T(n,k)x^k. \tag{2.6}$$

Setting $x = 1$, we obtain the *central Fubini-like numbers*,

$$\mathfrak{C}_n = \mathfrak{C}_n(1) := \sum_{k=0}^{n} k!T(n,k). \tag{2.7}$$

The first central polynomials $\mathfrak{C}_n(x)$ are given in Table 3.

n	$\mathfrak{C}_{2n}(x)$	$2^{2n}\mathfrak{C}_{2n+1}(x)$
0	1	x
1	$2x^2$	$x+24x^3$
2	$2x^2+24x^4$	$x+240x^3+1920x^5$
3	$2x^2+120x^4+720x^6$	$x+2184x^3+67200x^5+322560x^7$
4	$2x^2+504x^4+10080x^6+40320x^8$	$x+19680x^3+1854720x^5+27095040x^7+92897280x^9$

TABLE 3. First value of $\mathfrak{C}_n(x)$.

The first few central Fubini-like numbers are

$(\mathfrak{C}_{2n})_{n\geq 0}$: $1, 2, 26, 842, 50906, 4946282, 704888186, 138502957322, \ldots$

$(2^{2n}\mathfrak{C}_{2n+1})_{n\geq 0}$: $1, 25, 2161, 391945, 121866721, 57890223865, 38999338931281, \ldots$

2.1. *Exponential generating function*

We begin by establishing the exponential generating function of the central Fubini-like polynomials.

Theorem 1. *The polynomials $\mathfrak{C}_n(x)$ have the following exponential generating function*

$$G(x;t) := \sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n!} = \frac{1}{1-2x\sinh(t/2)}. \tag{2.8}$$

Proof. We have

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} k!T(n,k)x^k\frac{t^n}{n!} = \sum_{k=0} k!x^k \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!},$$

from [15, p. 214] , we have

$$\sum_{n=0} T(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(2\sinh(t/2)\right)^k,$$

therefore

$$\sum_{n=0} \mathfrak{C}_n(x)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\left(2\sinh(t/2)\right)^k x^k = \frac{1}{1-2x\sinh(t/2)}.$$

□

Corollary 1. *The sequence $(\mathfrak{C}_n)_{n\geq 0}$ has the following exponential generating function*

$$\sum_{k=0}^{n} \mathfrak{C}_n\frac{t^n}{n!} = \frac{1}{1-2\sinh(t/2)}. \tag{2.9}$$

2.2. *Explicit representations*

In this subsection we propose some explicit formulas for the central Fubini-like polynomials, we start by the derivative representation.

Proposition 1. *The polynomials $(\mathfrak{C}_n(x))_{n\geq 0}$ correspond to the higher derivative expression*

$$\mathfrak{C}_n(x) = \sum_{k=0}^{\infty}\frac{\partial^n}{\partial^n t}\left(2x\sinh(t/2)\right)^k\Big|_{t=0}.$$

Proof. Let

$$\frac{\partial^n}{\partial^n t}\left(\sum_{m=0}^{\infty}\mathfrak{C}_m(x)\frac{t^m}{m!}\right)\Bigg|_{t=0} = \sum_{m=n}^{\infty}\mathfrak{C}_m(x)\frac{t^{m-n}}{(m-n)!}\Bigg|_{t=0} = \sum_{m=0}^{\infty}\mathfrak{C}_{n+m}(x)\frac{t^m}{m!}\Bigg|_{t=0} = \mathfrak{C}_n(x).$$

Thus from Theorem 1 we get the result. □

From Formula (1.6), it is clear that the following proposition holds.

Proposition 2. *The central Fubini-like polynomials satisfy the following explicit formula*

$$\mathfrak{C}_n(x) = \sum_{k=0}^{n} x^k \sum_{j=0}^{k}(-1)^j\binom{k}{j}(k/2-j)^n.$$

Proof. It suffices to replace $T(n,k)$ in Equation (2.6) by its explicit formula (Equation (1.6)),

$$\mathfrak{C}_n(x) = \sum_{k=0}^{n} k!T(n,k)x^k = \sum_{k=0}^{n} x^k \sum_{j=0}^{k}(-1)^j\binom{k}{j}(k/2-j)^n.$$

□

Theorem 2. *For non-negative n, the following explicit representation holds true.*

$$\mathfrak{C}_n(x) = x\sum_{k=0}^{n-1}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\mathfrak{C}_j(x) = x\sum_{j=0}^{n-1}\binom{n}{j}\delta[0^{n-j}]\mathfrak{C}_j(x), \tag{2.10}$$

where $\delta[0^{n-j}] = (1/2)^{n-j} - (-1/2)^{n-j}$.

