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OPTIMIZATION THROUGH BEST PROXIMITY POINTS FOR MULTIVALUED F-CONTRACTIONS

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Abstract. Best proximity point theorems ensure the existence of an approximate optimal solution to the equations of the type $f(x) = x$ when f is not a self-map and a solution of the same does not necessarily exist. Best proximity points theorems, therefore, serve as a powerful tool in the theory of optimization and approximation. The aim of this article is to consider a global optimization problem in the context of best proximity points in a complete metric space. We establish an existence of best proximity result for multivalued mappings using Wardowski's technique.

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1. INTRODUCTION AND PRELIMINARIES

Nadler [9] defined a Hausdorff concept by considering the distance between two arbitrary sets as follows.

Let (Ω, η) be a complete metric space (in short, MS) and let $CB(\Omega)$ be the family of all nonempty closed and bounded subsets of the nonempty set Ω. For $\mathcal{M}, \mathcal{N} \in CB(\Omega)$, define the map $\mathcal{H} : CB(\Omega) \times CB(\Omega) \to [0, \infty)$ by

$$\mathcal{H}(\mathcal{M}, \mathcal{N}) = \max\{\sup_{\xi \in \mathcal{N}} \Delta(\xi, \mathcal{M}), \sup_{\delta \in \mathcal{M}} \Delta(\delta, \mathcal{N})\},$$

where $\Delta(\delta, \mathcal{N}) = \inf_{\xi \in \mathcal{N}} \eta(\delta, \xi)$. Then $(CB(\Omega), \mathcal{H})$ is an MS induced by η.

Let $\mathcal{M}, \mathcal{N}$ be any two nonempty subsets of the MS (Ω, η). The following notations will be used throughout:

$$\mathcal{M}_0 = \{\mu \in \mathcal{M} : \eta(\mu, \nu) = \eta(\mathcal{M}, \mathcal{N}) \text{ for some } \nu \in \mathcal{N}\},$$
$$\mathcal{N}_0 = \{\nu \in \mathcal{N} : \eta(\mu, \nu) = \eta(\mathcal{M}, \mathcal{N}) \text{ for some } \mu \in \mathcal{M}\},$$

where $\eta(\mathcal{M}, \mathcal{N}) = \inf\{\eta(\mu, \nu) : \mu \in \mathcal{M}, \nu \in \mathcal{N}\}$.

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For $\mathcal{M},\mathcal{N} \in CB(\Omega)$, we have

$$\eta(\mathcal{M},\mathcal{N}) \leq \mathcal{H}(\mathcal{M},\mathcal{N}).$$

We say that $\mu \in \mathcal{M}$ is a best proximity point (in short, BPP) of the multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ if $\Delta(\mu,\Gamma\mu) = \eta(\mathcal{M},\mathcal{N})$. $\upsilon \in \Omega$ is said to be a fixed point of the multivalued map $\Gamma : \Omega \to CB(\Omega)$ if $\upsilon \in \Gamma\upsilon$.

Remark 1.

(1) In the MS $(CB(\Omega),\mathcal{H})$, $\upsilon \in \Omega$ is a fixed point of Γ if and only if $\Delta(\upsilon,\Gamma\upsilon) = 0$.
(2) If $\eta(\mathcal{M},\mathcal{N}) = 0$, then a fixed point and a BPP are identical.
(3) The metric function $\eta : \Omega \times \Omega \to [0,\infty)$ is continuous in the sense that if $\{\upsilon_n\},\{\xi_n\}$ are two sequences in Ω with $(\upsilon_n,\xi_n) \to (\upsilon,\xi)$ for some $\upsilon,\xi \in \Omega$, as $n \to \infty$, then $\eta(\upsilon_n,\xi_n) \to \eta(\upsilon,\xi)$ as $n \to \infty$. The function Δ is continuous in the sense that if $\upsilon_n \to \upsilon$ as $n \to \infty$, then $\Delta(\upsilon_n,\mathcal{M}) \to \Delta(\upsilon,\mathcal{M})$ as $n \to \infty$ for any $\mathcal{M} \subseteq \Omega$.

