PURELY COSMETIC SURGERIES AND PRETZEL KNOTS

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Abstract. We show that all pretzel knots satisfy the (purely) cosmetic surgery conjecture, i.e. Dehn surgeries with different slopes along a pretzel knot provide different oriented three-manifolds.

1. Introduction

Suppose that $K \subset S^3$ is a knot in the three-sphere and $r \in \mathbb{Q}$ a rational number. Let $S^3_r(K)$ denote the effect of Dehn surgery along K with coefficient r. The Purely Cosmetic Surgery Conjecture (PCSC for short) asserts:

Conjecture 1.1 (PCSC)**.** *For every nontrivial knot K, the orientation-preserving diffeomorphism $S^3_s(K) \cong S^3_r(K)$ for $s, r \in \mathbb{Q}$ implies that $s = r$.*

The conjecture has been verified for 2-bridge knots [4], for connected sums [12], for 3-braid knots [14], for knots of Seifert genus one [16] and for prime knots with at most 16 crossings [3]. By the classification of Seifert fibered spaces, the conjecture also holds for torus knots. Note that K and its mirror image $m(K)$ satisfies the conjecture at the same time, since $S^3_r(m(K)) = -S^3_{-r}(K)$.

When we relax the condition that the diffeomorphism is orientation-preserving, there are some examples of knots admitting diffeomorphic surgeries with different slopes: for example, for an amphichiral knot K we have that $S^3_r(K)$ and $S^3_{-r}(K)$ are diffeomorphic. See [15] for further results, including theorems for preztel knots.

Suppose that $P = P(a_1, \dots, a_n)$ is a pretzel knot with n strands, where a_i denotes the number of half-twists (right-handed for positive and left-handed for negative a_i) on the i^{th} strand, see Figure 1 for an illustration.

Our main result is the verification of PCSC for pretzel knots:

Theorem 1.2. *The Purely Cosmetic Surgery Conjecture holds for pretzel knots.*

In the following we will always assume that P is a knot, implying that either

- all a_i are odd and n is odd, or
- exactly one a_i (which can be assumed to be a_1) is even, and n is odd, or
- exactly one a_i (which can be assumed to be a_1) is even, and n is even.

Note that the order of the a_i's in defining the pretzel knot $P = P(a_1, \dots, a_n)$ is important, and in general can be changed only by the action of the dihedral group (when P is viewed in the isotopic position shown by Figure 2). One noteable exception is that if $a_i = \pm 1$ then it can be commuted with any other strand (by rotating the two strands together), hence these can be collected at the end of the

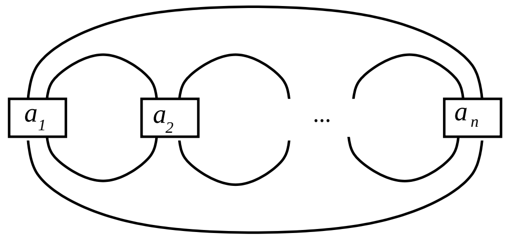

FIGURE 1. **The pretzel knot** $P(a_1, \ldots, a_n)$**.** In the following we will assume that $a_2, \ldots, a_n$ are odd, and a_1 is either even or odd. In order to have a knot, if a_1 is odd, then n must also be odd.

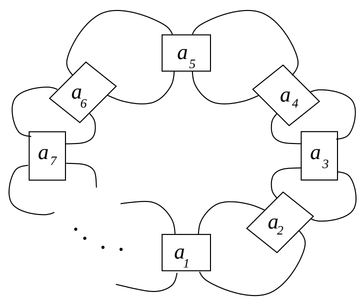

FIGURE 2. **The dihedral action is more visible in this diagram of** $P(a_1, \ldots, a_n$**.** The boxes are positioned at the vertices of a regular n-gon.

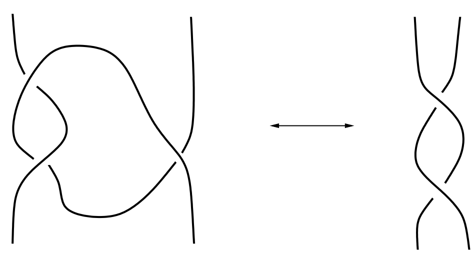

FIGURE 3. **The isotopy above shows that** $(2, -1)$ **in any string** $(a_1, \ldots, a_n)$ **defining the pretzel knot** $P(a_1, \ldots, a_n)$ **can be replaced by** (-2)**.**

string. In addition, there are two cases when the number of strands can be reduced: if $a_i = 1$ and $a_{i+1} = -1$ then these two strands can be eliminated by a simple isotopy (a Reidemeister 2 move); and if $a_1 = 2$ and $a_2 = -1$ (or if $a_1 = -2$ and $a_2 = 1$) then the isotopy shown by Figure 3 reduces the number of strands by one. For this reason, in the following we will always assume that $\{1, -1\}, \{2, -1\}$ and $\{-2, 1\}$ are not subsets of $\{a_i\}_{i=1}^n$. Furthermore we will always assume that $a_i \neq 0$, since when $a_1 = 0$, the knot P is the connected sum of alternating torus knots, and for connected sums the conjecture has already been verified [12]. In a similar manner, we will always assume that $n \geq 3$, since two-strand pretzel knots are (alternating) torus knots, and for those the conjecture is known to hold true.