The proof will depend on Lemma 1, Lemma 2 and Relation (1.8).

Lemma 1. *For all polynomials $p_n(x)$ the following relation holds true.*

$$\mathcal{Z}(p_n(x)) = x\mathcal{Z}(\delta p_n(x)).$$

Proof. We have

$$\mathcal{Z}(x^{[n]}) = n!x^n = xn(n-1)!x^{n-1} = x\mathcal{Z}(nx^{[n-1]}) = x\mathcal{Z}(\delta x^{[n]}),$$

as any polynomial can be written as sums of central factorials $x^{[n]}$. Thus, we have the result. □

Lemma 2 (Tanny [17]). *For all polynomials $p_n(x)$ we have*

$$\Delta p_n(x) = \sum_{k=0}^{n-1}\binom{n}{k}p_k(x). \tag{2.11}$$

Now we give the proof of Theorem 2,

Proof of Theorem 2. Using Lemma 1, Lemma 2 and setting $p_n(x) = x^n$, we get

$$\begin{aligned}\mathcal{Z}(x^n) &= x\mathcal{Z}(\delta x^n) = x\mathcal{Z}\left(\Delta E^{-1/2}x^n\right) = x\mathcal{Z}\left(\Delta\left(x-\frac{1}{2}\right)^n\right)\\ &= x\mathcal{Z}\left(\sum_{k=0}^{n-1}\binom{n}{k}\left(x-\frac{1}{2}\right)^k\right) = x\mathcal{Z}\left(\sum_{k=0}^{n-1}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}x^j\right)\\ &= x\sum_{k=0}^{n-1}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\mathfrak{C}_j(x).\end{aligned}$$

Using binomial product identity $\binom{n}{k}\binom{k}{j} = \binom{n-j}{k-j}\binom{n}{j}$, we get the result. □

Corollary 2. *The central Fubini-like numbers satisfy*

$$\mathfrak{C}_n = \sum_{j=0}^{n-1}\binom{n}{j}\delta[0^{n-j}]\mathfrak{C}_j. \tag{2.12}$$

Now we give an explicit formula connecting the central Fubini-like polynomials with Stirling numbers of the second kind $S(n,k)$,

Theorem 3. *The central Fubini-like polynomials $\mathfrak{C}_n(x)$ satisfy*

$$\mathfrak{C}_n(x) = \sum_{k=0}^{n} k! x^k \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-k}{2}\right)^j S(n-j,k). \tag{2.13}$$

Proof. From Theorem 1, we have

$$\sum_{n=0} \mathfrak{C}_n(x) \frac{t^n}{n} = \frac{1}{1-2x\sinh(t/2)}.$$

Using the exponential form of $2x\sinh(t/2)$ we get

$$\sum_{n=0} \mathfrak{C}_n(x) \frac{t^n}{n} = \frac{1}{1-xe^{(-t/2)}(e^t-1)} = \sum_{k=0} x^k e^{(-kt/2)} (e^t-1)^k.$$

It is also known that

$$\sum_{n=0} S(n,k) \frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}.$$

Therefore

$$\sum_{n=0} \mathfrak{C}_n(x) \frac{t^n}{n} = \sum_{k=0} x^k k! \sum_{n=0} \left(\frac{-k}{2}\right)^n \frac{t^n}{n!} \sum_{n=0} S(n,k) \frac{t^n}{n!}.$$

Then Cauchy's product implies the identity. □

Corollary 3. *The central Fubini-like numbers $\mathfrak{C}_n$ satisfy*

$$\mathfrak{C}_n = \sum_{k=0}^{n} k! \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-k}{2}\right)^j S(n-j,k). \tag{2.14}$$

2.3. *Umbral representation*

Umbral (or Blissard or symbolic) calculus originated as a method for discovering and proving combinatorial identities in which subscripts are treated as powers. Bell in [1] gave a postulational bases of this calculus. In this section we use the following property given by Riordan [16]. As specified by the author in [16], *"A sequence $a_0, a_1, \ldots$ may be replaced by $a^0, a^1, \ldots$ with the exponents are treated as powers during all formal operations, and only restored as indexes when operations are completed"*. Then when we have

$$a_n = \sum_{k=0} \binom{n}{k} b_k c_{n-k}$$

we can write it as

$$a_n = (b+c)^n,$$

where $b^n \equiv b_n$ and $c^n \equiv c_n$. We note that b^0 and c^0 is not necessary equal to 1.

