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GLOBAL ATTRACTOR FOR THE TIME DISCRETIZED MODIFIED THREE-DIMENSIONAL BÉNARD SYSTEMS

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Abstract. In this paper, we aim to study the existence of global attractors for the time discretized modified three-dimensional (3D) Bénard systems. Using the backward implicit Euler scheme, we obtain the time discretization systems of 3D Bénard systems. Then, by the Galerkin method and the Brouwer fixed point theorem, we prove the existence of the solution to this time-discretized systems. On this basis, we proved the existence of the attractor by the compact embedding theorem of Sobolev. Finally, we discuss the limiting behavior of the solution as N tends to infinity.

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1. INTRODUCTION

In this work, we study the following 3D Bénard system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|\nabla u\|)(u\cdot\nabla)u + \xi\omega = f(x) - \nabla p, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - \Delta\omega + (u\cdot\nabla)\omega = g(x), \\ \omega|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain; $u = u(t,x)$, $\omega = \omega(t,x)$ and $p = p(t,x)$ denote velocity, temperature and pressure of the fluid, respectively; $\nu > 0$, $\xi \in \mathbb{R}^3$ are constants; f: $\Omega \to \mathbb{R}^3$, g: $\Omega \to \mathbb{R}$ are given functions, and for $N \geq 1$, $F_N(r) = \min\left\{1, \dfrac{N}{r}\right\}$.

It is well-known that the Bénard system is a dynamic model describing the rate, pressure and temperature of incompressible fluids that are coupled by Navier-Stokes

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equations and convection diffusion equations. This problem is fundamentally important and of both theoretical and practical interest. In recent years, many important achievements have been made in the study of the Bénard system, of which the study of the solution and attractor of the Bénard system is a very important part. For example, in [10], the authors proved the existence of global solution of the equation (1.1)-(1.2) on channel-like domains by the Galerkin method, then they constructed the global φ attractor of this systems. To date, many studies have investigated the case of $F_N = 1$ in the equation (1.1)-(1.2), for example, see [8, 9, 11, 12]. In [8], authors introduced a class of functions which are strongly continuous with respect to the second component of the vector. Then they prove the existence of solutions for the 3D Bénard system, and construct a multi-valued semi-flow generated by such solutions. Moreover, they obtain the existence of a global φ attractor for the weak-strong topology. In [12], authors investigate the regularized 3D Bénard problem. Using the averaging technique which will give us the properties of the mean characteristics of the flow, they prove that the global existence and uniqueness of the solutions, and then obtain the existence of the global attractor. In [9], authors study the asymptotic behaviour of weak solutions for the 3D Bénard problem. They first show some regularity properties of the weak solutions of this systems. Then they construct a one parameter family of multi-valued semi-flow and obtain the existence of a global attractor with respect to the weak topology of the phase space. In [11], authors first establish an energy inequality in the space L^4 for a broader class of weak solutions. Using this inequality, they prove the existence and connectedness of a global attractor in the space $H_w \times L^2$ for the corresponding m-semi-flow.

It is well-known that the discretization method is the basic method to solve the problems of continuum mechanics, which is a method to approximate the physical quantities in continuum mechanics with finite parameters. The laws of continuum mechanics are generally described by differential equations and integral equations. The discretization method approximates the original problem by transforming it into an algebraic equation with finite parameters. The discretization of differential equation mainly refers to the discretization of time and space. The usual discretization methods include finite difference method, finite element method, weighted residual method and so on (see [1, 2, 4, 6, 7, 14, 16]). In [5], the modified 3D Navier-Stokes equations were discretized on the time by finite difference method, then the existence of the global attractor was proved. In the literature [15], the Benjamin-Bona-Mahony equation was discretized on the time by the Crank-Nicolson scheme. Then, using the Galerkin method and the Brouwer fixed point theorem, authors proved that the existence of the solution to this time discretized system. Furthermore, authors showed that the existence of attractor by Sobolev's compact embedding theorem.

The main purpose of this paper is to investigate the long time dynamical behavior of the solution of the discretized, modified 3D Bénard system (1.1)-(1.2) by the idea in [5, 15].

Let us firstly introduce some notations. Set

$$\mathbb{H} = \Big\{u \in (L^2(\Omega))^3, \text{div}\, u = 0,\ u\cdot n = 0 \text{ on } \partial\Omega\Big\},$$
$$\mathbb{V} = \Big\{u \in (H_0^1(\Omega))^3, \text{div}\, u = 0\Big\},$$

with norms $\|\cdot\|$, $|||\cdot|||$ and scalar products $(\cdot,\cdot)$, $((\cdot,\cdot))$ (the same notations for norms and scalar products also apply to $L^2(\Omega)$, $H_0^1(\Omega)$), where, n is the unit outward normal on $\partial\Omega$. Let

$$b(u,v,z) = \int_\Omega \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} z_j dx, \qquad c(u,\omega,\eta) = \int_\Omega \sum_{i=1}^{3} u_i \frac{\partial \omega}{\partial x_i} \eta dx,$$

and $b_N(u,v,z) = F_N(\|\nabla v\|)b(u,v,z)$. Thanks to Poincaré inequality, we can put

$$((u,v)) = (\nabla u, \nabla v), \qquad |||u||| = \|\nabla u\|.$$

We denote by $\mathcal{P}$ the Leray projection of $L^2(\Omega))^d$ onto $\mathbb{H}$ and by $\mathcal{T}$ the Leray projection of $L^2(\Omega))^d$ onto $L^2(\Omega)$. And we denote by $D(A_1)$ the domain of the Stokes operator $A_1 = -\mathcal{P}\Delta$ in $\mathbb{H}$, and by $D(A_2)$ the domain of $A_2 = -\mathcal{T}\Delta$ in $L^2(\Omega)$. Obviously, $A_1 : \mathbb{V} \to \mathbb{V}^*$, $A_2 : H_0^1(\Omega) \to H^{-1}(\Omega)$ are linear continuous operators and such that

$$\langle A_1 u, v\rangle_{\mathbb{V},\mathbb{V}^*} = (\nabla u, \nabla v), \qquad \langle A_2 \omega, \eta\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = (\nabla \omega, \nabla \eta),$$

where, $u, v \in \mathbb{V}, \omega, \eta \in H_0^1(\Omega)$. From the regularity theory for the Stokes equation, it is proved in [13] that $D(A_1) = H^2(\Omega)^3 \cap \mathbb{V}$, $D(A_2) = H^2(\Omega) \cap H_0^1(\Omega)$, and the following holds true

$$D(A_1) \subset \mathbb{V} \subset \mathbb{H}, \qquad D(A_2) \subset H_0^1(\Omega) \subset L^2(\Omega).$$

Therefore,

$$|||u||| \le \frac{1}{\sqrt{\lambda_1}}\|A_1 u\|,\ \ \forall u \in D(A_1), \qquad |||\,\omega||| \le \frac{1}{\sqrt{\lambda_2}}\|A_2\omega\|,\ \ \forall \omega \in D(A_2),$$
$$\|u\| \le \frac{1}{\sqrt{\lambda_1}}|||u|||,\ \ \forall u \in \mathbb{V}, \qquad \|\omega\| \le \frac{1}{\sqrt{\lambda_2}}|||\omega|||,\ \ \forall \omega \in H_0^1(\Omega),$$

where, $\lambda_1 > 0$, $\lambda_2 > 0$ are the first eigenvalues of the Stokes operator A_1, A_2, respectively.

We introduce two bilinear operators $B : \mathbb{V}\times\mathbb{V} \to \mathbb{V}^*$ and $C : \mathbb{V}\times H_0^1(\Omega) \to H^{-1}(\Omega)$, defined as:

$$\langle B(u,v), z\rangle_{\mathbb{V},\mathbb{V}^*} = b(u,v,z), \qquad \langle C(u,\omega),\eta\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = c(u,\omega,\eta),$$

where, $u,v,z \in \mathbb{V}$, $\omega,\eta \in H_0^1(\Omega)$. From [5],

$$\begin{cases} |b(u,v,z)| \leq C_b \|u\|^{\frac{1}{4}} |||u|||^{\frac{3}{4}} |||v||| \|z\|^{\frac{1}{4}} |||z|||^{\frac{3}{4}}, & \forall u,v,z \in \mathbb{V}, \\ |b(u,v,z)| \leq C_b |||u|||^{\frac{1}{2}} \|A_1 u\|^{\frac{1}{2}} |||v||| \|z\|, & \forall u \in D(A_1), v \in \mathbb{V}, z \in \mathbb{H}, \\ |b(u,v,z)| \leq C_b \|u\|^{\frac{1}{4}} \|A_1 u\|^{\frac{3}{4}} |||v||| \|z\|, & \forall u \in D(A_1), v \in \mathbb{V}, z \in \mathbb{H}, \\ b(u,v,v) = 0, & \forall u,v \in \mathbb{V}. \end{cases} \tag{1.3}$$

Therefore,

$$b_N(u,v,v) = 0, \ \forall u,v \in \mathbb{V} \quad \text{and} \quad \langle B_N(u,v), z\rangle_{\mathbb{V},\mathbb{V}^*} = b_N(u,v,z), \ \forall u,v,z \in \mathbb{V}.$$

Since $\Omega \subset \mathbb{R}^3$ is bounded, there exists a constant $c > 0$, which is only related to Ω, such that for all $v \in H^1(\Omega)$ [5],

$$\|v\|_{L^3(\Omega)} \leq c\|v\|^{1/2} |||v|||^{1/2}, \qquad \|v\|_{L^6(\Omega)} \leq c|||v|||. \tag{1.4}$$

For M, N, p, $q \in \mathbb{R}_+$, there holds [5]

$$|F_N(p) - F_N(q)| \leq \frac{|p-q|}{q}, \qquad |F_M(p) - F_N(q)| \leq \frac{|M-N|}{q} + \frac{|p-q|}{q}. \tag{1.5}$$

By the notations above, the equations (1.1)-(1.2) can be rewritten in the weak form as

$$\begin{cases} u_t + \nu A_1 u + B_N(u,u) + \xi\omega = f(x), \\ \omega_t + A_2\omega + C(u,\omega) = g(x). \end{cases} \tag{1.6}$$

In this paper, we aim to study the existence of global attractors for the time discretized modified three-dimensional (3D) Bénard systems (1.1)-(1.2). To this end, using the *backward implicit Euler scheme*, we obtain the time discretization systems of (1.6):

$$\begin{cases} \dfrac{u^m - u^{m-1}}{k} + \nu A_1 u^m + B_N(u^m,u^m) + \xi\omega^m = f, & (1.7) \\ \dfrac{\omega^m - \omega^{m-1}}{k} + A_2\omega^m + C(u^m,\omega^m) = g, & (1.8) \end{cases}$$

where k is the time step, and $u^m \sim u(t^m)$, $\omega^m \sim \omega(t^m)$.

The main results of this paper are as follows. Firstly, by the Galerkin method and the Brouwer fixed point theorem, we prove the existence of the solution to this time-discretized systems (1.7)-(1.8).

Theorem 1. *Supposing that $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$ be given functions, and let $k > 0$. Then there is at least one set of solutions $\{u^m, \omega^m\} \in D(A_1) \times D(A_2)$ to (1.7)-(1.8) for $m \geq 1$ be integers.*

On this basis, by the compact embedding theorem of Sobolev, we proved the existence of the attractor.

Theorem 2. *Supposing that* $u_0 \in \mathbb{V}$, $\omega_0 \in H_0^1(\Omega)$. *Let* $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$ *be given functions and let* $k > 0$ *small enough. Then the* C^0 *semigroup* S^m *defined by the systems* (1.7)-(1.8) *has global attractors* $\mathcal{A}$ *in* $\mathbb{V} \times H_0^1(\Omega)$.

Finally, we discuss the limiting behavior of the solution to (1.7)-(1.8) as N tends to infinity.