The following Lemmas are noteworthy.

Lemma 1 ([2, 4])**.** *Let (Ω,η) be an MS and $\mathcal{M},\mathcal{N} \in CB(\Omega)$. Then*

(1) *$\Delta(\mu,\mathcal{N}) \leq \eta(\mu,\gamma)$ for any $\gamma \in \mathcal{N}$ and $\mu \in \Omega$;*
(2) *$\Delta(\mu,\mathcal{N}) \leq \mathcal{H}(\mathcal{M},\mathcal{N})$ for any $\mu \in \mathcal{M}$.*

Lemma 2 ([9])**.** *Let $\mathcal{M},\mathcal{N} \in CB(\Omega)$ and let $\upsilon \in \mathcal{M}$, then for any $r > 0$, there exists $\xi \in \mathcal{N}$ such that*

$$\eta(\upsilon,\xi) \leq \mathcal{H}(\mathcal{M},\mathcal{N}) + r.$$

But we may not have any $\xi \in \mathcal{N}$ such that

$$\eta(\upsilon,\xi) \leq \mathcal{H}(\mathcal{M},\mathcal{N}).$$

Further, when $\mathcal{N}$ is compact, there exists $\xi \in \Omega$ such that $\eta(\upsilon,\xi) \leq \mathcal{H}(\mathcal{M},\mathcal{N})$.

The concept of $\mathcal{H}$-continuity for multivalued maps is listed next.

Definition 1 ([5])**.** Let (Ω,η) be an MS. We say that a multivalued map $\Gamma : \Omega \to CB(\Omega)$ is $\mathcal{H}$-continuous at a point μ_0, if for each sequence $\{\mu_n\} \subset \Omega$, such that $\lim_{n\to\infty} \eta(\mu_n,\mu_0) = 0$, we have $\lim_{n\to\infty} \mathcal{H}(\Gamma\mu_n,\Gamma\mu_0) = 0$ (i.e., if $\mu_n \to \mu_0$, then $\Gamma\mu_n \to \Gamma\mu_0$ as $n \to \infty$).

Definition 2 ([9])**.** Let $\Gamma : \Omega \to CB(\Omega)$ be a multivalued map. We say that Γ is a multivalued contraction if $\mathcal{H}(\Gamma\mu,\Gamma\nu) \leq \lambda\eta(\mu,\nu)$ for all $\mu,\nu \in \Omega$, where $\lambda \in [0,1)$.

Remark 2.

(1) If Γ is $\mathcal{H}$-continuous on every point of $\mathcal{M} \subseteq \Omega$, then it is said to be continuous on $\mathcal{M}$.
(2) A multivalued contraction Γ is $\mathcal{H}$-continuous.

In 2012, Wardowski [16] defined the concept of F-contraction as follows.

Definition 3. Let $F : (0,+\infty) \to (-\infty,+\infty)$ be a function which satisfies the following:

(F1) F is strictly increasing;

(F2) For each sequence $\{u_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$,

$$\lim_{n\to+\infty} u_n = 0 \text{ if and only if } \lim_{n\to+\infty} F(u_n) = -\infty;$$

(F3) There is $t \in (0,1)$ such that $\lim_{u\to 0^+} u^t F(u) = 0$.

Let $\mathcal{F}$ denote the class of all such functions F. If (Ω,η) is an MS, then a self-map $T : \Omega \to \Omega$ is said to be an $F-$contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, such that for all $\mu,\nu \in \Omega$,

$$\eta(T\mu,T\nu) > 0 \Rightarrow \tau + F(\eta(T\mu,T\nu)) \leq F(\eta(\mu,\nu)).$$

Multivalued F-contractions were defined by Altun et al. [1] as follows.