In the following theorem we use the umbral notation $\mathfrak{C}_k(x) \equiv \mathfrak{C}^k(x)$ and $\mathfrak{C}_k \equiv \mathfrak{C}^k$.

Theorem 4. *Let n be a non-negative integer, for all real x we have*

$$\mathfrak{C}_n(x) = x\left[(\mathfrak{C}(x)+1/2)^n - (\mathfrak{C}(x)-1/2)^n\right].$$

Proof. From Theorem 2 and using the umbral notation, a simple calculation gives the umbral representation result. □

Corollary 4. *For non-negative integer n, we have*

$$\mathfrak{C}_n = (\mathfrak{C}+1/2)^n - (\mathfrak{C}-1/2)^n.$$

2.4. *Parity*

A function $f(x)$ is said to be even when $f(x) = f(-x)$ for all x and it is said to be odd when $f(x) = -f(-x)$.

Theorem 5. *For all non-negative n and real variable x we have*

$$\mathfrak{C}_n(x) = (-1)^n \mathfrak{C}_n(-x).$$

Proof. Using the fact that the function $f : t \mapsto \sinh(t)$ is odd, this gives $G(x;t) = G(-x;-t)$, then comparing the coefficients of $t^n/n!$ in $G(x;t)$ and $G(-x;-t)$ the theorem follows. □

Corollary 5. *The polynomials $\mathfrak{C}_n(x)$ are odd if and only if n is odd.*

Proof. Using Theorem 5, it suffices to replace n by $2k+1$ (resp. $2k$) and establish the property. □

2.5. *Recurrences and derivatives of higher order*

Now we are interested to derive some recurrences for $\mathfrak{C}_n(x)$ in terms of their derivatives.

First, we deal with a recurrence of second order.

Theorem 6. *For $n \geq 2$, the polynomials $\mathfrak{C}_n(x)$ satisfy the following recurrence relation*

$$\mathfrak{C}_n(x) = 2x^2\mathfrak{C}_{n-2}(x) + \left(\frac{x}{4} + 4x^3\right)\mathfrak{C}'_{n-2}(x) + \left(\frac{x^2}{4} + x^4\right)\mathfrak{C}''_{n-2}(x).$$

Here $\mathfrak{C}'_n(x)$ and $\mathfrak{C}''_n(x)$ are respectively the first and second derivative of $\mathfrak{C}_n(x)$.

Proof. From Equation (1.4) we have

$$\begin{aligned}
\mathfrak{C}_n(x) &= \sum_{k=0}^{n} k!T(n,k)x^k \\
&= \sum_{k=2}^{n} k!T(n-2,k-2)x^k + \frac{1}{4}\sum_{k=0}^{n} k^2 k!T(n-2,k)x^k \\
&= \sum_{k=0}^{n} (k+2)!T(n-2,k)x^{k+2} + \frac{x}{4}\left(\sum_{k=0}^{n} kk!T(n-2,k)x^k\right)'
\end{aligned}$$

$$= x^2\left(x^2\sum_{k=0}^{n} k!T(n-2,k)x^k\right)'' + \frac{x}{4}\left(x\left(\sum_{k=0}^{n} k!T(n-2,k)x^k\right)'\right)'$$
$$= x^2\left(x^2\mathfrak{C}_{n-2}(x)\right)'' + \frac{x}{4}\left(x\mathfrak{C}'_{n-2}(x)\right)'$$
$$= 2x^2\mathfrak{C}_{n-2}(x) + \left(\frac{x}{4}+4x^3\right)\mathfrak{C}'_{n-2}(x) + \left(\frac{x^2}{4}+x^4\right)\mathfrak{C}''_{n-2}(x),$$

this concludes the proof. □

In the next theorem we give a recurrence formula for the *r-th* derivative of $\mathfrak{C}_n(x)$.