Theorem 3. *Supposing that* $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. *Let* $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$ *be given functions, and let* $k > 0$. *Then, for* $m > 1$ *be integers, the solution sequence* $\{u_N^m, \omega_N^m\}_N$ *of* (1.7)-(1.8) *converges to the weak solution of the following equations when* $N \to \infty$,

$$\begin{cases} \dfrac{u^m - u^{m-1}}{k} + \nu A_1 u^m + B(u^m, u^m) + \xi \omega^m = f, & (1.9) \\ \dfrac{\omega^m - \omega^{m-1}}{k} + A_2 \omega^m + C(u^m, \omega^m) = g. & (1.10) \end{cases}$$

This paper is organized as follows. Section 2 proves the existence of solutions and completes the proof of Theorem 1. Section 3 proves the boundedness of solution in phase space. Section 4 proves the continuous dependence of solution on initial value and parameter N , and establishes a discrete semigroup S^m to complete the proof of Theorem 2. Section 5 discusses the limit behavior of $\{u_N^m, \omega_N^m\}$ as N tends to infinity and completes the proof of Theorem 3.

2. Existence of solutions

In this section, we construct a weak solution of (1.7)-(1.8) by the Faedo-Galerkin method and the following Brouwer fixed point principle (see [3], 24-29).

Lemma 1 ([3]). *Let* X *be a finite-dimensional space endowed with a scalar product* $[\cdot,\cdot]$ *and consider a continuous mapping* $F\colon X \to X$. *Suppose that there exists* $R_0 > 0$ *such that* $[F(U_0), U_0] > 0$ *for all* $U_0 \in X$ *with* $[U_0, U_0] = R_0^2$. *Then there exists* U *with* $[U,U] \le R_0^2$ *such that* $F(U) = 0$.

To prove the existence of the solution for (1.7)-(1.8), the following three steps are required:

Step 1: Construct an approximate solution. Let $p \ge 1$ be an integer. For $u^1, \cdots, u^{m-1}, \omega^1, \cdots, \omega^{m-1}$, we can define the approximate solutions of (1.7)-(1.8) by $u_p^m = \sum_{i=1}^{p} g_{ip}{}^m e_i$ and $\omega_p^m = \sum_{i=1}^{p} h_{ip}{}^m \bar{e}_i$:

$$\begin{cases} \dfrac{u_p^m - u^{m-1}}{k} + \nu A_1 u_p^m + B_N\left(u_p^m, u_p^m\right) + \xi \omega_p^m = f, & (2.1) \\ \dfrac{\omega_p^m - \omega^{m-1}}{k} + A_2 \omega_p^m + C\left(u_p^m, \omega_p^m\right) = g, & (2.2) \end{cases}$$

where ${g_{ip}}^m \in \mathbb{R}$, $\{e_i\}_{i=1}^{\infty} \subset D(A_1)$, corresponding to the eigenvectors of the operator A_1, which are ortho-normal base in $\mathbb{H}$ and orthogonal in $\mathbb{V}$; and ${h_{ip}}^m \in \mathbb{R}$, $\{\bar{e}_i\}_{i=1}^{\infty} \subset D(A_2)$, corresponding to the eigenvectors of the operator A_2, which are ortho-normal base in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$. Let $K_p = \langle e_1, e_2, \cdots, e_p \rangle$ is the space generated by e_1, e_2, $\cdots$, e_p and $M_p = \langle \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_p \rangle$ is the space generated by $\bar{e}_1$, $\bar{e}_2$, $\cdots$, $\bar{e}_p$, we define operator Q_1: $K_p \to K_p$ and Q_2: $M_p \to M_p$ satisfy

$$((Q_1(u), v_1)) = (u, v_1) + \nu k(\nabla u, \nabla v_1) + kb_N(u, u, v_1) + k(\xi\omega, v_1) - (u^{m-1}, v_1) - k(f, v_1);$$
$$((Q_2(\omega), v_2)) = (\omega, v_2) + k(\nabla\omega, \nabla v_2) + kc(u, \omega, v_2) - (\omega^{m-1}, v_2) - k(g, v_2).$$

To apply Lemma 1, we introduce the operator $F(u, \omega) = (Q_1(u), Q_2(\omega))^\top$:

$$\begin{aligned}\Big(F(u,\omega),(v_1,v_2)^\top\Big) &= \Big((Q_1(u),Q_2(\omega))^\top,(v_1,v_2)^\top\Big)\\ &= ((Q_1(u),v_1)) + ((Q_2(\omega),v_2)).\end{aligned}$$

Now we need to prove that $F(u, \omega)$ is continuous in $\mathbb{V}$. To this end, let $u_1, u_2, v_1 \in K_p$ and $\omega_1, \omega_2, v_2 \in M_p$, we have

$$\begin{aligned}&\Big(F(u_1,\omega_1) - F(u_2,\omega_2),(v_1,v_2)^\top\Big)\\ &= \Big((Q_1(u_1),Q_2(\omega_1))^\top,(v_1,v_2)^\top\Big) + \Big((Q_1(u_2),Q_2(\omega_2))^\top,(v_1,v_2)^\top\Big)\\ &= (u_1,v_1) + \nu k(\nabla u_1,\nabla v_1) + kb_N(u_1,u_1,v_1) + k(\xi\omega_1,v_1) - (u^{m-1},v_1)\\ &\quad - k(f,v_1) + (\omega_1,v_2) + k(\nabla\omega_1,\nabla v_2) + kc(u_1,\omega_1,v_2) - (\omega^{m-1},v_2) - k(g,v_2)\\ &\quad - \Big[(u_2,v_1) + \nu k(\nabla u_2,\nabla v_1) + kb_N(u_2,u_2,v_1) + k(\xi\omega_2,v_1) - (u^{m-1},v_1)\\ &\quad - k(f,v_1) + (\omega_2,v_2) + k(\nabla\omega_2,\nabla v_2) + kc(u_2,\omega_2,v_2) - (\omega^{m-1},v_2) - k(g,v_2)\Big]\\ &= (u_1-u_2,v_1) + \nu k(\nabla(u_1-u_2),\nabla v_1) + k(\xi(\omega_1-\omega_2),v_1)\\ &\quad + k(\nabla(\omega_1-\omega_2),\nabla v_2) + (\omega_1-\omega_2,v_2)\\ &\quad + k\Big[b_N(u_1,u_1,v_1) - b_N(u_2,u_2,v_1)\Big] + k\Big[c(u_1,\omega_1,v_2) - c(u_2,\omega_2,v_2)\Big].\end{aligned} \tag{2.3}$$

Here, by using Poincaré inequality, we can get

$$\begin{aligned}&(u_1-u_2,v_1) + \nu k(\nabla(u_1-u_2),\nabla v_1) + k(\xi(\omega_1-\omega_2),v_1)\\ &+ k(\nabla(\omega_1-\omega_2),\nabla v_2) + (\omega_1-\omega_2,v_2)\\ &\le \Big[C\|v_1\| + \nu k|||v_1|||\Big]|||u_1-u_2||| + \Big[C\|v_2\| + kC|\xi|\,\|v_1\| + k|||v_2|||\Big]|||\omega_1-\omega_2|||.\end{aligned} \tag{2.4}$$

And by the definition of F_N and (1.3), we obtain

$$\begin{aligned}&b_N(u_1,u_1,v_1) - b_N(u_2,u_2,v_1) = F_N(|||u_1|||)b(u_1,u_1,v_1) - F_N(|||u_2|||)b(u_2,u_2,v_1)\\ &= F_N(|||u_1|||)b(u_1-u_2,u_1,v_1) + F_N(|||u_2|||)b(u_2,u_1-u_2,v_1)\end{aligned}$$

$$
\begin{aligned}
&+\Big[F_N(|||u_1|||)-F_N(|||u_2|||)\Big]b(u_2,u_1,v_1)\\
&\le CN|||u_1-u_2|||\,|||v_1|||+C|||u_1-u_2|||\,|||u_1|||\,|||v_1|||\\
&\le \Big[CN|||v_1|||+C|||u_1|||\,|||v_1|||\Big]|||u_1-u_2|||. \qquad (2.5)
\end{aligned}
$$

On the other hand, for $c(u,\omega,\eta)$, we have

$$
|c(u,\omega,\eta)|\le \|u\|_{L^6(\Omega)}\|\nabla\omega\|_{L^2(\Omega)}\|\eta\|_{L^3(\Omega)}\le C|||u|||\,|||\omega|||\,|||\eta|||,
$$

and $c(u,\omega,\omega)=0$, so we can get

$$
\begin{aligned}
c(u_1,\omega_1,v_2)-c(u_2,\omega_2,v_2)&=c(u_1-u_2,\omega_1,v_2)+c(u_2,\omega_1-\omega_2,v_2)\\
&\le C|||\omega_1|||\,|||v_2|||\,|||u_1-u_2|||+C|||u_2|||\,|||v_2|||\,|||\omega_1-\omega_2|||. \qquad (2.6)
\end{aligned}
$$

Thus, by (2.3)-(2.6), we obtain

$$
\begin{aligned}
&\Big(F(u_1,\omega_1)-F(u_2,\omega_2),(v_1,v_2)^\top\Big)\\
&\le \Big[C\|v_1\|+\nu k|||v_1|||+CNk|||v_1|||+Ck|||u_1|||\,|||v_1|||+C|||\omega_1|||\,|||v_2|||\Big]|||u_1-u_2|||\\
&\quad+\Big[C\|v_2\|+kC\,|\xi|\,\|v_1\|+k|||v_2|||+Ck|||u_2|||\,|||v_2|||\Big]|||\omega_1-\omega_2|||.
\end{aligned}
$$

It is easy to know that $F(u,\omega)$ is continuous in $\mathbb{V}$. Next, let k be small enough such that $1-\frac{k}{2}>0$ and $1-\frac{k}{2}|\xi|^2>0$. For $\{u,\omega\}\in K_p\times M_p$, by Cauchy-Schwarz and Poincaré inequality, we find

$$
\begin{aligned}
&\Big(F(u,\omega),(u,\omega)^\top\Big)=\Big((Q_1(u),Q_2(\omega))^\top,(u,\omega)^\top\Big)\\
&=(u,u)+\nu k(\nabla u,\nabla u)+kb_N(u,u,u)+k(\xi\omega,u)-(u^{m-1},u)-k(f,u)\\
&\quad+(\omega,\omega)+k(\nabla\omega,\nabla\omega)+kc(u,\omega,\omega)-(\omega^{m-1},\omega)-k(g,\omega)\\
&=\|u\|^2+\nu k|||u|||^2+k(\xi\omega,u)-(u^{m-1},u)-k(f,u)\\
&\quad+\|\omega\|^2+k|||\omega|||^2-(\omega^{m-1},\omega)-k(g,\omega)\\
&\ge \|u\|^2+\nu k|||u|||^2+\|\omega\|^2+k|||\omega|||^2-k\|\xi\omega\|\,\|u\|\\
&\quad-\|u^{m-1}\|\,\|u\|-k\|f\|\,\|u\|-\|\omega^{m-1}\|\,\|\omega\|-k\|g\|\,\|\omega\|\\
&\ge \|u\|^2+\nu k|||u|||^2+\|\omega\|^2+k|||\omega|||^2-\frac{k}{2}\Big[\|\xi\omega\|^2+\|u\|^2\Big]\\
&\quad-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}|||u|||-k\frac{\|f\|}{\sqrt{\lambda_1}}|||u|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}|||\omega|||-k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega|||\\
&=\Big[1-\frac{k}{2}\Big]\|u\|^2+\nu k|||u|||^2+\Big[1-\frac{k}{2}|\xi|^2\Big]\|\omega\|^2+k|||\omega|||^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}|||u|||-k\frac{\|f\|}{\sqrt{\lambda_1}}|||u|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}|||\omega|||-k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega||| \\
&\geq \nu k|||u|||^2+k|||\omega|||^2-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}|||u|||-k\frac{\|f\|}{\sqrt{\lambda_1}}|||u|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}|||\omega|||-k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega||| \\
&=|||u|||\left[\nu k|||u|||-\frac{\|u^{m-1}\|}{\sqrt{\lambda_1}}-k\frac{\|f\|}{\sqrt{\lambda_1}}\right]+|||\omega|||\left[k|||\omega|||-\frac{\|\omega^{m-1}\|}{\sqrt{\lambda_2}}-k\frac{\|g\|}{\sqrt{\lambda_2}}\right].
\end{aligned}
$$

Let $r_1 > \frac{\|u^{m-1}\|+k\|f\|}{\nu k\sqrt{\lambda_1}}$ and $r_2 > \frac{\|\omega^{m-1}\|+k\|g\|}{k\sqrt{\lambda_2}}$, for any $\{u,\omega\} \in K_p \times M_p$ with $|||u||| = r_1$ and $|||\omega||| = r_2$, one has $\left(F(u,\omega),(u,\omega)^\top\right) > 0$. Thus, from Lemma 1, we can find (u^*,ω^*) satisfy $F(u^*,\omega^*) = 0$, which is $(Q_1(u^*),Q_2(\omega^*))^\top = \mathbf{0}$, and so $\begin{cases} Q_1(u^*) = 0, \\ Q_2(\omega^*) = 0. \end{cases}$ Therefore, the approximate solution $\{u_p^m,\omega_p^m\}$ exists.