The paper is organized as follows. In Section 2 we collect some obstructions stemming from the Alexander and Jones polynomials for knots to support purely cosmetic surgeries. In Section 3 we observe that pretzel knots have (knot Floer homology) thickness at most one. In Section 4 some background regading Seifert genera of pretzel knots is given. (In the light of a recent result of Hanselman [3] to be discussed later, Seifert genera are of central importance in deriving statements regarding cosmetic surgeries.) In Section 5 we deal with n-strand pretzel knots with $n \neq 5$, and in Section 6 we deal with five-strand pretzel knots and complete the proof of Theorem 1.2. We include a short Appendix providing a computational scheme for the Jones polynomial of some pretzel knots.

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2. Obstructions for purely cosmetic surgeries

A general result of Ni-Wu [9, Theorem 1.2] provides strong constraints on the surgery coefficients potentially providing cosmetic surgeries.

Theorem 2.1 (Ni-Wu)**.** *Suppose that $K \subset S^3$ is a nontrivial knot and for $r, s \in \mathbb{Q}$ we have that $S^3_r(K)$ and $S^3_s(K)$ are orientation preserving diffeomorphic. Then $s = -r$ and if $r = \frac{p}{q}$ with $p, q > 0$ relatively prime integers, then $q^2 \equiv -1 \pmod p$.* □

The Casson-Walker invariants of the three-manifolds $S^3_r(K)$ and $S^3_{-r}(K)$ can be shown to be different (hence distinguish these oriented three-manifolds) provided the Alexander polynomial $\Delta_K(t)$ of K satisfies a certain condition. More precisely, $\Delta_K(t)$ provides the following obstruction for K to admit purely cosmetic surgeries.

Theorem 2.2. *(*[1, Proposition 5.1]*) If $K \subset S^3$ admits purely cosmetic surgeries, then for the Alexander polynomial $\Delta_K(t)$ we have $\Delta''_K(1) = 0$.* □

Here $\Delta_K(t)$ is defined by the skein relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{L_0}(t) \tag{2.1}$$

with (L_+, L_-, L_0) forming a usual skein triple, and Δ being normalized to 1 on the unknot. (Then Δ_K satisfies that $\Delta_K(1) = 1, \Delta'_K(1) = 0$ and $\Delta_K(t^{-1}) = \Delta_K(t)$.) Indeed, this obstruction can be conveniently reformulated in terms of the Conway polynomial $\nabla_K(z)$ of K, where ∇_K can be described by the identity

$$\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t).$$

In fact, the Conway polynomial can also be defined by a skein relation:

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z)$$

for the skein triple (L_+, L_-, L_0), normalized as 1 on the unknot. For a knot K, we have that $\nabla_K(z) = 1 + \sum_{i=1}^{d} a_{2i}(K)z^{2i}$, and it is easy to see that $2a_2(K) =$

$\Delta''_K(1)$. For a two-component (oriented) link $L = K_1 \cup K_2$ we have that $\nabla_L(z) = \sum_{i=0}^d a_{2i+1}(L) z^{2i+1}$, and $a_1(L) = \ell k(K_1, K_2)$, the linking number of the two components, cf. [8, Proposition 8.7].

The three-manifold invariant λ_2 discussed in [7], together with the surgery formula of [7, Theorem 7.1] for $\lambda_2(S^3_r(K))$ in terms of the knot invariant $w_3(K)$ also provides an obstruction for cosmetic surgeries, leading to the following result:

Theorem 2.3. *(*[5, Proposition 3.4]*) Suppose that $K \subset S^3$ is a knot with $a_2(K) = 0$ and p, q are postive integers with $q^2 \equiv -1 \pmod p$. Then $\lambda_2(S^3_{\frac{p}{q}}(K)) = \lambda_2(S^3_{-\frac{p}{q}}(K))$ if and only if $w_3(K) = 0$.* □

The invariant $w_3(K)$ satisfies the following crossing change formula: if $(K_+, K_-, K' \cup K'')$ is a skein triple involving two knots $K_\pm$ and the two-component link $K' \cup K''$, then

$$w_3(K_+) - w_3(K_-) = \frac{1}{2}(a_2(K') + a_2(K'')) - \frac{1}{4}(a_2(K_+) + a_2(K_-) + \ell k^2(K', K'')),$$

where (as usual) $\ell k(K', K'')$ is the linking number of the two (oriented) knots K', K''.