Proposition 3. *The r-th derivative of $G(x;t)$, defined in (2.8), is given by*

$$\frac{\partial^r}{\partial^r x}G(x;t) = \frac{r!}{x^r}G(x;t)(G(x;t)-1)^r.$$

Proof. Induction on r implies the equality. □

Theorem 7. *Let $\mathfrak{C}_n^{(r)}(x)$ be the r-th derivative of $\mathfrak{C}_n(x)$. Then $\mathfrak{C}_n^{(r)}(x)$ is given by*

$$\mathfrak{C}_n^{(r)}(x) = \frac{r!}{x^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\sum_{j_0+j_1+\cdots+j_k=n}\binom{n}{j_0,j_1,\ldots,j_k}\mathfrak{C}_{j_0}(x)\mathfrak{C}_{j_1}(x)\cdots\mathfrak{C}_{j_k}(x).$$

Proof. Using Proposition 3, by applying Cauchy product and comparing the coefficients of $t^n/n!$, we get the result. □

Corollary 6. *The following equality holds for any real x:*

$$x\mathfrak{C}'_n(x) = \sum_{k=0}^{n-1}\binom{n}{k}\mathfrak{C}_k(x)\mathfrak{C}_{n-k}(x).$$

Proof. Setting $r=1$ in Proposition 3, we get the first derivative of $G(x;t)$ as

$$\frac{\partial}{\partial x}G(x;t) = \frac{2\sinh\left(\frac{t}{2}\right)}{\left(1-2x\sinh(\frac{t}{2})\right)^2} = \frac{G(x;t)}{x}\left(G(x;t)-1\right),$$
$$x\frac{\partial}{\partial x}G(x;t) = G(x;t)^2 - G(x;t),$$
$$x\sum_{n=0}\mathfrak{C}'_n(x)\frac{t^n}{n!} = \left(\sum_{n=0}\mathfrak{C}_n(x)\frac{t^n}{n!}\right)^2 - \sum_{n=0}\mathfrak{C}_n(x)\frac{t^n}{n!},$$

then applying the Cauchy product in the right hand side and comparing the coefficients of $t^n/n!$ we get the result. □

2.6. *Integral representation*

Integral representation is a fundamental property in analytic combinatorics. The central Fubini-like polynomials can be represented as well.

Theorem 8. *The polynomials $\mathfrak{C}_n(x)$ satisfy*

$$\mathfrak{C}_n(x) = \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \frac{\sin(n\theta)}{1-2x\sinh\left(e^{i\theta}/2\right)}\partial\theta.$$

Proof. We will use here the known identity, see [5],

$$k^n = \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \exp\left(ke^{i\theta}\right)\sin(n\theta)\partial\theta.$$

We have

$$\begin{aligned}
\mathfrak{C}_n(x) &= \sum_{k=0}^{\infty} k!T(n,k)x^k \\
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j}\left(\frac{k}{2}-j\right)^n \\
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j}\frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \exp\left((k/2-j)e^{i\theta}\right)\sin(n\theta)\partial\theta \\
&= \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \sin(n\theta)\sum_{k=0}^{\infty} x^k \exp\left(-\frac{k}{2}e^{i\theta}\right)\left(\exp\left(e^{i\theta}\right)-1\right)^k\partial\theta \\
&= \frac{2n!}{\pi}\mathbf{Im}\int_0^{\pi} \frac{\sin(n\theta)}{1-2x\sinh\left(e^{i\theta}/2\right)}\partial\theta.
\end{aligned}$$

□

2.7. *Determinantal representation*

Several papers have been published on determinantal representations of many sequences as Bernoulli numbers, Euler numbers, ordered Bell numbers (or Fubini numbers), etc.

Komatsu and Ramírez in a recent paper gives the following theorem.

Theorem 9 (Komatsu & Ramírez [10]). *Let $(R(j))_{j\geq 0}$ be a sequence, and let α_n be defined by the following determinantal expression for all $n \geq 1$:*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}. \tag{2.15}$$

Then we have

$$\alpha_n = \sum_{j=1}^{n} (-1)^{j-1} R(j) \alpha_{n-j} \qquad (n \geq 1). \tag{2.16}$$

We set $\alpha_0 = 1$.