Step 2: Some priori estimates. For k and m are fixed, we want to get a priori estimates independent of p. Multiplying the equation (2.1) by u_p^m and the equation (2.2) by ω_p^m, we obtain

$$
\begin{aligned}
&\|u_p^m\|^2+\|\omega_p^m\|^2+\|u_p^m-u^{m-1}\|^2+\|\omega_p^m-\omega^{m-1}\|^2+2\nu k|||u_p^m|||^2+2k|||\omega_p^m|||^2 \\
&=2k\left(f,u_p^m\right)+2k\left(g,\omega_p^m\right)+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2-2k\left(\xi\omega_p^m,u_p^m\right) \\
&\leq 2k\|f\|\|u_p^m\|+2k\|g\|\|\omega_p^m\|+2k|\xi|\,\|\omega_p^m\|\|u_p^m\|+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2 \\
&\leq \frac{k}{\nu\lambda_1}\|f\|^2+\nu k|||u_p^m|||^2+\frac{k}{\lambda_2}\|g\|^2+k|||\omega_p^m|||^2+k|\xi|\,\|\omega_p^m\|^2 \\
&\quad +k|\xi|\,\|u_p^m\|^2+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(1-k|\xi|)\,\|u_p^m\|^2+(1-k|\xi|)\,\|\omega_p^m\|^2 \\
&\quad +\|u_p^m-u^{m-1}\|^2+\|\omega_p^m-\omega^{m-1}\|^2+\nu k|||u_p^m|||^2+k|||\omega_p^m|||^2 \\
&\leq \frac{k}{\nu\lambda_1}\|f\|^2+\frac{k}{\lambda_2}\|g\|^2+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2.
\end{aligned}
$$

Let k be small enough such that $1-k|\xi| > 0$, one has

$$
\begin{aligned}
&\|u_p^m\|^2+\|\omega_p^m\|^2+\frac{1}{1-k|\xi|}\|u_p^m-u^{m-1}\|^2 \\
&\quad +\frac{1}{1-k|\xi|}\|\omega_p^m-\omega^{m-1}\|^2+\frac{\nu k}{1-k|\xi|}|||u_p^m|||^2+\frac{k}{1-k|\xi|}|||\omega_p^m|||^2 \\
&\leq \frac{1}{1-k|\xi|}\left[\frac{k}{\nu\lambda_1}\|f\|^2+\frac{k}{\lambda_2}\|g\|^2+\|u^{m-1}\|^2+\|\omega^{m-1}\|^2\right]. \qquad (2.7)
\end{aligned}
$$

Now taking the L^2 inner product of the equation (2.1) with $A_1 u_p^m$ and of the equation (2.2) with $A_2\omega_p^m$, we obtain

$$
\begin{aligned}
&\nu k\|A_1 u_p^m\|^2 + k\|A_2\omega_p^m\|^2 \\
&= k\left(f, A_1 u_p^m\right) + k\left(g, A_2\omega_p^m\right) - \left(u_p^m - u^{m-1}, A_1 u_p^m\right) - \left(\omega_p^m - \omega^{m-1}, A_2\omega_p^m\right) \\
&\quad - kb_N\left(u_p^m, u_p^m, A_1 u_p^m\right) - kc\left(u_p^m, \omega_p^m, A_2\omega_p^m\right) - k\left(\xi\omega_p^m, A_1 u_p^m\right) \\
&\le k\|f\|\|A_1 u_p^m\| + k\|g\|\|A_2\omega_p^m\| + \|u_p^m - u^{m-1}\|\|A_1 u_p^m\| + \|\omega_p^m - \omega^{m-1}\|\|A_2\omega_p^m\| \\
&\quad + k\frac{N}{|||u_p^m|||}\|\left(u_p^m\cdot\nabla\right)u_p^m\|\|A_1 u_p^m\| + k\left|c\left(u_p^m, \omega_p^m, A_2\omega_p^m\right)\right| + k\|\xi\omega_p^m\|\|A_1 u_p^m\| \\
&\le k\|f\|\|A_1 u_p^m\| + k\|g\|\|A_2\omega_p^m\| + \|u_p^m - u^{m-1}\|\|A_1 u_p^m\| + \|\omega_p^m - \omega^{m-1}\|\|A_2\omega_p^m\| \\
&\quad + k\frac{N}{|||u_p^m|||}C|||u_p^m|||\,|||u_p^m|||^{\frac{1}{2}}\|A_1 u_p^m\|^{\frac{3}{2}} + k\left|c\left(u_p^m, \omega_p^m, A_2\omega_p^m\right)\right| + k|\xi|\,\|\omega_p^m\|\|A_1 u_p^m\| \\
&\le k\|f\|\|A_1 u_p^m\| + k\|g\|\|A_2\omega_p^m\| + \|u_p^m - u^{m-1}\|\|A_1 u_p^m\| + \|\omega_p^m - \omega^{m-1}\|\|A_2\omega_p^m\| \\
&\quad + CNk|||u_p^m|||^{\frac{1}{2}}\|A_1 u_p^m\|^{\frac{3}{2}} + k\left|c\left(u_p^m, \omega_p^m, A_2\omega_p^m\right)\right| + k|\xi|\,\|\omega_p^m\|\|A_1 u_p^m\|.
\end{aligned}
$$

Since

$$
|c(u,\omega,A_2\omega)| \le \|u\|_{L^\infty(\Omega)}\|\nabla\omega\|_{L^2(\Omega)}\|A_2\omega\|_{L^2(\Omega)} \le C\left[\frac{\varepsilon}{2}|||u|||^2|||\omega|||^2 + \frac{1}{2\varepsilon}\|A_2\omega\|^2\right],
$$

we can get

$$
\begin{aligned}
\|A_1 u_p^m\|^2 + \|A_2\omega_p^m\|^2 &\le C\|f\|^2 + C\|g\|^2 + C\|u_p^m - u^{m-1}\|^2 + C\|\omega_p^m - \omega^{m-1}\|^2 \\
&\quad + CNk|||u_p^m|||^2 + ck|\xi|\,\|\omega_p^m\|^2 + c|||u_p^m|||^2|||\omega_p^m|||^2. \qquad (2.8)
\end{aligned}
$$

By (2.7)-(2.8), we can get

$$
\|A_1 u_p^m\|^2 + \|A_2\omega_p^m\|^2 \le C\left(\|f\|, \|g\|, \nu, \lambda_1, \lambda_2, k, \|u^{m-1}\|, \|\omega^{m-1}\|, N, |\xi|\right).
$$

Step 3: Passage to the limit. For k and m fixed, from the above inequality we can see $\left\{u_p^m\right\}_p$, $\left\{\omega_p^m\right\}_p$ are bounded in $D(A_1)$ and $D(A_2)$, respectively. Thus one can extract from $\left\{u_p^m\right\}_p$ and $\left\{\omega_p^m\right\}_p$ subsequences respectively, denoted also by $\left\{u_p^m\right\}_p$, $\left\{\omega_p^m\right\}_p$, such that $u_p^m \rightharpoonup u^m$, $p\to\infty$ in $D(A_1)$, and $\omega_p^m \rightharpoonup \omega^m$, $p\to\infty$ in $D(A_2)$. But, $D(A_1)\hookrightarrow \mathbb{V}$ and $D(A_2)\hookrightarrow H_0^1(\Omega)$ are compact, so $u_p^m \to u^m$, $p\to\infty$ in $\mathbb{V}$, and $\omega_p^m \to \omega^m$, $p\to\infty$ in $H_0^1(\Omega)$. Next, we prove that $\left\{u^m, \omega^m\right\}$ is the solution of (2.1)-(2.2). For the purpose, it is enough to show that

$$
\lim_{p\to\infty} b_N\left(u_p^m, u_p^m, v_1\right) = b_N\left(u^m, u^m, v_1\right) \quad \text{and} \quad \lim_{p\to\infty} c\left(u_p^m, \omega_p^m, v_2\right) = c\left(u^m, \omega^m, v_2\right).
$$

To this end, we calculate as follows

$$
\begin{aligned}
&b_N\left(u_p^m, u_p^m, v_1\right) - b_N\left(u^m, u^m, v_1\right) = F_N\left(|||u_p^m|||\right) b\left(u_p^m, u_p^m, v_1\right) - F_N\left(|||u^m|||\right) b\left(u^m, u^m, v_1\right) \\
&= F_N\left(|||u_p^m|||\right)\left[b\left(u_p^m, u_p^m, v_1\right) - b\left(u^m, u^m, v_1\right)\right]
\end{aligned}
$$

$$+F_N\left(|||u_p^m|||\right)b\left(u^m,u^m,v_1\right)-F_N\left(|||u^m|||\right)b\left(u^m,u^m,v_1\right)$$
$$=F_N\left(|||u_p^m|||\right)\left[b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right]+\left[F_N\left(|||u_p^m|||\right)-F_N\left(|||u^m|||\right)\right]b\left(u^m,u^m,v_1\right)$$
$$\leq\left|b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right|+\frac{\left||||u_p^m|||-|||u^m|||\right|}{|||u^m|||}\left|b\left(u^m,u^m,v_1\right)\right|.$$

Following [13], one has $\left|b\left(u_p^m,u_p^m,v_1\right)-b\left(u^m,u^m,v_1\right)\right|\to 0,\ p\to\infty$. And since $\left|b\left(u^m,u^m,v_1\right)\right|$ is bounded uniformly with respect to p, one sees that

$$\frac{\left||||u_p^m|||-|||u^m|||\right|}{|||u^m|||}\left|b\left(u^m,u^m,v_1\right)\right|\to 0, p\to\infty.$$

Therefore, $\lim_{p\to\infty} b_N\left(u_p^m,u_p^m,v_1\right)=b_N\left(u^m,u^m,v_1\right)$. Similarly, we can get

$$\lim_{p\to\infty} c\left(u_p^m,\omega_p^m,v_2\right)=c\left(u^m,\omega^m,v_2\right).$$

So $\left\{u^m,\omega^m\right\}$ is the solution of (1.7)-(1.8). The proof of Theorem 1 is completed. □

3. Boundedness

Let $\left\{u^m,\omega^m\right\}$ be the solution sequence of (1.7)-(1.8), we are going to show that the boundedness of $\left\{u^m,\omega^m\right\}$ in $\mathbb{H}\times L^2(\Omega)$, $\mathbb{V}\times H_0^1(\Omega)$ and $D(A_1)\times D(A_2)$ respectively.

3.1. *Boundedness in* $\mathbb{H}\times L^2(\Omega)$

Lemma 2. *Let $\left\{u^m,\omega^m\right\}_m$ be the solution sequence of (1.7)-(1.8), constructed in Theorem 1. Then for all integers $m\geq 1$, $\left\{u^m,\omega^m\right\}$ remain bounded in $\mathbb{H}\times L^2(\Omega)$, in the following sense,*

$$\|\omega^m\|^2\leq K_1, \qquad \forall m\geq 1; \tag{3.1}$$
$$\|u^m\|^2\leq K_1^*, \qquad \forall m\geq 1; \tag{3.2}$$
$$\sum_{m=i}^{L-1}\|\omega^m-\omega^{m-1}\|^2+\sum_{m=i}^{L-1}k|||\omega^m|||^2\leq\frac{k}{\lambda_2}\|g\|^2(L-i)+K_1, \qquad L\geq i; \tag{3.3}$$
$$\sum_{m=i}^{L-1}\|u^m-u^{m-1}\|^2+\sum_{m=i}^{L-1}\nu k|||u^m|||^2$$
$$\leq K_1^*+(L-i)\left[\frac{2k}{\nu\lambda_1}\|f\|^2+\frac{2k|\xi|^2}{\nu\lambda_1}K_1\right], \qquad L\geq i, \tag{3.4}$$

where $K_1\triangleq\|\omega^0\|^2+\frac{\|g\|^2}{\lambda_2^2}$, $\quad K_1^*\triangleq\|u^0\|^2+\frac{2\|f\|^2}{\nu^2\lambda_1^2}+\frac{2|\xi|^2}{\nu^2\lambda_1^2}K_1$.