Definition 4 ([1])**.** Let (Ω,η) be an MS. A multivalued map $\Gamma : \Omega \to CB(\Omega)$ is said to be a multivalued F-contraction (MVFC, in short) if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq F(\eta(\mu,\nu)) \tag{1.1}$$

for all $\mu,\nu \in \Omega$ with $\Gamma\mu \neq \Gamma\nu$.

Remark 3. An MVFC is $\mathcal{H}$-continuous.

We can find the concept of P-property in [12], whereas the notion of weak P property was defined by Zhang et al. [18].

Definition 5 ([12])**.** Let (Ω,η) be an MS and $\mathcal{M},\mathcal{N}$ be two non-empty subsets of Ω such that $\mathcal{M}_0 \neq \phi$. The pair $(\mathcal{M},\mathcal{N})$ is said to have the P-property if and only if $\eta(\mu_1,\nu_1) = \eta(\mathcal{M},\mathcal{N}) = \eta(\mu_2,\nu_2)$ implies $\eta(\mu_1,\mu_2) = \eta(\nu_1,\nu_2)$, where $\mu_1,\mu_2 \in \mathcal{M}_0$ and $\nu_1,\nu_2 \in \mathcal{N}$.

Definition 6 ([18])**.** Let (Ω,η) be an MS and $\mathcal{M},\mathcal{N}$ be two non-empty subsets of Ω such that $\mathcal{M}_0 \neq \phi$. The pair $(\mathcal{M},\mathcal{N})$ is said to have the weak P-property if and only if $\eta(\mu_1,\nu_1) = \eta(\mathcal{M},\mathcal{N}) = \eta(\mu_2,\nu_2)$ implies $\eta(\mu_1,\mu_2) \leq \eta(\nu_1,\nu_2)$, where $\mu_1,\mu_2 \in \mathcal{M}_0$ and $\nu_1,\nu_2 \in \mathcal{N}_0$.

BPP theorems for F-contractive non-self mappings were studied by Omidvari et al. [11] with the help of P-property. Later, Nazari [10] investigated BPPs for a particular type of generalized multivalued contractions by using the weak P-property.

Srivastava et al. [13, 14] presented Krasnosel'skii type hybrid fixed point theorems and found their very interesting applications to fractional integral equations. Xu et al. [17] proved Schwarz lemma that involves boundary fixed point. Very recently, Debnath and Srivastava [6] investigated common BPPs for multivalued contractive pairs of mappings in connection with global optimization. Debnath and Srivastava [7] also proved new extensions of Kannan's and Reich's theorems in the context of multivalued mappings using Wardowski's technique. Further, a very significant application of fixed points of $F(\psi,\varphi)$-contractions to fractional differential equations was recently provided by Srivastava et al. [15].

In this paper, we introduce a best proximity result for multivalued mappings with the help of F-contraction and the weak P property. Also we provide an example where the P-property is not satisfied but the weak P-property holds.

2. Best proximity point for MVFC

In this section, with the help of the notion of F-contraction, we show that an MVFC satisfying certain conditions admits a BPP.

Theorem 1. *Let (Ω,η) be a complete MS and $\mathcal{M},\mathcal{N}$ be two non-empty closed subsets of Ω such that $\mathcal{M}_0 \neq \phi$ and that the pair $(\mathcal{M},\mathcal{N})$ has the weak P-property. Suppose $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ be a MVFC such that $\Gamma\mu$ is compact for each $\mu \in \mathcal{M}$ and $\Gamma\mu \subseteq \mathcal{N}_0$ for all $\mu \in \mathcal{M}_0$. Then Γ has a BPP.*