By applying the previous theorem we get

Theorem 10. *For* $n \geq 1$, *we have*

$$\frac{\mathfrak{C}_n(x)}{n!} = \begin{vmatrix} R(1) & 1 & & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}, \tag{2.17}$$

where

$$R(j) = x\frac{(-1)^{j-1}}{j!}\delta[0^j] = x\frac{(-1)^{j-1}}{j!}\left(\left(\frac{1}{2}\right)^j - \left(-\frac{1}{2}\right)^j\right).$$

Proof. From Theorem 2 we have,

$$\mathfrak{C}_n(x) = x\sum_{j=0}^{n-1}\binom{n}{j}\delta[0^{n-j}]\mathfrak{C}_j(x) = x\sum_{j=1}^{n}\binom{n}{j}\delta[0^j]\mathfrak{C}_{n-j}(x)$$

$$\frac{\mathfrak{C}_n(x)}{n!} = \sum_{j=1}^{n}\frac{x}{j!}\delta[0^j]\frac{\mathfrak{C}_{n-j}(x)}{(n-j)!}.$$

It suffices to set $\alpha_n = \frac{\mathfrak{C}_n(x)}{n!}$ and $R(j) = x\frac{(-1)^{j-1}}{j!}\delta[0^j]$ to get the result. □

Remark 1. The function $R(j) = 0$ for j even.

Using Remark 1, we establish the following binomial convolution for the polynomials $\mathfrak{C}_n(x)$.

Theorem 11. *For* $n \geq 0$ *we have*

$$\mathfrak{C}_{n+1}(x) = x\sum_{k=0}^{\lfloor n/2\rfloor} 4^{-k}\binom{n+1}{2k+1}\mathfrak{C}_{n-2k}(x). \tag{2.18}$$

Proof. From Remark 1 and using Formula (2.16) with $\alpha_n = \mathfrak{C}_n(x)/n!$ and $R(j) = x\frac{(-1)^{j-1}}{j!}\left(\left(\frac{1}{2}\right)^j - \left(-\frac{1}{2}\right)^j\right)$ we get the result. □

Remark 2. Formula (2.18) is better than result of Theorem 2 from a computational point of view.

2.8. *Asymptotic result with respect to $\mathfrak{C}_n$*

Find an asymptotic behaviour of a sequence $(a_n)_{n\geq 0}$ means to find a second function depending on n simple than the expression of a_n which gives a good approximation to the values of a_n when n is large.

In this subsection, we are interested to obtaining the asymptotic behaviour of the central Fubini-like numbers.

Let $(a_n)_{n\geq 0}$ be a sequence of non-negative real numbers, the asymptotic behaviour a_n is closely tied to the poles in $G(z)$, where $G(z)$ is the generating function of a_n,

$$G(z) = \sum_{n=0} a_n z^n.$$

Wilf, in his book [19] and Flajolet et al. in [7] gave a method to determine the asymptotic behaviour a_n which can be summarized in the following steps:

(1) Find the poles $z_0, z_1, \ldots, z_s$ in $G(z)$.

(2) Calculate the principal parts $P(G(z), z_i)$ at the dominant singularities z_i (which have the smallest modulus R) as

$$P(G(z), z_i) = \frac{Res(G(z), z_i)}{(z - z_i)},$$

where $Res(G(z), z_i)$ is the residue of $G(z)$ at the pole z_i.

(3) Set $H(z) = \sum_{i=0}^{s} P(G(z), z_i)$ then write $H(z)$ as the expansion below,

$$H(z) = \sum_{n=0} b_n z^n.$$

(4) The sequence $(b_n)_{n=0}$ is the asymptotic behaviour of a_n when n is big enough,

$$a_n \sim b_n + O\left(\left(\frac{1}{R'} + \varepsilon\right)^n\right), \quad n \longmapsto \infty.$$

where R' is the next smallest modulus of the poles.

For more details about singularities analysis method we refer to [7].

Remark 3. Poles $z_0, z_1, \ldots, z_s$ are considered as simple poles (has a multiplicity equal to 1).

Analytic methods of determining the asymptotic behavior of a sequence $(a_n)_n$ are widely discussed on [2, 7, 13, 14, 19].

Theorem 12. *The asymptotic behaviour of the $\mathfrak{C}_n$ is given by*

$$\mathfrak{C}_n \sim \frac{n!}{2^n \sqrt{5} \log^{n+1}(\phi)} + O\left((0.15732 + \varepsilon)^n\right), \quad n \longmapsto \infty$$

where ϕ is the Golden ratio.

Proof. Applying the previous steps in the generating function $G(z) = \frac{1}{1 - 2\sinh(z/2)}$ gives

(1) The poles of $G(z)$ are

$$z_0 = -2\log\left(\frac{1+\sqrt{5}}{2}\right) + 2i\pi + 4i\pi k \text{ and } z_1 = 2\log\left(\frac{1+\sqrt{5}}{2}\right) + 4i\pi k,$$

with $k \in \mathbb{Z}$.