Proof. Taking the L^2 inner product of the equation (1.8) with $2k\omega^m$, we obtain

$$\|\omega^m\|^2+\|\omega^m-\omega^{m-1}\|^2+2k|||\omega^m|||^2=2k(g,\omega^m)+\|\omega^{m-1}\|^2$$

$$\leq 2k\frac{\|g\|}{\sqrt{\lambda_2}}|||\omega^m||| + \|\omega^{m-1}\|^2 \leq \frac{k}{\lambda_2}\|g\|^2 + k|||\omega^m|||^2 + \|\omega^{m-1}\|^2.$$

Thus, we can get

$$\|\omega^m\|^2 + \|\omega^m - \omega^{m-1}\|^2 + k|||\omega^m|||^2 \leq \frac{k}{\lambda_2}\|g\|^2 + \|\omega^{m-1}\|^2. \tag{3.5}$$

By Poincaré inequality, we have

$$\|\omega^m\|^2 \leq \frac{1}{1+k\lambda_2}\|\omega^{m-1}\|^2 + \frac{1}{1+k\lambda_2}\frac{k}{\lambda_2}\|g\|^2.$$

Using the above inequality recursively, we find

$$\|\omega^m\|^2 \leq \frac{1}{(1+k\lambda_2)^m}\|\omega^0\|^2 + \frac{\|g\|^2}{\lambda_2^2}\left[1 - \frac{1}{(1+k\lambda_2)^m}\right] \leq \|\omega^0\|^2 + \frac{\|g\|^2}{\lambda_2^2} \triangleq K_1, \tag{3.6}$$

that is, equation (3.1) holds. On the other hand, taking the L^2 inner product of the equation (1.7) with $2ku^m$, we obtain

$$\begin{aligned}\|u^m\|^2 + \|u^m - u^{m-1}\|^2 + 2\nu k|||u^m|||^2 &= 2k(f,u^m) + \|u^{m-1}\|^2 - 2k(\xi\omega^m, u^m)\\ &\leq 2k\|f\|\|u^m\| + \|u^{m-1}\|^2 + 2k\|\xi\omega^m\|\|u^m\|\\ &\leq \frac{2k}{\sqrt{\lambda_1}}\|f\||||u^m||| + \|u^{m-1}\|^2 + \frac{2k|\xi|}{\sqrt{\lambda_1}}\|\omega^m\||||u^m|||\\ &\leq \frac{2k}{\nu\lambda_1}\|f\|^2 + \nu k|||u^m|||^2 + \|u^{m-1}\|^2 + \frac{2k|\xi|^2}{\nu\lambda_1}\|\omega^m\|^2.\end{aligned}$$

Combination (3.6) imply that

$$\|u^m\|^2 + \|u^m - u^{m-1}\|^2 + \nu k|||u^m|||^2 \leq \|u^{m-1}\|^2 + \frac{2k}{\nu\lambda_1}\|f\|^2 + \frac{2k|\xi|^2}{\nu\lambda_1}K_1. \tag{3.7}$$

Using Poincaré inequality again, we can get

$$(1+\nu k\lambda_1)\|u^m\|^2 \leq \|u^{m-1}\|^2 + \frac{2k}{\nu\lambda_1}\|f\|^2 + \frac{2k|\xi|^2}{\nu\lambda_1}K_1.$$

Using the above inequality recursively, we find

$$\begin{aligned}\|u^m\|^2 &\leq \frac{1}{(1+\nu k\lambda_1)^m}\|u^0\|^2 + \left[1 - \frac{1}{(1+\nu k\lambda_1)^m}\right]\left[\frac{2\|f\|^2}{\nu^2\lambda_1^2} + \frac{2|\xi|^2}{\nu^2\lambda_1^2}K_1\right]\\ &\leq \|u^0\|^2 + \frac{2\|f\|^2}{\nu^2\lambda_1^2} + \frac{2|\xi|^2}{\nu^2\lambda_1^2}K_1 \triangleq K_1^*,\end{aligned} \tag{3.8}$$

that is, the equation (3.2) holds. Adding up (3.5) with m from i to $L-1$, we find

$$\|\omega^{L-1}\|^2 + \sum_{m=i}^{L-1}\|\omega^m - \omega^{m-1}\|^2 + \sum_{m=i}^{L-1}k|||\omega^m|||^2 \leq \frac{k}{\lambda_2}\|g\|^2(L-i) + \|\omega^{i-1}\|^2.$$

Combination (3.6) imply that (3.3) holds. Adding up (3.7) with m from i to $L-1$, we find

$$\|u^{L-1}\|^2+\sum_{m=i}^{L-1}\|u^m-u^{m-1}\|^2+\sum_{m=i}^{L-1}\nu k|||u^m|||^2\le\|u^{i-1}\|^2+(L-i)\left[\frac{2k}{\nu\lambda_1}\|f\|^2+\frac{2k|\xi|^2}{\nu\lambda_1}K_1\right].$$

Combination (3.8) imply that (3.4) holds. The proof of Lemma 2 is completed. □

3.2. *Boundedness in* $\mathbb{V}\times H_0^1(\Omega)$

Lemma 3 ([5]). *Let* $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ *be non-negative sequences. Assume that there are integers* m_0, m_1 *such that, for* $k>0$,

$$ky_m<\frac{1}{2},\qquad \forall m\ge m_0,$$
$$(1-ky_m)x_m\le x_{m-1}+kz_m,\qquad \forall m>m_0+m_1.$$

and that for all integers $m*\ge m_0$, $k\sum_{m=m*}^{m*+m_1}y_m\le a_1$, $k\sum_{m=m*}^{m*+m_1}z_m\le a_2$, $k\sum_{m=m*}^{m*+m_1}x_m\le a_3$. *Then*

$$x_m\le\left[\frac{a_3}{km_1}+a_2\right]e^{4a_1},\qquad \forall m>m_0+m_1.$$

Lemma 4. *Let* $\{u^m,\omega^m\}_m$ *be the solutions sequence of* (1.7)-(1.8), *constructed in Theorem 1. Then for all integers* $m\ge1$, $\{u^m,\omega^m\}$ *remain bounded in* $\mathbb{V}\times H_0^1(\Omega)$, *in the following sense, there exists positive constants* C, a_1, a_2, a_3, *such that*

$$|||\omega^m|||^2\le K_3,\qquad \forall m\ge1; \tag{3.9}$$

$$|||u^m|||^2\le K_2,\qquad \forall m\ge1; \tag{3.10}$$

$$\begin{aligned}&\sum_{m=i}^{L-1}|||u^m-u^{m-1}|||^2+\sum_{m=i}^{L-1}\frac{\nu k}{16}\|A_1u^m\|^2\\&\le K_2+C\left[\frac{\|f\|^2}{\nu}+\frac{N^8}{\nu^7}K_1^*+\frac{|\xi|^2}{\nu}K_1\right](L-i)k,\qquad L\ge i\ge1;\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\sum_{m=i}^{L-1}|||\omega^m-\omega^{m-1}|||^2+\sum_{m=i}^{L-1}\frac{1}{2}k\|A_2\omega^m\|^2\\&\le K_3+\left[k\|g\|^2+CK_2\left(\frac{k}{\lambda_2}\|g\|^2+K_1\right)\right](L-i),\qquad L\ge i\ge1,\end{aligned} \tag{3.12}$$

where

$$K_2\triangleq|||u^0|||^2+\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{N^8C}{\nu^8\lambda_1}K_1^*+\frac{C|\xi|^2}{\nu^2\lambda_1}K_1,$$

$$K_3\triangleq\max\left\{|||\omega^0|||^2+\frac{\|g\|^2}{\frac{1}{2}\lambda_2-CK_2},\ \left[\frac{a_3}{km_1}+a_2\right]e^{4a_1},\ C|||\omega^0|||^2+\frac{C}{K_2-\lambda_2}\|g\|^2\right\},$$

$L \geq i \geq 1$.

Proof. Taking the L^2 inner product of the equation (1.7) with $2kA_1u^m$, we obtain

$$|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + 2\nu k\|A_1u^m\|^2$$
$$= |||u^{m-1}|||^2 + 2k(f, A_1u^m) - 2kb_N(u^m, u^m, A_1u^m) - 2k(\xi\omega^m, A_1u^m).$$

Each term of the right hand side of the above equation can be majorize by (1.3) as follows

$$2k(f, A_1u^m) \leq 2k\|f\|\|A_1u^m\| \leq \frac{k}{\nu}\|f\|^2 + \nu k\|A_1u^m\|^2;$$
$$2k\,|(\xi\omega^m, A_1u^m)| \leq \frac{16k}{\nu}\,|\xi|^2\,\|\omega^m\|^2 + \frac{\nu k}{16}\|A_1u^m\|^2;$$
$$2k\,|b_N(u^m, u^m, A_1u^m)| = 2kF_N(|||u^m|||)\,|b(u^m, u^m, A_1u^m)|$$
$$\leq 2k\frac{N}{|||u^m|||}C\|u^m\|^{\frac{1}{4}}\|A_1u^m\|^{\frac{3}{4}}|||u^m|||\|A_1u^m\|$$
$$= CNk\|u^m\|^{\frac{1}{4}}\|A_1u^m\|^{\frac{7}{4}} \leq \frac{7}{8}\nu k\|A_1u^m\|^2 + \frac{N^8C^8k}{8\nu^7}\|u^m\|^2.$$

Then

$$|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + 2\nu k\|A_1u^m\|^2$$
$$\leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \nu k\|A_1u^m\|^2 + \frac{7}{8}\nu k\|A_1u^m\|^2$$
$$+ \frac{N^8C^8k}{8\nu^7}\|u^m\|^2 + \frac{16k}{\nu}\,|\xi|^2\,\|\omega^m\|^2 + \frac{\nu k}{16}\|A_1u^m\|^2.$$

By (3.1) and (3.2) gives

$$|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + \frac{\nu k}{16}\|A_1u^m\|^2$$
$$\leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}\|u^m\|^2 + \frac{16k}{\nu}\,|\xi|^2\,\|\omega^m\|^2$$
$$\leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}\,|\xi|^2\,K_1, \tag{3.13}$$

which together with $|||u^m||| \leq \frac{1}{\sqrt{\lambda_1}}\|A_1u^m\|$ gives

$$|||u^m|||^2 + |||u^m - u^{m-1}|||^2 + \frac{\nu k\lambda_1}{16}|||u^m|||^2 \leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}\,|\xi|^2\,K_1.$$

Therefore

$$\left[1 + \frac{\nu k\lambda_1}{16}\right]|||u^m|||^2 \leq |||u^{m-1}|||^2 + \frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}\,|\xi|^2\,K_1,$$

which is

$$|||u^m|||^2 \le \frac{1}{1+\frac{\nu k\lambda_1}{16}}|||u^{m-1}|||^2 + \frac{1}{1+\frac{\nu k\lambda_1}{16}}\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2 K_1\right].$$

Using the above inequality recursively, we find

$$\begin{aligned}|||u^m|||^2 &\le \frac{1}{(1+\frac{\nu k\lambda_1}{16})^m}|||u^0|||^2 + \frac{16}{\nu k\lambda_1}\left[1-\frac{1}{(1+\frac{\nu k\lambda_1}{16})^m}\right]\\ &\quad\times\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2 K_1\right]\\ &\le |||u^0|||^2 + \frac{16}{\nu k\lambda_1}\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2 K_1\right]\\ &\le |||u^0|||^2 + \frac{C}{\nu^2\lambda_1}\|f\|^2 + \frac{N^8C}{\nu^8\lambda_1}K_1^* + \frac{C|\xi|^2}{\nu^2\lambda_1}K_1 \triangleq K_2. \end{aligned} \tag{3.14}$$

Therefore, we get (3.10), and u^m is bounded in $\mathbb{V}$.