Remark 2.4. *Indeed, both knot invariants above can be conveniently presented in terms of the Jones polynomial $V_K(t)$ of the knot K. (Here we consider the Jones polynomial satisfying the skein relation $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t)$, normalized as 1 on the unknot.) Indeed, since $6a_2(K) = 3\Delta''_K(1) = -V''_K(1)$ and by* [5, Lemma 2.2]

$$w_3(K) = \frac{1}{72}V'''_K(1) + \frac{1}{24}V''_K(1)$$

holds, the above obstructions can be summarized as was done in [5, Theorem 1.1]*: if $K \subset S^3$ admits purely cosmetic surgeries then $V''_K(1) = 0$ and $V'''_K(1) = 0$.*

3. Knot Floer homology of pretzel knots

Heegaard Floer homology can be used in more than one way to verify that a knot satisfies PCSC. The concordance invariant τ (introduced in [11]) provides the following obstruction:

Theorem 3.1. *(*[9, Theorem 1.2(c)]*) If the tau-invariant $\tau(K)$ of the knot $K \subset S^3$ derived from knot Floer homology is not equal to 0, then K satisfies PCSC.* □

The hat version of knot Floer homology (over the field $\mathbb{F}$ of two elements) of a knot $K \subset S^3$ is a finite dimensional bigraded vector space $\widehat{\mathrm{HFK}}(K) = \sum_{M,A} \widehat{\mathrm{HFK}}_M(K, A)$. By collapsing the two gradings to $\delta = A - M$, we get the δ-graded invariant $\widehat{\mathrm{HFK}}^\delta(K)$.

Definition 3.2. *The **thickness** $th(K)$ of the knot $K \subset S^3$ is the maximal value of the difference $|\delta(x) - \delta(x')|$ for homogeneous elements $x, x' \in \widehat{\mathrm{HFK}}^\delta(K)$. In particular, if $th(K) = 0$ then K is called **thin**.*

Examples of thin knots are provided by alternating knots, where the difference $A-M$ of a homogeneous element is equal to half the negative of the signature of the knot.

Work of Hanselman [3] regarding PCSC is crucial in our discussions. In particular, a direct consequence of [3, Theorem 2] is

Corollary 3.3 (Hanselman [3])**.** *If a nontrivial knot $K \subset S^3$ has thickness $th(K) \leq 5$ and $g(K) \neq 2$, then PCSC holds for K.*

Proof. By the result of Wang [16] (see Theorem 4.1), together with [3, Theorem 2] of Hanselman, the orientation-preserving diffeomorphism $S^3_s(K) \cong S^3_r(K)$ for a nontrivial knot K and $r \neq s$ implies that $g(K) > 1$ and

- either $\{r, s\} = \{\pm 2\}$ and $g(K) = 2$, or
- $\{r, s\} = \{\pm \frac{1}{q}\}$ for some positive integer q which satisfies $q \leq \frac{th(K)+2g(K)}{2g(K)(g(K)-1)}$.

For a knot with $g(K) \neq 2$ the first option is not possible, and if $th(K) \leq 5$ and $g(K) \geq 3$, we get that the positive integer q satisfies $q \leq \frac{11}{12}$, concluding the proof. $\square$

Proposition 3.4. *Suppose that $P = P(a_1, \dots, a_n)$ is an n-strand pretzel knot. Then the thickness $th(P)$ of P is at most 1.*

Proof. We will show that there is a δ-graded chain complex computing $\widehat{\mathrm{HFK}}^\delta$ for which the thickness is at most 1, hence the same applies to the homologies. This chain complex is generated by the Kauffman states of the usual diagram of the pretzel knot $P = P(a_1, \dots, a_n)$; we only need to determine the δ-gradings of these generators. (For the definition and basic properties of Kauffman states, as well as that they span a chain complex computing knot Floer homology, see [10].) There are three types of domains in the diagram of P from which the contributions should be counted: bigons in the strands, domains between the strands, and the 'top domain'. (Notice that the 'bottom domain' and the outside unbounded domain does not have to be considered, since these are occupied by the marking, which is placed on the lower arc of the diagram.) Since the orientation of the strands is important in these calculations, we distinguish three cases. These combinatorially different cases (together with the markings, symbolized by a heavy dot) and the orientations are shown by Figure 4.

Consider now a Kauffman state κ. The local contributions to δ are shown by Figure 5; notice that the orientations of the strands are important in these calculations, hence the three cases shown by Figure 4 should be discussed separately.

Case I: *All a_i are odd.* In this case the orientation of P can be chosen as shown by Figure 4(a). (Since P is a knot, n is odd.) The contribution of the marking of the Kauffman state κ in the top domain, as well as in all bigons is 0. The domains between the strands, on the other hand, contribute either $\frac{1}{2}$ or $-\frac{1}{2}$, depending whether the marking is on the strand with positive or negative twisting. The fact whether the marking of such a domain is on the left or right strand is determined by the strand distinguished by the marking in the top domain. Therefore the sign of this distinguished strand determines how many $\frac{1}{2}$ or $-\frac{1}{2}$ contributions do we