Proof. Fix $\mu_0 \in \mathcal{M}_0$ and choose $\nu_0 \in \Gamma\mu_0 \subseteq \mathcal{N}_0$. By the definition of $\mathcal{N}_0$, we can select $\mu_1 \in \mathcal{M}_0$ such that

$$\eta(\mu_1,\nu_0) = \eta(\mathcal{M},\mathcal{N}). \tag{2.1}$$

If $\nu_0 \in \Gamma\mu_1$, then

$$\eta(\mathcal{M},\mathcal{N}) \leq \Delta(\mu_1,\Gamma\mu_1) \leq \eta(\mu_1,\nu_0) = \eta(\mathcal{M},\mathcal{N}).$$

Thus $\eta(\mathcal{M},\mathcal{N}) = \Delta(\mu_1,\Gamma\mu_1)$, i.e., μ_1 is a BPP of Γ. Therefore, assume that $\nu_0 \notin \Gamma\mu_1$. Since $\Gamma\mu_1$ is compact, by Lemma 2, there exists $\nu_1 \in \Gamma\mu_1$ such that

$$0 < \eta(\nu_0,\nu_1) \leq \mathcal{H}(\Gamma\mu_0,\Gamma\mu_1).$$

Since F is strictly increasing, the last inequality implies that

$$\begin{aligned} F(\eta(\nu_0,\nu_1)) &\leq F(\mathcal{H}(\Gamma\mu_0,\Gamma\mu_1)) \\ &\leq F(\eta(\mu_0,\mu_1)) - \tau. \end{aligned} \tag{2.2}$$

Since $\nu_1 \in \Gamma\mu_1 \subseteq \mathcal{N}_0$, there exists $\mu_2 \in \mathcal{M}_0$ such that

$$\eta(\mu_2,\nu_1) = \eta(\mathcal{M},\mathcal{N}). \tag{2.3}$$

From (2.1) and (2.3) and using weak $P-$property , we have that

$$\eta(\mu_1,\mu_2) \leq \eta(\nu_0,\nu_1). \tag{2.4}$$

From (2.2) and (2.4), we have

$$F(\eta(\mu_1,\mu_2)) \leq F(\eta(\nu_0,\nu_1)) \leq F(\eta(\mu_0,\mu_1)) - \tau. \tag{2.5}$$

If $\nu_1 \in \Gamma\mu_2$, then

$$\eta(\mathcal{M},\mathcal{N}) \leq \Delta(\mu_2,\Gamma\mu_2) \leq \eta(\mu_2,\nu_1) = \eta(\mathcal{M},\mathcal{N}).$$

Thus $\eta(\mathcal{M},\mathcal{N}) = \Delta(\mu_2,\Gamma\mu_2)$, i.e., μ_1 is a BPP of Γ. So, assume that $\nu_1 \notin \Gamma\mu_2$.

Since $\Gamma\mu_2$ is compact, by Lemma 2, there exists $\nu_2 \in \Gamma\mu_2$ such that

$$0 < \eta(\nu_1, \nu_2) \leq \mathcal{H}(\Gamma\mu_1, \Gamma\mu_2).$$

Using the fact that F is strictly increasing, we have that

$$\begin{aligned} F(\eta(\nu_1, \nu_2)) &\leq F(\mathcal{H}(\Gamma\mu_1, \Gamma\mu_2)) \\ &\leq F(\eta(\mu_1, \mu_2)) - \tau \\ &\leq F(\eta(\mu_0, \mu_1)) - 2\tau \text{ (using 2.5)}. \end{aligned}$$

Since $\nu_2 \in \Gamma\mu_2 \subseteq \mathcal{N}_0$, there exists $\mu_3 \in \mathcal{M}_0$ such that

$$\eta(\mu_3, \nu_2) = \eta(\mathcal{M}, \mathcal{N}). \tag{2.6}$$

From (2.5) and (2.6) and using weak property P, we have that

$$\eta(\mu_2, \mu_3) \leq \eta(\nu_1, \nu_2). \tag{2.7}$$

From (2.6) and (2.7), we have

$$F(\eta(\mu_2, \mu_3)) \leq F(\eta(\nu_1, \nu_2)) \leq F(\eta(\mu_0, \mu_1)) - 2\tau. \tag{2.8}$$