(2) By setting $k = 0$, the dominant singularity is $z_1 = 2\log(\phi)$ (the modulus $R = 0.96$), then,

$$P(G(z), z_1) = -\frac{2}{\sqrt{5}(z - 2\log(\phi))}.$$

(3) Set $H(z) = -\frac{2}{\sqrt{5}(z-2\log(\phi))}$, if we write $H(z)$ as the expansion we get

$$H(z) = \sum_{n=0} \frac{1}{2^n\sqrt{5}\log^{n+1}(\phi)} z^n.$$

(4) The the next smallest modulus of the poles $R' = 6.356...$, then the asymptotic behaviour of $\mathfrak{C}_n$ when n is big enough is,

$$\mathfrak{C}_n \sim \frac{n!}{2^n\sqrt{5}\log^{n+1}(\phi)} + O\left((0.15732+\varepsilon)^n\right), \quad n \longmapsto \infty.$$

□

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References

[1] E. T. Bell, "Postulational bases for the umbral calculus," *American Journal of Mathematics*, vol. 62, no. 1, pp. 717–724, 1940, doi: 10.2307/2371481.

[2] E. A. Bender, "Asymptotic methods in enumeration," *SIAM review*, vol. 16, no. 4, pp. 485–515, 1974, doi: 10.1137/1016082.

[3] K. N. Boyadzhiev, "A series transformation formula and related polynomials," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 23, pp. 3849–3866, 2005, doi: 10.1155/IJMMS.2005.3849.

[4] P. L. Butzer, K. Schmidt, E. Stark, and L. Vogt, "Central factorial numbers; their main properties and some applications." *Numerical Functional Analysis and Optimization*, vol. 10, no. 5-6, pp. 419–488, 1989, doi: 10.1080/01630568908816313.

[5] D. Callan, "Cesaro's integral formula for the Bell numbers (corrected)," *arXiv preprint arXiv:0708.3301*, 2007.

[6] A. Dil and V. Kurt, "Investigating geometric and exponential polynomials with Euler-Seidel matrices," *J. Integer Seq*, vol. 14, no. 4, 2011.

[7] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge University Press, 2009.

[8] O. A. Gross, "Preferential arrangements," *The American Mathematical Monthly*, vol. 69, no. 1, pp. 4–8, 1962, doi: 10.1080/00029890.1962.11989826.

[9] C. Jordan and K. Jordán, *Calculus of finite differences*. American Mathematical Soc., 1965, vol. 33.
[10] T. Komatsu and J. L. Ramírez, "Some determinants involving incomplete Fubini numbers," *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 26, no.3*, 2018, doi: 10.2478/auom-2018-0038.
[11] N. Krzywonos and F. Alayont, "Rook polynomials in three and higher dimensions," *Involve*, vol. 6, no. 1, pp. 35–52, 2013, doi: 10.2140/involve.2013.6.35.
[12] I. Mező, "Periodicity of the last digits of some combinatorial sequences," *J. Integer Seq*, vol. 17, pp. 1–18, 2014.
[13] A. M. Odlyzko, "Asymptotic enumeration methods," *Handbook of combinatorics*, vol. 2, no. 1063, p. 1229, 1995.
[14] J. Plotkin and J. Rosenthal, "Some asymptotic methods in combinatorics," *Journal of the Australian Mathematical Society*, vol. 28, no. 4, pp. 452–460, 1979, doi: 10.1017/S1446788700012593.
[15] J. Riordan, *Combinatorial identities*. Wiley New York, 1968, vol. 6.
[16] J. Riordan, *Introduction to combinatorial analysis*. Courier Corporation, 2012.
[17] S. M. Tanny, "On some numbers related to the Bell numbers," *Canadian Mathematical Bulletin*, vol. 17, no. 5, pp. 733–738, 1975, doi: 10.4153/CMB-1974-132-8.
[18] W. A. Whitworth, *Choice and chance: with 1000 exercises*. D. Bell and Company;[etc., etc.,], 1901.
[19] H. S. Wilf, *generatingfunctionology*. AK Peters/CRC Press, 2005.
[20] D. Zeitlin, "Remarks on a formula for preferential arrangements," *The American Mathematical Monthly*, vol. 70, no. 2, pp. 183–187, 1963, doi: 10.2307/2312890.

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