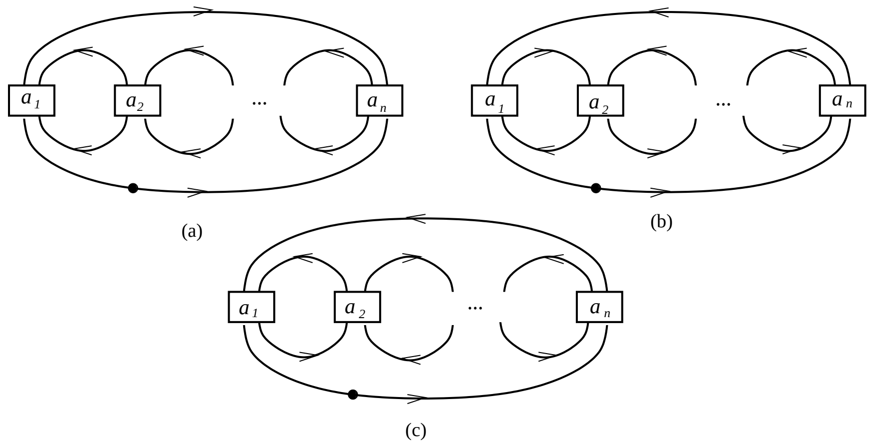

FIGURE 4. **Orientation on P.** The three diagrams indicate the three combinatorially different orientations: in (a) we show the case when all a_i are odd (hence n is odd), in (b) the case when a_1 is even and n is odd, and finally in (c) the case when a_1 is even and n is even. (The difference between the two last cases is the orientation at the first strand.)

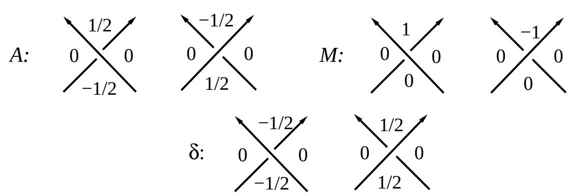

FIGURE 5. **The local contributions for A, M and δ at a crossing.** The Kauffman state distinguishes a corner at the crossing, and we take the value in that corner as a contribution of the crossing in A, M or δ of the Kauffman state at hand.

get. Consequently, if there are k negative and ℓ positive coefficients among the parameters a_i of the pretzel knot P, the δ-grading of κ is either $\frac{1}{2}(k-\ell-1)$ (if the marking of the top domain is at a strand with negative parameter) or $\frac{1}{2}(k-\ell+1)$ (if the marking in the top domain is at a strand with positive parameter). In conclusion there are at most two δ-gradings, which are one apart, hence the thickness of the knot is at most 1. Indeed, if all a_i have the same sign, then the knot is thin, in accordance with the fact that in that case the knot is alternating.

Case II: *Assume now that a_1 is even and n is odd*, shown by the diagram of Figure 4(b). In this case the first strand (with the even parameter a_1) is special. Bigons in the first strand contribute 0, while in the other strands bigons contribute $\pm\frac{1}{2}$ (the sign depending on the sign of the parameter of the strand). Consequently the bigons contribute to the δ-grading of κ a fix value independent of the Kauffman state, determined by the diagram only. The top domain provides 0 if the marking is at the first strand, and all the other domains give further 0's. If the marking in the top domain is not at the first strand, then its contribution is $\pm\frac{1}{2}$ (the sign depending on the sign of the parameter), while now the domain between the first and the second strand will have a nonzero contribution (which is again $\pm\frac{1}{2}$, depending on the sign of a_1); call this contribution c. Then the total contributions from the top domain and the ones between the strands is either 0, or $-\frac{1}{2}+c$ or $\frac{1}{2}+c$. Since $c=\pm\frac{1}{2}$, the δ-grading still takes two possible values which are one apart, implying that $th \leq 1$.

Case III: *Finally, assume that a_1 is even and n is even*, cf. the diagram of Figure 4(c). The only difference between this and the previous case is that the orientation along the first strand (with a_1 twists) is different. This case is similar to **Case I**: all bigons contribute $\pm\frac{1}{2}$ (sign depending on the sign of the parameter of the strand), the top domain contributes $\pm\frac{1}{2}$ (depending on the fact whether the marking is on the top of a positive or a negative strand), while the contribution of the domains between the strands is all 0. Once again, there are two possible δ-values, which are 1 apart, verifying the claim. □

As a direct consequence of Corollary 3.3 we have

Corollary 3.5. *Suppose that $P=P(a_1,\ldots,a_n)$ is an n-strand pretzel knot. If the Seifert genus $g(P)\neq 2$ then the purely cosmetic surgery conjecture holds for P.* □

4. Genera of pretzel knots

The Seifert genera of knots play an important role in understanding cosmetic surgeries on them. Regarding low genus knots, the following general result of Wang provides relevant information.

Theorem 4.1. *(*[16, Theorem 1.3]*) If $g(K)=1$ for a knot K then PCSC holds for K.* □

For Seifert genera of pretzel knots, we quote three results, detailed below. As before, we will assume that for the pretzel knot $P(a_1,\ldots,a_n)$ we have that $\{1,-1\},\{-2,1\}$ and $\{2,-1\}$ are not subsets of $\{a_i\}_{i=1}^n$.