Continuing in this manner, we obtain two sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $\mathcal{M}_0$ and $\mathcal{N}_0$ respectively, satisfying

- **(B1)** $\nu_n \in \Gamma\mu_n \subseteq \mathcal{N}_0$,
- **(B2)** $\eta(\mu_{n+1}, \nu_n) = \eta(\mathcal{M}, \mathcal{N})$,
- **(B3)** $F(\eta(\mu_n, \mu_{n+1})) \leq F(\eta(\nu_{n-1}, \nu_n)) \leq F(\eta(\mu_0, \mu_1)) - n\tau$,

for each $n = 0, 1, 2, \ldots$.

Put $\alpha_n = \eta(\mu_n, \mu_{n+1})$ for each $n = 0, 1, 2, \ldots$. Taking limit on both sides of **(B3)** as $n \to \infty$, we have

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

Using **(F2)**, we obtain

$$\lim_{n\to\infty} \alpha_n = 0. \tag{2.9}$$

Using **(F3)**, there exists $k \in (0, 1)$ such that

$$\alpha_n^k F(\alpha_n) \to 0 \text{ as } n \to \infty. \tag{2.10}$$

From **(B3)**, for each $n \in \mathbb{N}$, we have that

$$F(\alpha_n) - F(\alpha_0) \leq -n\tau.$$

This implies

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -n\alpha_n^k \tau \leq 0. \tag{2.11}$$

Letting $n \to \infty$ in (2.11) and using (2.9), (2.10), we obtain

$$\lim_{n\to\infty} n\alpha_n^k = 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_0$, i.e., $\alpha_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_0$.

Let $m,n \in \mathbb{N}$ with $m > n \geq n_0$. Then

$$\eta(\mu_m,\mu_n) \leq \sum_{i=n}^{m-1} \eta(\mu_i,\mu_{i+1}) = \sum_{i=n}^{m-1} \alpha_i \\ \leq \sum_{i=n}^{\infty} \alpha_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent for $k \in (0,1)$, we have $\eta(\mu_m,\mu_n) \to 0$ as $m,n \to \infty$. Hence $\{\mu_n\}$ is Cauchy in $\mathcal{M}_0 \subseteq \mathcal{M}$. Since (Ω,η) is complete and $\mathcal{M}$ is closed, we have $\lim_{n\to\infty} \mu_n = \theta$ for some $\theta \in \mathcal{M}$.
Since Γ is $\mathcal{H}$-continuous (for it is an MVFC), we have

$$\lim_{n\to\infty} \mathcal{H}(\Gamma\mu_n,\Gamma\theta) = 0. \tag{2.12}$$

Exactly in the similar manner as above, using **(B3)**, we can prove that $\{\nu_n\}$ is Cauchy in $\mathcal{N}$ and since $\mathcal{N}$ is closed, there exists $\xi \in B$ such that $\lim_{n\to\infty} \nu_n = \xi$.
Since $\eta(\mu_{n+1},\nu_n) = \eta(\mathcal{M},\mathcal{N})$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \eta(\mu_{n+1},\nu_n) = \eta(\theta,\xi) = \eta(\mathcal{M},\mathcal{N}).$$

We claim that $\xi \in \Gamma\theta$. Indeed, since $\nu_n \in \Gamma\mu_n$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \Delta(\nu_n,\Gamma\theta) \leq \lim_{n\to\infty} \mathcal{H}(\Gamma\mu_n,\Gamma\theta) = 0.$$

Therefore, $\Delta(\xi,\Gamma\theta) = 0$. Since $\Gamma\theta$ is closed, we have $\xi \in \Gamma\theta$.
Now,

$$\eta(\mathcal{M},\mathcal{N}) \leq \Delta(\theta,\Gamma\theta) \leq \eta(\theta,\xi) = \eta(\mathcal{M},\mathcal{N}).$$