Taking the L^2 inner product of the equation (1.8) with $2kA_2\omega^m$, we find

$$\begin{aligned}&|||\omega^m|||^2 + |||\omega^m-\omega^{m-1}|||^2 + 2k\|A_2\omega^m\|^2\\ &= |||\omega^{m-1}|||^2 + 2k(g,A_2\omega^m) - 2kc(u^m,\omega^m,A_2\omega^m)\\ &\le |||\omega^{m-1}|||^2 + k\|g\|^2 + k\|A_2\omega^m\|^2 + CK_2k|||\omega^m|||^2 + \frac{1}{2}k\|A_2\omega^m\|^2.\end{aligned}$$

Taking k small enough such that $1-CK_2k>0$, we obtain

$$(1-CK_2k)|||\omega^m|||^2 + \frac{1}{2}k\|A_2\omega^m\|^2 + |||\omega^m-\omega^{m-1}|||^2 \le |||\omega^{m-1}|||^2 + k\|g\|^2, \tag{3.15}$$

which, together with $|||\omega^m||| \le \frac{1}{\sqrt{\lambda_2}}\|A_2\omega^m\|$, leads to

$$\left[1+\frac{1}{2}\lambda_2 k - CK_2k\right]|||\omega^m|||^2 + |||\omega^m-\omega^{m-1}|||^2 \le |||\omega^{m-1}|||^2 + k\|g\|^2. \tag{3.16}$$

There are two cases to discuss the above inequality (3.16):

Case 1: If $\frac{1}{2}\lambda_2 > CK_2$, which is $1+\frac{1}{2}\lambda_2 k - CK_2k > 1$, then from (3.16), one has

$$|||\omega^m|||^2 \le \frac{1}{1+\frac{1}{2}\lambda_2k-CK_2k}|||\omega^{m-1}|||^2 + \frac{k}{1+\frac{1}{2}\lambda_2k-CK_2k}\|g\|^2.$$

Using the above inequality recursively, we find

$$\begin{aligned}|||\omega^m|||^2 &\le \frac{1}{(1+\frac{1}{2}\lambda_2k-CK_2k)^m}|||\omega^0|||^2\\ &\quad + \frac{1}{\frac{1}{2}\lambda_2-CK_2}\|g\|^2\left[1-\frac{1}{(1+\frac{1}{2}\lambda_2k-CK_2k)^m}\right]\end{aligned}$$

$$\leq |||\omega^0|||^2 + \frac{1}{\frac{1}{2}\lambda_2 - CK_2}\|g\|^2.$$

Case 2: If $\frac{1}{2}\lambda_2 \leq CK_2$, which is $1+\frac{1}{2}\lambda_2 k - CK_2 k \leq 1$, then from (3.16), one gets

$$\left[1 - k\left(CK_2 - \tfrac{1}{2}\lambda_2\right)\right] |||\omega^m|||^2 \leq |||\omega^{m-1}|||^2 + k\|g\|^2.$$

In the following, we will use Lemma 3 to discuss the above inequality. Let $x_m = |||\omega^m|||^2$, $y_m = CK_2 - \frac{1}{2}\lambda_2$, $z_m = \|g\|^2$. Obviously, $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ are non-negative sequences, and there exists m_0, m_1 such that

$$\begin{aligned} ky_m = CK_2 k - \frac{1}{2}k\lambda_2 < \frac{1}{2}, \qquad & \forall m \geq m_0, \\ (1-ky_m)x_m \leq x_{m-1} + kz_m, \qquad & \forall m > m_0 + m_1, \end{aligned}$$

and that for all integers $m* \geq m_0$, from (3.3), we get

$$\begin{aligned} k\sum_{m=m*}^{m*+m_1} y_m &= k\sum_{m=m*}^{m*+m_1}\left[CK_2 - \tfrac{1}{2}\lambda_2\right] = k(m_1+1)\left[CK_2 - \tfrac{1}{2}\lambda_2\right] \leq a_1, \\ k\sum_{m=m*}^{m*+m_1} z_m &= k\sum_{m=m*}^{m*+m_1} \|g\|^2 = k(m_1+1)\|g\|^2 \leq a_2, \\ k\sum_{m=m*}^{m*+m_1} x_m &= k\sum_{m=m*}^{m*+m_1} |||\omega^m|||^2 \leq \frac{k}{\lambda_2}\|g\|^2(m_1+1) + K_1 \leq a_3. \end{aligned}$$

By Lemma 3, we get

$$|||\omega^m|||^2 \leq \left[\frac{a_3}{km_1} + a_2\right] e^{4a_1}, \qquad \forall m > m_0 + m_1.$$

When $m \leq m_0 + m_1$, there is $\left[1+\frac{1}{2}\lambda_2 k - CK_2 k\right]^m \geq \left[1+\frac{1}{2}\lambda_2 k - CK_2 k\right]^{m_0+m_1}$, thus

$$\begin{aligned} |||\omega^m|||^2 &\leq \frac{1}{(1+\frac{1}{2}\lambda_2 k - CK_2 k)^{m_0+m_1}} |||\omega^0|||^2 \\ &\quad + \frac{1}{CK_2 - \frac{1}{2}\lambda_2}\|g\|^2 \left[\frac{1}{(1+\frac{1}{2}\lambda_2 k - CK_2 k)^{m_0+m_1}} - 1\right] \\ &\leq C|||\omega^0|||^2 + \frac{C}{K_2 - \lambda_2}\|g\|^2, \qquad \forall m \leq m_0 + m_1. \end{aligned}$$

Above all, we get (3.9), that is, ω^m is bounded in $H_0^1(\Omega)$.

Next, adding up (3.13) with $m = i, i+1, \cdots, L-1$, we find

$$|||u^{L-1}|||^2 + \sum_{m=i}^{L-1} |||u^m - u^{m-1}|||^2 + \sum_{m=i}^{L-1} \frac{\nu k}{16}\|A_1 u^m\|^2$$

$$\leq |||u^{i-1}|||^2 + (L-i)\left[\frac{k}{\nu}\|f\|^2 + \frac{N^8C^8k}{8\nu^7}K_1^* + \frac{16k}{\nu}|\xi|^2 K_1\right].$$

Together with (3.14), leads to (3.11). Using (3.5)-(3.6) and (3.15), we obtain

$$\begin{aligned}
&|||\omega^m|||^2 + \frac{1}{2}k\|A_2\omega^m\|^2 + |||\omega^m - \omega^{m-1}|||^2 \\
&\leq |||\omega^{m-1}|||^2 + k\|g\|^2 + CK_2k|||\omega^m|||^2 \\
&\leq |||\omega^{m-1}|||^2 + k\|g\|^2 + CK_2\left[\frac{k}{\lambda_2}\|g\|^2 + K_1\right].
\end{aligned}$$

Adding up the above inequality with $m = i, i+1, \cdots, L-1$, we get

$$\begin{aligned}
&|||\omega^{L-1}|||^2 + \sum_{m=i}^{L-1}|||\omega^m - \omega^{m-1}|||^2 + \sum_{m=i}^{L-1}\frac{1}{2}k\|A_2\omega^m\|^2 \\
&\qquad \leq |||\omega^{i-1}|||^2 + (L-i)\left[k\|g\|^2 + CK_2\left(\frac{k}{\lambda_2}\|g\|^2 + K_1\right)\right].
\end{aligned}$$

Together with (3.9), one obtains (3.12). Above all, we have $\{u^m, \omega^m\}$ is bounded in $\mathbb{V} \times H_0^1(\Omega)$. The proof of Lemma 4 is completed. □

3.3. *Boundedness in $D(A_1) \times D(A_2)$*

Lemma 5. *Assuming that $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$, $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $\{u^m, \omega^m\}_{m\geq 1}$ be the solutions sequence of (1.7)-(1.8). Then there exist positive constants $C_0 \equiv C_0(\nu, N, |\xi|, K_1, K_2, K_3, \|A_1u^0\|, \|A_2\omega^0\|, \|f\|, \|g\|)$ and $K_4 \equiv K_4\left(k, \nu, \lambda_1, \lambda_2, N, |\xi|, \|u^0\|, \|\omega^0\|, \|f\|, \|g\|\right)$ such that*

$$\left\|\frac{u^1 - u^0}{k}\right\| + \left\|\frac{\omega^1 - \omega^0}{k}\right\| \leq C_0; \tag{3.17}$$

$$\left\|\frac{u^m - u^{m-1}}{k}\right\|^2 + \left\|\frac{\omega^m - \omega^{m-1}}{k}\right\|^2 \leq K_4, \quad m > 1. \tag{3.18}$$

Proof. Let $u = u^1 - u^0$, $\omega = \omega^1 - \omega^0$, then from (1.7)-(1.8), one obtains

$$\begin{cases}
\frac{u}{k} + \nu A_1u + \nu A_1u^0 + B_N(u+u^0, u+u^0) + \xi(\omega + \omega^0) = f, & (3.19) \\
\frac{\omega}{k} + A_2\omega + A_2\omega^0 + C(u+u^0, \omega+\omega^0) = g. & (3.20)
\end{cases}$$

Taking the scalar product of the equation (3.19) with A_1u, we obtain

$$\frac{|||u|||^2}{k} + \nu\|A_1u\|^2 = (f - \nu A_1u^0, A_1u) - b_N(u+u^0, u+u^0, A_1u) - (\xi\omega + \xi\omega^0, A_1u). \tag{3.21}$$

Taking the scalar product of the equation (3.20) with $A_2\omega$, we find

$$\frac{|||\omega|||^2}{k}+\|A_2\omega\|^2=(g-A_2\omega^0,A_2\omega)-c(u+u^0,\omega+\omega^0,A_2\omega). \tag{3.22}$$

Putting together (3.21) and (3.22), one obtains

$$\begin{aligned}&\frac{|||u|||^2}{k}+\frac{|||\omega|||^2}{k}+\nu\|A_1u\|^2+\|A_2\omega\|^2\\&=(f-\nu A_1u^0,A_1u)+(g-A_2\omega^0,A_2\omega)-(\xi\omega+\xi\omega^0,A_1u)\\&\quad-b_N(u+u^0,u+u^0,A_1u)-c(u+u^0,\omega+\omega^0,A_2\omega).\end{aligned}$$

By (3.1) and (3.9)-(3.10) gives

$$\begin{aligned}|b_N(u+u^0,u+u^0,A_1u)|&=F_N(|||u+u^0|||)|b(u+u^0,u+u^0,A_1u)|\\&\le F_N(|||u+u^0|||)\|(u+u^0)\cdot\nabla(u+u^0)\|\|A_1u\|\\&\le F_N(|||u+u^0|||)\|u+u^0\|_{L_6}\|\nabla(u+u^0)\|_{L_3}\|A_1u\|\\&\le\frac{CN}{|||u+u^0|||}|||u+u^0|||\,|||u+u^0|||^{\frac12}\|A_1(u+u^0)\|^{\frac12}\|A_1u\|\\&\le CN|||u+u^0|||^{\frac12}\|A_1(u+u^0)\|^{\frac12}\|A_1u\|\\&\le CNK_2^{\frac14}\|A_1u\|^{\frac32}+CNK_2^{\frac14}\|A_1u^0\|^{\frac12}\|A_1u\|;\\(f-\nu A_1u^0,A_1u)&\le\|f\|\|A_1u\|+\nu\|A_1u^0\|\|A_1u\|;\\(g-A_2\omega^0,A_2\omega)&\le\|g\|\|A_2\omega\|+\|A_2\omega^0\|\|A_2\omega\|;\\|c(u+u^0,\omega+\omega^0,A_2\omega)|&\le C|||u^1|||\,|||\omega^1|||\|A_2\omega\|\le CK_2^{\frac12}K_3^{\frac12}\|A_2\omega\|;\\|(\xi\omega+\xi\omega^0,A_1u)|&\le|\xi|\|\omega^1\|\|A_1u\|\le|\xi|K_1^{\frac12}\|A_1u\|.\end{aligned}$$