4.1. Three-strand pretzel knots.

Theorem 4.2. *(Kim-Lee,* [6, Corollary 2.7]*) The Seifert genus $g(P(p,q,r))$ of the three-strand pretzel knot $P(p,q,r)$ with parameters $p,q,r\in\mathbb{Z}\setminus\{0\}$ (also satisfying that $\{1,-1\},\{2,-1\}$ and $\{-2,1\}$ are not subsets of $\{p,q,r\}$) is equal to*

(1) *1 if all p,q,r are odd,*
(2) *$\frac{1}{2}(|q|+|r|)$ if p is even and q,r have the same sign, and*
(3) *$\frac{1}{2}(|q|+|r|-2)$ if p is even and q,r have opposite signs.* □

A three-strand pretzel knot $P=P(p,q,r)$ with all odd coefficients therefore satisfies PCSC by Theorem 4.1. For $P=P(2\ell,q,r)$ with q,r odd then we have the following simple consequence of the above statement:

Corollary 4.3. *For a three-stand pretzel knot P either the genus $g(P)$ is different from 2, or up to mirroring it is $P(2\ell, 3, 1)$, $P(2\ell, 3, -3)$ or $P(2\ell, -5, 1)$ for some $\ell \in \mathbb{Z}$.* □

4.2. **All a_i's are odd.** The following theorem of Gabai describes the genus of an n-strand pretzel knot with all coefficients odd for a general (odd) n. Recall that we always assume that $\{a_i\}_{i=1}^n$ cannot contain both 1 and -1.

Theorem 4.4. *(Gabai,* [2, Theorem 3.2]*) Suppose that $P = P(a_1, \dots, a_n)$ is an n-strand pretzel knot with $n \geq 3$ and all a_i odd, and there are no two indices i, j with $a_i a_j = -1$. Then the genus $g(P)$ is equal to $\frac{1}{2}(n-1)$. In particular, $g(P) = 2$ if and only if $n = 5$.* □

4.3. **The first coefficient a_1 is even.** In this case, work of Kim-Lee provides a bound (and often a formula) for the genus of $P = P(a_1, \dots, a_n)$ (with a_1 even and all a_i with $i > 1$ odd). We will again assume that $\{a_i\}_{i=1}^n$ does not contain both 1 and -1, $a_1 \neq 0$ and if $a_1 = \pm 2$ then there is no further a_i which is equal to ∓ 1. By determining the Alexander-Conway polynomial $\nabla_P(z)$ of P and identifying its leading coefficient, the following bound on the Seifert genus $g(P)$ has been proved:

Theorem 4.5. *(Kim-Lee,* [6, Theorem 4.1]*) Suppose that the pretzel knot $P = P(a_1, \dots, a_n)$ has a_1 even ($\neq 0$), which (by possibly taking the mirror) can be assumed to be positive. Let $\alpha = \sum_{i=2}^n \operatorname{sign}(a_i)$ and $\delta = \sum_{i=2}^n (|a_i| - 1)$. Then the genus $g(P)$ of P is bounded from below by*

- $\frac{1}{2}(\delta + 2)$ *if n is odd and $\alpha \neq 0$.*
- $\frac{1}{2}\delta$ *if n is odd and $\alpha = 0$.*
- $\frac{1}{2}(a_1 + \delta)$ *if n is even and $\alpha \neq -1$.*
- $\frac{1}{2}(a_1 + \delta) - 1$ *if n is even and $\alpha = -1$.*

In addition, if none of the a_i are equal to ± 1, then the bounds above provide the precise value of the genus $g(P)$. □

A simple consequence of the above result is:

Corollary 4.6. *The pretzel knot $P = P(a_1, \dots, a_n)$ with $a_1 \neq 0$ even and a_i odd ($i > 1$) and with $n \geq 4$ has genus > 2 unless*

(1) *all a_i with $i > 1$ is either 1 or -1 (all these with the same sign),*
(2) *n odd, $\alpha \neq 0$, $a_1 = 2\ell$, $a_2 = \pm 3$ and for $i > 2$ all $a_i = \pm 1$ (all these with the same sign);*
(3) *n odd, $\alpha = 0$, $a_1 = 2\ell$, $a_2 = \pm 3$, $a_3 = \pm 3$ and for $i > 3$ all $a_i = \pm 1$ (all with the same sign),*
(4) *n even, $\alpha \neq -1$, $a_1 = 2$, $a_2 = \pm 3$ and for $i > 2$ all $a_i = \pm 1$ (all with the same sign),*
(5) *n even, $\alpha = -1$, $a_1 = 4$, $a_2 = \pm 3$, and for $i > 2$ all $a_i = \pm 1$ (all with the same sign),*
(6) *n even, $\alpha = -1$, $a_1 = 2$, $a_2 = \pm 5$, and for $i > 2$ all $a_i = \pm 1$ (all with the same sign).*
(7) *n even, $\alpha = -1$, $a_1 = 2$, $a_2 = \pm 3$, $a_3 = \pm 3$, and for $i > 3$ all $a_i = \pm 1$ (all with the same sign).* □

5. PCSC for pretzel knots with $n \neq 5$ strands

In this section we start proving Theorem 1.2. First we deal with those pretzel knots where $n \neq 5$, or when $n = 5$ and the first coefficient a_1 is even.