Hence $\Delta(\theta,\Gamma\theta) = \eta(\mathcal{M},\mathcal{N})$, i.e., θ is a BPP of Γ. □

A Geraghty type [8] result follows as a consequence of our previous theorem. Let $\mathcal{G}$ be the class of functions $g : [0,\infty) \to [0,1)$ satisfying the condition: $g(\xi_n) \to 1$ implies $\xi_n \to 0$. An example of such a map is $g(\xi) = (1+\xi)^{-1}$ for all $\xi > 0$ and $g(0) \in [0,1)$.

Definition 7. Let $\mathcal{M},\mathcal{N}$ be two non-empty subsets of a MS (Ω,η). A multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ is said to be a multivalued Geraghty-type F-contraction (MVGFC, in short) if there exist $\tau > 0$, $F \in \mathcal{F}$ and $g \in \mathcal{G}$ such that

$$\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq g(\eta(\mu,\nu)) \cdot F(\eta(\mu,\nu)) \tag{2.13}$$

for all $\mu,\nu \in \Omega$ with $\Gamma\mu \neq \Gamma\nu$.

Corollary 1. *Let (Ω,η) be a complete MS and $\mathcal{M},\mathcal{N}$ be two non-empty closed subsets of Ω such that $\mathcal{M}_0 \neq \phi$ and that the pair $(\mathcal{M},\mathcal{N})$ satisfies the weak P-property. Suppose $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ be a MVGFC such that $\Gamma\mu$ is compact for each $\mu \in \mathcal{M}$ and $\Gamma\mu \subseteq \mathcal{N}_0$ for all $\mu \in \mathcal{M}_0$. Then Γ has a BPP.*

Proof. Since $g(t) \in [0,1)$ for all $t \in [0,\infty)$, from (2.13), we have that

$$\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq F(\eta(\mu,\nu)) \tag{2.14}$$

for all $\mu,\nu \in \mathcal{M}$ with $\Gamma\mu \neq \Gamma\nu$. Thus, Γ is an MVFC and hence from Theorem 1 it follows that Γ has a BPP. □

Remark 4. Corollary 1 extends the results due to Caballero et al. [3] and Zhang et al. [18] to their multivalued analogues using F-contraction.

Next, we provide some examples in support of our main result.

Example 1. Consider $\Omega = \mathbb{R}$ with usual metric $\eta(\mu,\nu) = |\mu-\nu|$ for all $\mu,\nu \in \Omega$. Let $\mathcal{M} = [5,6]$ and $\mathcal{N} = [-6,-5]$. Then $\eta(\mathcal{M},\mathcal{N}) = 10$ and $\mathcal{M}_0 = \{5\}$, $\mathcal{N}_0 = \{-5\}$. Define the multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ such that

$$\Gamma\mu = [\frac{-\mu-5}{2}, -5] \text{ for all } \mu \in [5,6].$$

Therefore $\Gamma(5) = \{-5\}$ (i.e., $\Gamma\mu \subseteq \mathcal{N}_0$ for all $\mu \in \mathcal{M}_0$).
We claim that Γ is a MVFC. Let $\mathcal{H}(\Gamma\mu,\Gamma\nu) > 0$. Then we have

$$\begin{aligned}\mathcal{H}(\Gamma\mu,\Gamma\nu) &= \mathcal{H}([\frac{-\mu-5}{2},-5],[\frac{-\nu-5}{2},-5]) \\ &= |(\frac{-\mu-5}{2}) - (\frac{-\nu-5}{2})| \\ &= \frac{|\nu-\mu|}{2} \\ &= \frac{\eta(\mu,\nu)}{2} \\ &< \eta(\mu,\nu).\end{aligned}$$

From the last inequality, we have that $\ln(\mathcal{H}(\Gamma\mu,\Gamma\nu)) < \ln(\eta(\mu,\nu))$, and further, $\tau + \ln(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq \ln(\eta(\mu,\nu))$, for any $\tau \in (0,\ln 2]$. Therefore, we have that $\tau + F(\mathcal{H}(\Gamma\mu,\Gamma\nu)) \leq F(\eta(\mu,\nu))$, for any $\tau \in (0,\ln 2]$, where $F(t) = \ln t, t > 0$.