Therefore, one has

$$\begin{aligned}&\frac{|||u|||^2}{k}+\frac{|||\omega|||^2}{k}+\nu\|A_1u\|^2+\|A_2\omega\|^2\\&\le\|f\|\|A_1u\|+\nu\|A_1u^0\|\|A_1u\|+\|g\|\|A_2\omega\|+\|A_2\omega^0\|\|A_2\omega\|\\&\quad+CNK_2^{\frac14}\|A_1u\|^{\frac32}+CNK_2^{\frac14}\|A_1u^0\|^{\frac12}\|A_1u\|\\&\quad+CK_2^{\frac12}K_3^{\frac12}\|A_2\omega\|+|\xi|K_1^{\frac12}\|A_1u\|,\end{aligned}$$

which, together with Young's inequality, leads to

$$\begin{aligned}&\frac{|||u|||^2}{k}+\frac{|||\omega|||^2}{k}+C\|A_1u\|^2+C\|A_2\omega\|^2\\&\le C\|f\|^2+C\nu\|A_1u^0\|^2+C\|g\|^2+C\|A_2\omega^0\|^2\end{aligned}$$

$$+CNK_2+CNK_2^{\frac{1}{2}}\|A_1u^0\|+CK_2K_3+C|\xi|^2K_1. \tag{3.23}$$

From (1.7)-(1.8), one has

$$\begin{cases} \dfrac{u^1-u^0}{k}=-\nu A_1u^1-B_N(u^1,u^1)-\xi\omega^1+f, \\ \dfrac{\omega^1-\omega^0}{k}=-A_2\omega^1-C(u^1,\omega^1)+g. \end{cases}$$

Thus

$$\begin{aligned} &\left\|\frac{u^1-u^0}{k}\right\|+\left\|\frac{\omega^1-\omega^0}{k}\right\| \\ &\leq \nu\|A_1u^1\|+\|A_2\omega^1\|+\|B_N(u^1,u^1)\|+\|C(u^1,\omega^1)\|+\|\xi\omega^1\|+\|f\|+\|g\| \\ &\leq \nu\|A_1u\|+\nu\|A_1u^0\|+\|A_2\omega\|+\|A_2\omega^0\| \\ &\quad +\|B_N(u^1,u^1)\|+\|C(u^1,\omega^1)\|+|\xi|\|\omega^1\|+\|f\|+\|g\| \\ &\leq \nu\|A_1u\|+\nu\|A_1u^0\|+\|A_2\omega\|+\|A_2\omega^0\| \\ &\quad +CN\||u^1\||^{\frac{1}{2}}\|A_1u^1\|^{\frac{1}{2}}+\||u^1\|\|\|\omega^1\||+|\xi|\|\omega^1\|+\|f\|+\|g\| \\ &\leq \nu\|A_1u\|+\nu\|A_1u^0\|+\|A_2\omega\|+\|A_2\omega^0\|+CNK_2^{\frac{1}{4}}\|A_1u\|^{\frac{1}{2}} \\ &\quad +CNK_2^{\frac{1}{4}}\|A_1u^0\|^{\frac{1}{2}}+K_2^{\frac{1}{2}}K_3^{\frac{1}{2}}+|\xi|K_1^{\frac{1}{2}}+\|f\|+\|g\|. \end{aligned}$$

Which together with (3.23), gives the desired result, that is

$$\left\|\frac{u^1-u^0}{k}\right\|+\left\|\frac{\omega^1-\omega^0}{k}\right\|\leq C_0\left(\nu,N,|\xi|,K_1,K_2,K_3,\|A_1u^0\|,\|A_2\omega^0\|,\|f\|,\|g\|\right).$$

Then (3.17) is holds.

For $m>1$, let $u_*^m=\dfrac{u^m-u^{m-1}}{k}$, $\omega_*^m=\dfrac{\omega^m-\omega^{m-1}}{k}$ in (1.7)-(1.8), we obtain

$$\begin{cases} \dfrac{u_*^m-u_*^{m-1}}{k}+\nu A_1u_*^m+\dfrac{1}{k}\left[B_N(u^m,u^m)-B_N(u^{m-1},u^{m-1})\right]+\xi\omega_*^m=0, & (3.24) \\ \dfrac{\omega_*^m-\omega_*^{m-1}}{k}+A_2\omega_*^m+\dfrac{1}{k}\left[C(u^m,\omega^m)-C(u^{m-1},\omega^{m-1})\right]=0. & (3.25) \end{cases}$$

Taking the scalar product of the equation (3.24) with $2ku_*^m$, and taking the scalar product of the equation (3.25) with $2k\omega_*^m$, we obtain

$$\begin{aligned} &\|u_*^m\|^2+\|u_*^m-u_*^{m-1}\|^2+2k\nu\||u_*^m\||^2+\|\omega_*^m\|^2+\|\omega_*^m-\omega_*^{m-1}\|^2+2k\||\omega_*^m\||^2 \\ &=\|u_*^{m-1}\|^2-2b_N(u^m,u^m,u_*^m)+2b_N(u^{m-1},u^{m-1},u_*^m)-2k(\xi\omega_*^m,u_*^m) \\ &\quad +\|\omega_*^{m-1}\|^2-2c(u^m,\omega^m,\omega_*^m)+2c(u^{m-1},\omega^{m-1},\omega_*^m). \end{aligned} \tag{3.26}$$

We now majorize the right-hand side of (3.26). By (1.3) and (3.10), one gets

$$
\begin{aligned}
&2b_N(u^{m-1},u^{m-1},u_*^m)-2b_N(u^m,u^m,u_*^m)\\
&=2F_N(|||u^{m-1}|||)b(u^{m-1},u^{m-1},u_*^m)-2F_N(|||u^m|||)b(u^m,u^m,u_*^m)\\
&=2\left[F_N(|||u^{m-1}|||)-F_N(|||u^m|||)\right]b(u^{m-1},u^{m-1},u_*^m)\\
&\quad+2F_N(|||u^m|||)\left[b(u^{m-1},u^{m-1},u_*^m)-b(u^m,u^m,u_*^m)\right]\\
&\le 2\left[F_N(|||u^{m-1}|||)-F_N(|||u^m|||)\right]b(u^{m-1},u^{m-1},u_*^m)\\
&\quad+2F_N(|||u^m|||)|kb(u_*^m,u^m,u_*^m)|\\
&\le C\frac{|||u^m-u^{m-1}|||}{|||u^{m-1}|||}|||u^{m-1}|||\,|||u^{m-1}|||\,\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac34}\\
&\quad+\frac{CNk}{|||u^m|||}\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac34}|||u^m|||\,\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac34}\\
&\le CK_2^{\frac12}k\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac74}+CNk\|u_*^m\|^{\frac12}|||u_*^m|||^{\frac32},\\
&\quad-2k(\xi\omega_*^m,u_*^m)\le 2k|(\xi\omega_*^m,u_*^m)|\le 2k|\xi|\|\omega_*^m\|\|u_*^m\|.
\end{aligned}
$$

Together with (3.9), one has

$$
\begin{aligned}
2c(u^{m-1},\omega^{m-1},\omega_*^m)-2c(u^m,\omega^m,\omega_*^m)&\le 2|c(ku_*^m,\omega^m,\omega_*^m)|\\
&\le Ck|||u_*^m|||\,|||\omega^m|||\,\|\omega_*^m\|\le CK_3^{\frac12}k|||u_*^m|||\,\|\omega_*^m\|.
\end{aligned}
$$

Thus, from (3.26) and above inequality, by Young's inequality, we obtain

$$
\begin{aligned}
&\|u_*^m\|^2+\|u_*^m-u_*^{m-1}\|^2+2k\nu|||u_*^m|||^2+\|\omega_*^m\|^2+\|\omega_*^m-\omega_*^{m-1}\|^2+2k|||\omega_*^m|||^2\\
&\le\|u_*^{m-1}\|^2+CK_2^{\frac12}k\|u_*^m\|^{\frac14}|||u_*^m|||^{\frac74}+CNk\|u_*^m\|^{\frac12}|||u_*^m|||^{\frac32}\\
&\quad+2k|\xi|\|\omega_*^m\|\|u_*^m\|+\|\omega_*^{m-1}\|^2+CK_3^{\frac12}k|||u_*^m|||\,\|\omega_*^m\|\\
&\le\|u_*^{m-1}\|^2+\|\omega_*^{m-1}\|^2+Ck\|u_*^m\|^2+2k\nu|||u_*^m|||^2+Ck\|\omega_*^m\|^2.
\end{aligned}
$$

Therefore

$$
(1-Ck)\left(\|u_*^m\|^2+\|\omega_*^m\|^2\right)\le\|u_*^{m-1}\|^2+\|\omega_*^{m-1}\|^2. \tag{3.27}
$$

Using the above inequality (3.27) recursively, we find

$$
\|u_*^m\|^2+\|\omega_*^m\|^2\le\frac{1}{(1-Ck)^{m-1}}\left(\|u_*^1\|^2+\|\omega_*^1\|^2\right).
$$

Obviously, when $m\le M_0=ent\left\{\frac{T}{k}\right\}$, then $\|u_*^m\|^2+\|\omega_*^m\|^2$ is bounded, where $ent\left\{\frac{T}{k}\right\}$ is the entire part of $\frac{T}{k}$ with T an arbitrarily fixed constant. Let $x_m=\|u_*^m\|^2+\|\omega_*^m\|^2$, $y_m=C$, $z_m=0$, then by Lemma 2-3 and equation (3.27), we see that $\|u_*^m\|^2+\|\omega_*^m\|^2$

is bounded as $m > M_0$. Thus we have

$$\left\|\frac{u^m-u^{m-1}}{k}\right\|^2+\left\|\frac{\omega^m-\omega^{m-1}}{k}\right\|^2 \le K_4\left(\nu,\lambda_1,\lambda_2,N,|\xi|,\|u^0\|,\|\omega^0\|,\|f\|,\|g\|\right). \tag{3.28}$$

Then (3.18) is holds. The proof of Lemma 5 is completed. □

Theorem 4. *Assuming that $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$, $u_0 \in D(A_1)$, $\omega_0 \in D(A_2)$. Let $\{u^m,\omega^m\}_{m\ge1}$ be the solution sequence of (1.7)-(1.8). Then there exists a positive constant C such that, $\forall m \ge 1$,*

$$\|A_1u^m\|^2+\|A_2\omega^m\|^2 \le K_5. \tag{3.29}$$

Proof. From (1.7)-(1.8), we obtain

$$\begin{cases} \nu A_1u^m = -\dfrac{u^m-u^{m-1}}{k} - B_N(u^m,u^m) - \xi\omega^m + f, & (3.30)\\[2mm] A_2\omega^m = -\dfrac{\omega^m-\omega^{m-1}}{k} - C(u^m,\omega^m) + g. & (3.31)\end{cases}$$

Two sides of equation (3.30) and (3.31) multiply by A_1u^m and $A_2\omega^m$ respectively, and then integrate gives

$$\begin{aligned}
&\nu\|A_1u^m\|^2+\|A_2\omega^m\|^2\\
&= -\left[\frac{u^m-u^{m-1}}{k},A_1u^m\right]-\left[\frac{\omega^m-\omega^{m-1}}{k},A_2\omega^m\right]-b_N(u^m,u^m,A_1u^m)\\
&\quad -c(u^m,\omega^m,A_2\omega^m)-(\xi\omega^m,A_1u^m)+(f,A_1u^m)+(g,A_2\omega^m)\\
&\le \left\|\frac{u^m-u^{m-1}}{k}\right\|\|A_1u^m\|+\left\|\frac{\omega^m-\omega^{m-1}}{k}\right\|\|A_2\omega^m\|\\
&\quad +\frac{CN}{|||u^m|||}\|(u^m\cdot\nabla)u^m\|\|A_1u^m\|+C|||u^m|||\,|||\omega^m|||\,\|A_2\omega^m\|\\
&\quad +\|\xi\omega^m\|\|A_1u^m\|+\|f\|\|A_1u^m\|+\|g\|\|A_2\omega^m\|\\
&\le \left\|\frac{u^m-u^{m-1}}{k}\right\|\|A_1u^m\|+\left\|\frac{\omega^m-\omega^{m-1}}{k}\right\|\|A_2\omega^m\|+CN|||u^m|||^{\frac12}\|A_1u^m\|^{\frac32}\\
&\quad +C|||u^m|||\,|||\omega^m|||\,\|A_2\omega^m\|+\|\xi\omega^m\|\|A_1u^m\|+\|f\|\|A_1u^m\|+\|g\|\|A_2\omega^m\|.
\end{aligned}$$

By Young's inequality, one gets

$$\begin{aligned}
\|A_1u^m\|^2+\|A_2\omega^m\|^2 &\le C\left\|\frac{u^m-u^{m-1}}{k}\right\|^2+C\left\|\frac{\omega^m-\omega^{m-1}}{k}\right\|^2+C|||u^m|||^2\\
&\quad +C|||u^m|||^2|||\omega^m|||^2+C|\xi|^2\|\omega^m\|^2+C\|f\|^2+C\|g\|^2\\
&\le CK_4+CK_2+CK_2K_3+C|\xi|^2K_1+C\|f\|^2+C\|g\|^2 \triangleq K_5.
\end{aligned}$$

The proof of Theorem 4 is completed. □

4. Global Attractor

In this section, we first prove that continuous dependence of solutions on initial data and N.