5.1. **Three-strand pretzel knots.** Corollary 4.3 gave a list of those three-strand pretzel knots which have Seifert genus $g(P) = 2$.

Suppose that the three-strand pretzel knot has one even coefficient $a_1 = 2\ell$, which for simplicity is assumed to be negative. Then by the repeated application of the skein relation for the Conway polynomial ∇ we have that (with $\ell < 0$)

$$\nabla_{P(2\ell,q,r)}(z) = \nabla_{P(0,q,r)}(z) + |\ell| z \nabla_{T_{2,q+r}}(z).$$

(In the inductive step we used the fact that the 2-component link L_0 involved in the skein triple is the same torus link $T_{2,q+r}$ at every step.) Note that $P(0, q, r)$ is the connected sum of two torus knots $T_{2,q}$ and $T_{2,r}$. Since $a_2(T_{2,2n+1}) = \binom{n+1}{2}$ and for the torus link $a_1(T_{2,2m}) = \ell k(T_{2,2m}) = m$, it follows that for $\{q, r\} = \{\pm 3, \pm 1\}, \{\pm 3, \pm 3\}, \{\pm 5, \pm 1\}$ (including all the possible cases of Corollary 4.3) we get either $a_2(P) \neq 0$ or $|\ell|$ so small that $P(2\ell, q, r)$ is a knot with at most 16 crossing. Since for those the PCSC has been verified, we have

Proposition 5.1. *If $P = P(p, q, r)$ is a three-strand pretzel knot, then the purely cosmetic surgery conjecture holds for P.* □

5.2. **More than three strands.** We start with the case when a_1 is even (and nonzero).

Theorem 5.2. *Suppose that $P = P(a_1, \ldots, a_n)$ is an n-strand pretzel knot with $n \geq 4$ and a_1 even, while all a_i with $i > 1$ are odd. Then P satisfies PCSC.*

Proof. Most of these knots have genus more than 2, hence Proposition 3.4 provides the result. The exceptions (i.e. those pretzel knots considered by the theorem which have genus at most 2) are listed in Corollary 4.6, and they can be handled by similar means as we did in the case of three-strand knots: either they have low crossing number, or the second coefficient of the Conway polynomial provides the desired obstruction.

Indeed, if we have Case (1) of Corollary 4.6, then P is a two-bridge knot, and PCSC follows from [4].

For n odd (cases (2) and (3) in Corollary 4.6) the computation of the Conway polynomial proceeds exactly as for the three-strand case, providing that

$$\nabla_{P(2\ell,a_2,\ldots,a_n)}(z) = \prod_{i=2}^{n} \nabla_{T_{2,a_i}}(z) + |\ell| z \nabla_{P(a_2,\ldots,a_n)}(z).$$

By multiplicativity of ∇ under connected sum, we have that $a_2(\#_{i=2}^{n} T_{2,a_i}) = \sum_{i=2}^{n} a_2(T_{2,a_i})$ and $a_2(T_{2,a_i}) = \binom{\frac{|a_i|+1}{2}}{2}$. Furthermore, for the two-component link $Q = P(a_2, \ldots, a_n)$ we have $a_1(Q) = \ell k(Q) = -\frac{1}{2}\sum_{i=2}^{n} a_i$, where this latter term is the linking number of the two components of Q (both unknots). In the cases (2) and (3) the a_2-invariants of the torus knots are 1 (for $T_{2,3}$) and 0 (for the trivial knot), hence the same argument as for the three-strand case shows that either $a_2(P) \neq 0$, or the knot has crossing number at most 16, concluding the argument.

A similar argument works when n is even. Indeed, we can relate $\nabla_{P(2\ell, a_2, \ldots, a_n)}(z)$ to $\nabla_{P(0, a_2, \ldots, a_n)}(z)$ by the repeated application of the skein rule, although this case is slightly different. Because of the change of the orientation pattern on the strand with even coefficient, the link in the skein triple will be different in every step: in the i^{th} step it will be $P(2\ell - (2i-1), a_2, \ldots, a_n)$. The expression for $a_2(P(2\ell, a_2, \ldots, a_n))$ (just as before) will involve a term $a_2(P(0, a_2, \ldots, a_n))$, which (as before) is the sum of a_2-invariants of alternating torus knots — mostly the unknot. The other term now is a sum of the form $\sum_{i=1}^{\ell} a_1(P(2\ell - (2i-1), a_2, \ldots, a_n))$, and here the terms are equal to the linking numbers of components of the two-component links. In the cases listed under (4)-(7) in Corollary 4.6 the same scheme will be visible: there will be only few cases when a_2 is zero, and those correspond to knots with low crossing number, hence the argument is complete. □

We close this section with the case when all a_i are odd and $n \geq 6$.