Finally, it is easy to check that $(\mathcal{M},\mathcal{N})$ satisfies weak P-property. Thus, all conditions of Theorem 1 are satisfied and we observe that $\mu = 5$ is a BPP of Γ.

In fact, in Example 1, the pair $(\mathcal{M},\mathcal{N})$ satisfies P-property (and hence the weak P-property as well). Next, we present an example in which the pair $(\mathcal{M},\mathcal{N})$ satisfies only the weak P-property but not the P-property.

Example 2. Consider $\Omega = \mathbb{R}^2$ with the Euclidean metric η.
Let $\mathcal{M} = \{(-5,0),(0,1),(5,0)\}$ and $\mathcal{N} = \{(\mu,\nu) : \nu = 2+\sqrt{2-\mu^2}, \mu \in [-\sqrt{2},\sqrt{2}]\}$.
Then $\eta(\mathcal{M},\mathcal{N}) = \sqrt{3}$ and $\mathcal{M}_0 = \{(0,1)\}$, $\mathcal{N}_0 = \{(\sqrt{2},2),(-\sqrt{2},2)\}$.
Define the multivalued map $\Gamma : \mathcal{M} \to CB(\mathcal{N})$ such that

$$\Gamma(-5,0) = \{(-\sqrt{2},2),(-1,3)\},\ \Gamma(0,1) = \{(\sqrt{2},2)\},\ \Gamma(5,0) = \{(\sqrt{2},2),(1,3)\}.$$

It is easy to check that Γ is a MVFC with $\tau = \ln 2$ and $F(t) = \ln t, t > 0$.

Finally, we observe that

$$\eta((0,1),(\sqrt{2},2)) = \eta((0,1),(-\sqrt{2},2)) = \sqrt{3} = \eta(\mathcal{M},\mathcal{N}),$$

but

$$\eta((0,1),(0,1)) = 0 < \eta((\sqrt{2},2),(-\sqrt{2},2)) = 2\sqrt{2}.$$

Thus, $(\mathcal{M},\mathcal{N})$ satisfies weak P-property, but not the P-property. Therefore, all conditions of Theorem 1 are satisfied and since $\Delta((0,1),\Gamma(0,1)) = \sqrt{3} = \eta(\mathcal{M},\mathcal{N})$, we conclude that $(0,1)$ is a BPP of Γ.

3. Conclusion

We have proved our main result with a strong condition that images of the MVFC are compact sets. Relaxation of this compactness criterion is a suggested future work. We have shown the non-triviality of the assumption of the weak P-property by presenting an example which does not satisfy the P-property but satisfies only the weak P-property. The results due to Caballero et al. [3] and Zhang et al. [18] are also extended to their multivalued analogues as a consequence of our results.

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References

[1] I. Altun, G. Minak, and H. Dag, "Multivalued F-contractions on complete metric spaces," *J. Nonlinear Convex Anal.*, vol. 16, no. 4, pp. 659–666, 2015.

[2] M. Boriceanu, A. Petrusel, and I. Rus, "Fixed point theorems for some multivalued generalized contractions in b-metric spaces," *Internat. J. Math. Statistics*, vol. 6, pp. 65–76, 2010.

[3] J. Caballero, J. Harjani, and K. Sadarangani, "A best proximity point theorem for Geraghty-contractions," *Fixed Point Theory Appl.*, vol. 2012, no. 231, pp. 1–9, 2012, doi: 10.1186/1687-1812-2012-231.