Lemma 6. *Assuming that M, $N > 0$, $f \in L^2(\Omega)^3$, $g \in L^2(\Omega)$, u_1^0, $u_2^0 \in D(A_1)$, and ω_1^0, $\omega_2^0 \in D(A_2)$.*

Let $\{u_1^m, \omega_1^m\}_m$ be the solution of (1.7)-(1.8), with initial condition $\{u_1^0, \omega_1^0\}$ and parameter N.

Let $\{u_2^m, \omega_2^m\}_m$ be the solution of (1.7)-(1.8), with initial condition $\{u_2^0, \omega_2^0\}$ and parameter M.

Then there exists C, C_*, a_1^*, a_2^*, a_3^*, such that

$$|||u_1^m - u_2^m|||^2 + |||\omega_1^m - \omega_2^m|||^2 \le \frac{1}{{C_*}^{M_0}}\left[|||u_1^0 - u_2^0|||^2 + |||\omega_1^0 - \omega_2^0|||^2\right] + \frac{kCK_5}{1 - C_*}\left[\frac{1}{C_*^{M_0}} - 1\right]|M - N|^2, \quad \forall m \le M_0; \tag{4.1}$$

$$|||u_1^m - u_2^m|||^2 + |||\omega_1^m - \omega_2^m|||^2 \le \left[\frac{a_3^*}{k(M_0 - 3)} + a_2^*\right]e^{4a_1^*}, \qquad \forall m > M_0, \tag{4.2}$$

where $M_0 = ent\left(\frac{T}{k}\right)$, T is an arbitrarily fixed constant.

Proof. Let $u_*^m = u_1^m - u_2^m$, $\omega_*^m = \omega_1^m - \omega_2^m$ in (1.7)-(1.8), we obtain

$$\begin{cases} \dfrac{u_*^m - u_*^{m-1}}{k} + \nu A_1 u_*^m + B_N(u_1^m, u_1^m) - B_M(u_2^m, u_2^m) + \xi\omega_*^m = 0, & (4.3) \\ \dfrac{\omega_*^m - \omega_*^{m-1}}{k} + A_2\omega_*^m + C(u_1^m, \omega_1^m) - C(u_2^m, \omega_2^m) = 0. & (4.4) \end{cases}$$

Taking the scalar product of (4.3) with $2kA_1u_*^m$, we find

$$\begin{aligned} &|||u_*^m|||^2 - |||u_*^{m-1}|||^2 + |||u_*^m - u_*^{m-1}|||^2 + 2\nu k\|A_1 u_*^m\|^2 \\ &= 2k\left[b_M(u_2^m, u_2^m, A_1u_*^m) - b_N(u_1^m, u_1^m, A_1u_*^m)\right] - 2k(\xi\omega_*^m, A_1u_*^m) \\ &= 2kF_M(|||u_2^m|||)b(u_2^m, u_2^m, A_1u_*^m) \\ &\quad - 2kF_N(|||u_1^m|||)b(u_1^m, u_1^m, A_1u_*^m) - 2k(\xi\omega_*^m, A_1u_*^m) \\ &= 2k\left[F_M(|||u_2^m|||) - F_N(|||u_1^m|||)\right]b(u_1^m, u_2^m, A_1u_*^m) \\ &\quad - 2kF_M(|||u_2^m|||)b(u_*^m, u_2^m, A_1u_*^m) \\ &\quad - 2kF_N(|||u_1^m|||)b(u_1^m, u_*^m, A_1u_*^m) - 2k(\xi\omega_*^m, A_1u_*^m). \end{aligned} \tag{4.5}$$

From (1.3) and Lemma 1, one has

$$F_M(|||u_2^m|||)\,|b(u_*^m, u_2^m, A_1u_*^m)| \le \frac{M}{|||u_2^m|||}C|||u_*^m|||^{\frac{1}{2}}\,|||u_2^m|||\,\|A_1u_*^m\|^{\frac{1}{2}}\,\|A_1u_*^m\|$$

$$= CM|||u_*^m|||^{\frac{1}{2}}\|A_1u_*^m\|^{\frac{3}{2}};$$

$$F_N(|||u_1^m|||)\,|b(u_1^m,u_*^m,A_1u_*^m)| \le |b(u_1^m,u_*^m,A_1u_*^m)| \le C\|A_1u_1^m\|\,|||u_*^m|||\,\|A_1u_*^m\|;$$

and

$$\begin{aligned}
&[F_M(|||u_2^m|||)-F_N(|||u_1^m|||)]b(u_1^m,u_2^m,A_1u_*^m)\\
&\le \Big[\frac{|M-N|+|||u_1^m-u_2^m|||}{|||u_2^m|||}\Big]C\|A_1u_1^m\|\,|||u_2^m|||\,\|A_1u_*^m\|\\
&\le C\Big[|M-N|+|||u_*^m|||\Big]\|A_1u_1^m\|\|A_1u_*^m\|.
\end{aligned}$$

From (4.5) and above inequality, we obtain

$$\begin{aligned}
&|||u_*^m|||^2-|||u_*^{m-1}|||^2+|||u_*^m-u_*^{m-1}|||^2+2\nu k\|A_1u_*^m\|^2\\
&\le 2kC\,[|M-N|+|||u_*^m|||]\,\|A_1u_1^m\|\|A_1u_*^m\|+2kCM|||u_*^m|||^{\frac{1}{2}}\|A_1u_*^m\|^{\frac{3}{2}}\\
&\quad +2kC\|A_1u_1^m\|\,|||u_*^m|||\,\|A_1u_*^m\|+2k\|\xi\omega_*^m\|\|A_1u_*^m\|\\
&\le \Big[\frac{kC\,|M-N|^2}{\varepsilon_1}+\frac{kC|||u_*^m|||^2}{\varepsilon_2}\Big]\|A_1u_1^m\|^2+(kC\varepsilon_1+kC\varepsilon_2)\|A_1u_*^m\|^2\\
&\quad +\frac{kM^4C^4}{2\varepsilon_3}|||u_*^m|||^2+\frac{3k\varepsilon_3^3}{2}\|A_1u_*^m\|^2+\frac{kC^2}{\varepsilon_4}\|A_1u_1^m\|^2|||u_*^m|||^2\\
&\quad +k\varepsilon_4\|A_1u_*^m\|^2+\frac{k\,|\xi|^2}{\varepsilon_5\lambda_2}|||\omega_*^m|||^2+k\varepsilon_5\|A_1u_*^m\|^2.
\end{aligned}\tag{4.6}$$

Taking the scalar product of (4.4) with $2kA_2\omega_*^m$, we find

$$|||\omega_*^m|||^2-|||\omega_*^{m-1}|||^2+|||\omega_*^m-\omega_*^{m-1}|||^2+2k\|A_2\omega_*^m\|^2 = 2k[c(u_2^m,\omega_2^m,A_2\omega_*^m)-c(u_1^m,\omega_1^m,A_2\omega_*^m)].\tag{4.7}$$

By Young's inequality and (3.9)-(3.10) gives

$$\begin{aligned}
&|c(u_2^m,\omega_2^m,A_2\omega_*^m)-c(u_1^m,\omega_1^m,A_2\omega_*^m)|\\
&=|c(u_2^m,\omega_2^m,A_2\omega_*^m)-c(u_*^m,\omega_1^m,A_2\omega_*^m)-c(u_2^m,\omega_1^m,A_2\omega_*^m)|\\
&=|c(u_2^m,\omega_*^m,A_2\omega_*^m)+c(u_*^m,\omega_1^m,A_2\omega_*^m)|\\
&\le C|||u_2^m|||\,|||\omega_*^m|||\,\|A_2\omega_*^m\|+C|||u_*^m|||\,|||\omega_1^m|||\,\|A_2\omega_*^m\|\\
&\le C\Big[\frac{1}{2\varepsilon_6}|||u_2^m|||^2|||\omega_*^m|||^2+\frac{\varepsilon_6}{2}\|A_2\omega_*^m\|^2\Big]\\
&\quad +C\Big[\frac{1}{2\varepsilon_7}|||u_*^m|||^2|||\omega_1^m|||^2+\frac{\varepsilon_7}{2}\|A_2\omega_*^m\|^2\Big]\\
&\le C\Big[\frac{1}{2\varepsilon_6}K_2|||\omega_*^m|||^2+\frac{\varepsilon_6}{2}\|A_2\omega_*^m\|^2\Big]
\end{aligned}$$

$$+C\Big[\frac{1}{2\varepsilon_7}|||u_*^m|||^2K_3+\frac{\varepsilon_7}{2}\|A_2\omega_*^m\|^2\Big]$$
$$=\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2.$$

Thus, from (4.7), we have

$$|||\omega_*^m|||^2-|||\omega_*^{m-1}|||^2+|||\omega_*^m-\omega_*^{m-1}|||^2+2k\|A_2\omega_*^m\|^2 \le 2k\Big[\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2\Big]. \quad (4.8)$$

From (4.6) and (4.8), we get

$$\begin{aligned}
&|||u_*^m|||^2+|||\omega_*^m|||^2-|||u_*^{m-1}|||^2-|||\omega_*^{m-1}|||^2+|||u_*^m-u_*^{m-1}|||^2\\
&\quad+|||\omega_*^m-\omega_*^{m-1}|||^2+2\nu k\|A_1u_*^m\|^2+2k\|A_2\omega_*^m\|^2\\
&\le\Big[\frac{kC|M-N|^2}{\varepsilon_1}+\frac{kC|||u_*^m|||^2}{\varepsilon_2}\Big]\|A_1u_1^m\|^2+(kC\varepsilon_1+kC\varepsilon_2)\|A_1u_*^m\|^2\\
&\quad+\frac{kM^4C^4}{2\varepsilon_3}|||u_*^m|||^2+\frac{3k\varepsilon_3^3}{2}\|A_1u_*^m\|^2+\frac{kC^2}{\varepsilon_4}\|A_1u_1^m\|^2|||u_*^m|||^2\\
&\quad+k\varepsilon_4\|A_1u_*^m\|^2+\frac{k|\xi|^2}{\varepsilon_5\lambda_2}|||\omega_*^m|||^2+k\varepsilon_5\|A_1u_*^m\|^2\\
&\quad+2k\Big[\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2\Big]\\
&\le\Big[\frac{kC|M-N|^2}{\varepsilon_1}+\frac{kC|||u_*^m|||^2}{\varepsilon_2}\Big]K_5+(kC\varepsilon_1+kC\varepsilon_2)\|A_1u_*^m\|^2\\
&\quad+\frac{kM^4C^4}{2\varepsilon_3}|||u_*^m|||^2+\frac{3k\varepsilon_3^3}{2}\|A_1u_*^m\|^2+\frac{kC^2}{\varepsilon_4}K_5|||u_*^m|||^2\\
&\quad+k\varepsilon_4\|A_1u_*^m\|^2+\frac{k|\xi|^2}{\varepsilon_5\lambda_2}|||\omega_*^m|||^2+k\varepsilon_5\|A_1u_*^m\|^2\\
&\quad+2k\Big[\frac{CK_2}{\varepsilon_6}|||\omega_*^m|||^2+\frac{CK_3}{\varepsilon_7}|||u_*^m|||^2+C(\varepsilon_6+\varepsilon_7)\|A_2\omega_*^m\|^2\Big]\\
&=|||u_*^m|||^2kC\Big[\frac{K_5}{\varepsilon_2}+\frac{M^4}{\varepsilon_3}+\frac{K_5}{\varepsilon_4}+\frac{K_3}{\varepsilon_7}\Big]+\|A_1u_*^m\|^2kC(\varepsilon_1+\varepsilon_2+\varepsilon_3^3+\varepsilon_4+\varepsilon_5)\\
&\quad+|||\omega_*^m|||^2kC\Big[\frac{K_2}{\varepsilon_6}+\frac{|\xi|^2}{\varepsilon_5\lambda_2}\Big]+\|A_2\omega_*^m\|^2kC(\varepsilon_6+\varepsilon_7)+\frac{kCK_5}{\varepsilon_1}|M-N|^2.
\end{aligned}$$