Proposition 5.3. *If $n \geq 6$ odd and all a_i are odd, then the pretzel knot $P(a_1, \ldots, a_n)$ satisfies PCSC.*

Proof. In these cases Theorem 4.4 implies that the genus of the knot is $\frac{1}{2}(n-1) > 2$, hence Proposition 3.4 concludes the argument. □

6. Five-strand pretzel knots

Suppose now that $P = P(a_1, \ldots, a_5)$ is a five-strand pretzel knot with all a_i odd. Depending on the signs of the coefficients, we will distinguish two cases.

6.1. Among the a_i's there are 0,1,4 or 5 negative coefficients.

Lemma 6.1. *Suppose that the five-strand pretzel knot $P = P(a_1, \ldots, a_5)$ has only odd coefficients and among them 0,1,4 or 5 are negative. Then $\tau(P) \neq 0$.*

Proof. As the proof of Proposition 3.4 shows, in these cases the two possible δ-gradings are 3 and 2 (if there are only positive coefficients), 2 and 1 (if there is a unique negative coefficient), and symmetrically -2 and -1 in case of a unique positive coefficient, and -3 and -2 when there are five negative coefficients. Recall that $\tau(P)$ is the Alexander grading of one of the homogeneous elements of $\widehat{\mathrm{HFK}}(P)$ with Maslov grading 0. In case $\tau(P) = 0$, there should be an element with δ-grading 0, a contradiction. Therefore in these cases $\tau(P) \neq 0$. □

Proposition 6.2. *Suppose that the five-strand pretzel knot $P = P(a_1, \ldots, a_5)$ has only odd coefficients and among the five odd coefficients 0,1,4 or 5 are negative. Then P satisfies PCSC.*

Proof. Since in these cases by Proposition 6.1 we have that $\tau(P) \neq 0$, Theorem 3.1 implies the result. □

6.2. **There are 2 or 3 negative coefficients among the a_i's.** In this case our arguments will rest on the obstructions stemming from the coefficient a_2 of the Conway polynomial, together with the w_3-invariant introduced in Section 2. Since the coefficients of $P = P(a_1, \ldots, a_5)$ are all odd, there is an obvious Seifert surface of genus two associated to the diagram of the knot given in Figure 1. The Seifert matrix in the obvious basis is given in [15, Section 2.1], where it has been also shown that

Proposition 6.3. *(*[15, Lemma 2.2]*) Suppose that $P = P(a_1, \ldots, a_5)$ is a five-strand pretzel knot with $a_i = 2k_i + 1$ odd. Then*

$$a_2(P) = s_2 + 2s_1 + 3,$$

where s_i is the value of the i^{th} elementary symmetric polynomial in five variables evaluated on $\{k_1, \ldots, k_5\}$. □

Using the skein rule, a formula for $v_3(K) = -2w_3(K)$ has been given in [15, Lemma 2.2] for all pretzel knots with odd coefficients. For a five-strand pretzel knot $P = P(2k_1 + 1, \ldots, 2k_5 + 1)$ the result provides

Lemma 6.4. *(*[15, Lemma 2.2]*) $w_3(K) = \frac{1}{2}(5 + 3s_1 + s_1^2 + s_2 + \frac{1}{2}(s_3 + s_1 s_2))$, where the values of the elementary symmetric polynomials s_1, s_2, s_3 are as given in Proposition 6.3.* □

Remark 6.5. *The statements of Proposition 6.3 and Lemma 6.4 in* [15] *have been formulated for the case of $k_i \geq 0$; the proofs of these statements, however, hold in the wider generality we use them here.*

With these preparations in place, we can now turn to the verification of PCSC for five-strand pretzel knots.

Proposition 6.6. *Suppose that $P = P(a_1, a_2, a_3, a_4, a_5)$ is a five-strand pretzel knot with all coefficients odd. Then the purely cosmetic surgery conjecture holds for P.*

Proof. We can assume that there are two or three negative coefficients among the $\{a_i\}_{i=1}^5$, since (by Proposition 6.2) in the other cases PCSC holds. If P has $a_2(P) \neq 0$, then Theorem 2.2 implies the result. If $a_2(P) = 0$ and $w_3(P) \neq 0$, then Theorem 2.3 concludes the argument. Suppose therefore that $P = P(2k_1+1, \ldots, 2k_5+1)$ has $a_2(P) = 0$ (implying that $s_2 = -2s_1 - 3$) and $w_3(P) = 0$, implying in the light of Lemma 6.4 (after substituting $s_2 = -2s_1 - 3$) that $s_3 = s_1 + 2$.