[4] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti Sem. Mat. Univ. Modena*, vol. 46, pp. 263–276, 1998.

[5] P. Debnath and M. de La Sen, "Fixed points of eventually Δ-restrictive and $\Delta(\varepsilon)$-restrictive set-valued maps in metric spaces," *Symmetry*, vol. 12, no. 1, pp. 1–7, 2020, doi: 10.3390/sym12010127.

[6] P. Debnath and H. M. Srivastava, "Global optimization and common best proximity points for some multivalued contractive pairs of mappings," *Axioms*, vol. 9, no. 3, pp. 1–8, 2020, doi: 10.3390/axioms9030102.

[7] P. Debnath and H. M. Srivastava, "New extensions of Kannan's and Reich's fixed point theorems for multivalued maps using Wardowski's technique with application to integral equations," *Symmetry*, vol. 12, no. 7, pp. 1–9, 2020, doi: 10.3390/sym12071090.

[8] M. A. Geraghty, "On contractive mappings," *Proc. Amer. Math. Soc.*, vol. 40, pp. 604–608, 1973.

[9] S. B. Nadler, "Multi-valued contraction mappings," *Pac. J. Math.*, vol. 30, no. 2, pp. 475–488, 1969, doi: 10.2140/pjm.1969.30.475.

[10] E. Nazari, “Best proximity point theorems for generalized multivalued contractions in metric spaces,” *Miskolc Math. Notes*, vol. 16, no. 2, pp. 1055–1062, 2015, doi: 10.18514/MMN.2015.1329.

[11] M. Omidvari, S. M. Vaezpour, and R. Saadati, “Best proximity point theorems for F-contractive non-self mappings,” *Miskolc Math. Notes*, vol. 15, no. 2, pp. 615–623, 2014, doi: 10.18514/MMN.2014.1011.

[12] V. Sankar Raj, “A best proximity point theorem for weakly contractive non-self-mappings,” *Nonlinear Anal.*, vol. 74, no. 14, pp. 4804–4808, 2011, doi: 10.1016/j.na.2011.04.052.

[13] H. M. Srivastava, S. V. Bedre, S. M. Khairnar, and B. S. Desale, “Krasnosel’skii type hybrid fixed point theorems and their applications to fractional integral equations,” *Abstr. Appl. Anal.*, vol. 2014, no. Article ID: 710746, pp. 1–9, 2014, doi: 10.1155/2014/710746.

[14] H. M. Srivastava, S. V. Bedre, S. M. Khairnar, and B. S. Desale, “Corrigendum to ”Krasnosel’skii type hybrid fixed point theorems and their applications to fractional integral equations”,” *Abstr. Appl. Anal.*, vol. 2015, no. Article ID: 467569, pp. 1–2, 2015, doi: 10.1155/2015/467569.

[15] H. M. Srivastava, A. Shehata, and S. I. Moustafa, “Some fixed point theorems for $f(\psi,\varphi)$-contractions and their application to fractional differential equations,” *Russian J. Math. Phys.*, vol. 27, pp. 385–398, 2020, doi: 10.1134/S1061920820030103.

[16] D. Wardowski, “Fixed points of a new type of contractive mappings in complete metric space,” *Fixed Point Theory Appl.*, vol. 2012, no. 94, pp. 1–6, 2012, doi: 10.1186/1687-1812-2012-94.

[17] Q. Xu, Y. Tang, T. Yang, and H. M. Srivastava, “Schwarz lemma involving the boundary fixed point,” *Fixed Point Theory Appl.*, vol. 2016, no. Article ID: 84, pp. 1–8, 2016, doi: 10.1186/s13663-016-0574-8.

[18] J. Zhang, Y. Su, and Q. Cheng, “A note on ’A best proximity point theorem for Geraghty-contractions’,” *Fixed Point Theory Appl.*, vol. 2013, no. 99, pp. 1–4, 2013, doi: : 10.1186/1687-1812-2013-83.

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