That is

$$\big[1-kC(K_5+M^4+K_3)\big]|||u_*^m|||^2+\Big[1-kC\Big(K_2+\frac{|\xi|^2}{\lambda_2}\Big)\Big]|||\omega_*^m|||^2$$

$$+ |||u_*^m - u_*^{m-1}|||^2 + |||\omega_*^m - \omega_*^{m-1}|||^2 + kC\|A_1 u_*^m\|^2 + kC\|A_2\omega_*^m\|^2$$
$$\leq |||u_*^{m-1}|||^2 + |||\omega_*^{m-1}|||^2 + kCK_5\,|M-N|^2. \tag{4.9}$$

Let $C_* = \min\left\{1 - kC(K_5 + M^4 + K_3),\ \ 1 - kC\left(K_2 + \frac{|\xi|^2}{\lambda_2}\right)\right\}$, then

$$|||u_*^m|||^2 + |||\omega_*^m|||^2 \leq \frac{1}{C_*}\left[|||u_*^{m-1}|||^2 + |||\omega_*^{m-1}|||^2\right] + \frac{kCK_5}{C_*}\,|M-N|^2.$$

Using the above inequality recursively, we find

$$|||u_*^m|||^2 + |||\omega_*^m|||^2 \leq \frac{1}{C_*^m}\left[|||u_*^0|||^2 + |||\omega_*^0|||^2\right] + \frac{kCK_5}{1-C_*}\,|M-N|^2\left[\frac{1}{C_*^m} - 1\right].$$

Since $0 < C_* < 1$, for $m \leq M_0 = ent\left\{\frac{T}{k}\right\}$, one has

$$|||u_*^m|||^2 + |||\omega_*^m|||^2 \leq \frac{1}{C_*^{M_0}}\left[|||u_*^0|||^2 + |||\omega_*^0|||^2\right] + \frac{kCK_5}{1-C_*}\,|M-N|^2\left[\frac{1}{C_*^{M_0}} - 1\right].$$

For $m > M_0$, let $C_{**} = \max\left\{C(K_5 + M^4 + K_3),\ \ C\left(K_2 + \frac{|\xi|^2}{\lambda_2}\right)\right\}$. From (4.9), one has

$$(1 - kC_{**})|||u_*^m|||^2 + (1 - kC_{**})|||\omega_*^m|||^2 \leq |||u_*^{m-1}|||^2 + |||\omega_*^{m-1}|||^2 + kCK_5\,|M-N|^2.$$

Let $x_m = |||u_*^m|||^2 + |||\omega_*^m|||^2$, $y_m = C_{**}$, $z_m = CK_5\,|M-N|^2$. Obviously, $\{x_m\}$, $\{y_m\}$, $\{z_m\}$ are non-negative sequences, and for $k > 0$,

$$ky_m = kC_{**} < \frac{1}{2}, \qquad \forall m \geq 2,$$
$$(1 - ky_m)x_m \leq x_{m-1} + kz_m, \qquad \forall m \geq M-1.$$

For all integers $m_* \geq 2$, by Lemma 4, we get

$$k\sum_{m=m_*}^{m_*+m_1} y_m = k(M_0 - 2)C_{**} \leq a_1^*;$$
$$k\sum_{m=m_*}^{m_*+m_1} z_m = k(M_0 - 2)CK_5\,|M-N|^2 \leq a_2^*;$$
$$k\sum_{m=m_*}^{m_*+m_1} x_m = k\sum_{m=m_*}^{m_*+m_1}\left(|||u_1^m - u_2^m|||^2 + |||\omega_1^m - \omega_2^m|||^2\right)$$
$$\leq k\sum_{m=m_*}^{m_*+m_1}\left(C|||u_1^m|||^2 + C|||u_2^m|||^2 + C|||\omega_1^m|||^2 + C|||\omega_2^m|||^2\right)$$
$$\leq kC\sum_{m=m_*}^{m_*+m_1}(K_2 + K_2 + K_3 + K_3) \leq kC(M_0 - 2)(K_2 + K_3) \leq a_3^*.$$

Thus, $x_m \le \left[\frac{a_3^*}{k(M_0-3)} + a_2^*\right] e^{4a_1^*}$, which is

$$|||u_*^m|||^2 + |||\omega_*^m|||^2 \le \left[\frac{a_3^*}{k(M_0-3)} + a_2^*\right] e^{4a_1^*}.$$

The proof of Lemma 6 is completed. □

Proof of Theorem 2. Above, we show that the continuous dependence of solutions on initial value and parameter N. It can be seen under the above conditions, when determining the initial value and the parameter N, the system (1.7)-(1.8) has a unique solution. Therefore, we can define a C^0 semigroup S^m, acting on the phase space $\mathbb{V} \times H_0^1(\Omega)$, and defined as follows:

$$S^m\big(u^0, \omega^0\big) = \big(u^m, \omega^m\big), \ \ \forall m \ge 0.$$

From Lemma 4, the semigroup S^m has a bounded absorbing set in $\mathbb{V} \times H_0^1(\Omega)$:

$$B_{\mathbb{V}\times H_0^1(\Omega)} = \Big\{\big(u^m, \omega^m\big) \in \mathbb{V} \times H_0^1(\Omega), |||u^m|||^2 + |||\omega^m|||^2 \le K_2 + K_3\Big\}.$$

And from Theorem 4 we can know that S^m is bounded in $D(A_1) \times D(A_2)$, and using Sobolev embedding theorem to know that S^m is compact in $\mathbb{V} \times H_0^1(\Omega)$. Hence, S^m has a global attractor $\mathcal{A}$ in $\mathbb{V} \times H_0^1(\Omega)$. The proof of Theorem 2 is completed. □

5. Limiting behavior for $N \to \infty$

One sees from Lemma 2 that

$$\nu k |||u^m|||^2 + k|||\omega^m|||^2 \le \frac{2k}{\nu\lambda_1}\|f\|^2 + \frac{k}{\lambda_2}\|g\|^2 + \left[1 + \frac{2k|\xi|^2}{\nu\lambda_1}\right] K_1 + K_1^*.$$

For m and k fixed, let $\{u_N^m, \omega_N^m\}_N$ be the solution of (1.7)-(1.8), then

$$|||u_N^m|||^2 + |||\omega_N^m|||^2 \le \frac{C}{\nu^2\lambda_1}\|f\|^2 + \frac{C}{\nu\lambda_2}\|g\|^2 + \frac{1}{\nu k}\left[\left(1 + \frac{2k|\xi|^2}{\nu\lambda_1}\right) K_1 + K_1^*\right].$$

Thus the sequence $\{u_N^m, \omega_N^m\}_N$ is bounded in $\mathbb{V} \times H_0^1(\Omega)$ uniformly in N. Therefore, we can extract from $\{u_N^m, \omega_N^m\}_N$ a subsequence still denoted by $\{u_N^m, \omega_N^m\}_N$ such that $u_N^m \rightharpoonup u^m$, as $N \to \infty$ in $\mathbb{V}$, and $\omega_N^m \rightharpoonup \omega^m$, as $N \to \infty$ in $H_0^1(\Omega)$. As the injection $\mathbb{V} \hookrightarrow \mathbb{H}$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ both are compact, we have $u_N^m \to u^m$, as $N \to \infty$ in $\mathbb{H}$, and $\omega_N^m \to \omega^m$ as $N \to \infty$ in $L^2(\Omega)$.

We shall show that

$$\begin{cases} \lim\limits_{N\to\infty} F_N(|||u_N^m|||) b(u_N^m, u_N^m, v) = b(u^m, u^m, v), & \forall v \in D(A_1), \\ \lim\limits_{N\to\infty} C(u_N^m, \omega_N^m, v_*) = c(u^m, \omega^m, v_*), & \forall v_* \in H^2(\Omega). \end{cases}$$

Indeed, a simple computation gives

$$F_N(|||u_N^m|||)b(u_N^m,u_N^m,v)-b(u^m,u^m,v)$$
$$=[F_N(|||u_N^m|||)-1]\,b(u_N^m,u_N^m,v)+b(u_N^m,u_N^m,v)-b(u^m,u^m,v).$$

First, by the definition of F_N, we have $F_N(|||u_N^m|||)=\min\left\{1,\frac{N}{|||u_N^m|||}\right\}\leq 1$. And from the above inequality we can see

$$\frac{N}{|||u_N^m|||}\geq N\left\{\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{C}{\nu\lambda_2}\|g\|^2+\frac{1}{\nu k}\left[\left(1+\frac{2k|\xi|^2}{\nu\lambda_1}\right)K_1+K_1^*\right]\right\}^{-\frac{1}{2}}.$$

Hence, if $N>\left\{\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{C}{\nu\lambda_2}\|g\|^2+\frac{1}{\nu k}\left[\left(1+\frac{2k|\xi|^2}{\nu\lambda_1}\right)K_1+K_1^*\right]\right\}^{\frac{1}{2}}$, we find that $F_N(|||u_N^m|||)=1$. Therefore, $\lim_{N\to\infty}F_N(|||u_N^m|||)=1$. Next, from (1.3), we obtain

$$b(u_N^m,u_N^m,v)\leq C|||u_N^m|||\,|||u_N^m|||\,\|A_1v\|$$
$$\leq C\left(\frac{C}{\nu^2\lambda_1}\|f\|^2+\frac{C}{\nu\lambda_2}\|g\|^2+\frac{1}{\nu k}\left[\left(1+\frac{2k|\xi|^2}{\nu\lambda_1}\right)K_1+K_1^*\right]\right)\|A_1v\|,$$

showing that $b(u_N^m,u_N^m,v)$ is bounded uniformly with respect to N, so

$$\lim_{N\to\infty}[F_N(|||u_N^m|||)-1]b(u_N^m,u_N^m,v)=0.$$

Using the strong convergence of u_N^m in $\mathbb{H}$, we can prove as in [13], that $b(u_N^m,u_N^m,v)\to b(u^m,u^m,v)$, as $N\to\infty$. Thus $\lim_{N\to\infty}F_N(|||u_N^m|||)b(u_N^m,u_N^m,v)=b(u^m,u^m,v)$, $\forall v\in D(A_1)$. Similarly, we have $\lim_{N\to\infty}c(u_N^m,\omega_N^m,v_*)=c(u^m,\omega^m,v_*)$, $\forall v_*\in H^2(\Omega)$. Therefore, $\{u_N^m,\omega_N^m\}_N$ converges to the weak solution of the following equations when $N\to\infty$,

$$\begin{cases}\frac{u^m-u^{m-1}}{k}+\nu A_1u^m+B(u^m,u^m)+\xi\omega^m=f,\\ \frac{\omega^m-\omega^{m-1}}{k}+A_2\omega^m+C(u^m,\omega^m)=g.\end{cases}\tag{5.1}$$

Thus, we have completed the proof of Theorem 3. □

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