By using the standard identities

$$\sum_{i=1}^{5} k_i^2 = s_1^2 - 2s_2, \qquad \sum_{i=1}^{5} k_i^3 = s_1^3 - 3s_1 s_2 + 3s_3,$$

and substituting $s_2 = -2s_1 - 3$ and $s_3 = s_1 + 2$, we get

$$\sum_{i=1}^{5} k_i^2 = s_1^2 + 4s_1 + 6 = (s_1 + 2)^2 + 2, \qquad \sum_{i=1}^{5} k_i^3 = s_1^3 + 6s_1^2 + 12s_1 + 6 = (s_1 + 2)^3 - 2.$$

Let $S = \sum_{i=1}^{5} a_i$. Since $S = 2s_1 + 5$, we get that

$$\sum a_i^2 = S^2 + 4, \qquad \sum a_i^3 = S^3.$$

Let

$$\mathcal{P} = \{i \in \{1, \dots, 5\} \mid a_i > 0\}, \qquad \mathcal{N} = \{i \in \{1, \dots, 5\} \mid a_i < 0\}.$$

By our assumption on the signs of the k_i, we can assume that both $\mathcal{P}$ and $\mathcal{N}$ have two or three elements, implying that

$$\sum_{i\in\mathcal{P}} a_i^3 > 0, \qquad \sum_{i\in\mathcal{N}} a_i^3 < 0. \tag{6.1}$$

We can also assume that one of the two inequalities

$$\sum_{i\in\mathcal{P}} a_i^2 \leq S^2, \qquad \sum_{i\in\mathcal{N}} a_i^2 \leq S^2 \tag{6.2}$$

holds, since the violation of both would imply $2S^2 \leq S^2 + 4$, hence $S^2 \leq 4$, so $\sum_{i=1}^5 a_i^2 \leq 8$, therefore P is a knot of crossing number less than 16, for which PCSC holds true.

Assume first that *both* inequalities in Equation (6.2) are satisfied. In this case $|a_i| \leq |S|$, hence

$$\sum_{i\in\mathcal{P}} a_i^3 \leq \sum_{i\in\mathcal{P}} |S| a_i^2 \leq |S|^3$$

and

$$\sum_{i\in\mathcal{N}} a_i^3 \geq \sum_{i\in\mathcal{N}} -|S| a_i^2 \geq -|S|^3.$$

Combining these inequalities with the ones from Equation (6.1) we get

$$-|S|^3 < \sum_{i=1}^5 a_i^3 < |S|^3,$$

providing a contradiction to $\sum_{i=1}^5 a_i^3 = S^3$. This shows, that under the assumptions that both inequalities of Equation (6.2) hold, if $a_2(P) = 0$ then $w_3(P) \neq 0$.

Assume now that one of the inequalities of Equation (6.2) is false. This implies that terms in the other inequality sum up to at most 3, implying that all terms in this other inequality satisfy $a_i^2 = 1$, i.e. $a_i = \pm 1$ (with the same sign). By possibly mirroring the knot, we can assume that these terms are all equal to 1, hence the corresponding $k_i = 0$. By our previous assumption, there are two or three such coefficients.

Case I: Suppose first that there are three positive coefficients $a_1 = a_2 = a_3 = 1$ and $a_4, a_5 < 0$. This implies that $k_1 = k_2 = k_3 = 0$, hence when computing $a_2(P)$, we get that it is equal to $3 + 2(k_4 + k_5) + k_4 k_5$, while the expression $s_3 - s_1 - 2$ is equal to $-k_4 - k_5 - 2$. If the corresponding pretzel knot violates PCSC, both expressions need to be zero, and we get $k_4 k_5 = 1$, hence $k_4 = k_5 = -1$. Since $(a_1, \dots, a_5) = (1, 1, 1, -1, -1)$ gives the unknot, we can ignore this case.

Case II: Suppose that there are two positive coefficients $a_1 = a_2 = 1$, and $a_3, a_4, a_5 < 0$. With the usual definition of k_i as $a_i = 2k_i + 1$, we have that $k_1 = k_2 = 0$ and $a_2(P) = k_3k_4 + k_3k_5 + k_4k_5 + 2(k_3 + k_4 + k_5) + 3$ and $s_3 - s_1 - 2 = k_3k_4k_5 - k_3 - k_4 - k_5 - 2$. If one of them is nonzero, then P satisfies PCSC. If both are zero, then so is their sum:

$$2k_3k_4k_5 + k_3k_4 + k_3k_5 + k_4k_5 - 1 = 0.$$

Writing this sum as

$$k_3k_4(k_5+1)+k_3k_5(k_4+1)+k_4k_5-1, \tag{6.3}$$

the first two terms are negative unless $k_5=-1$ or $k_4=-1$, in which cases the knot has (at most) three strands; the same applies if $k_3=-1$. Since $k_3(k_5+1)>|k_5|$ or $k_3(k_4+1)>|k_4|$ once $k_3<-1$, the expression of Equation 6.3 is negative, providing the desired contradiction. □

Proof of Theorem 1.2. The proof of the theorem for the case of $n=3$ is provided by Proposition 5.1. When a_1 is even and $n\geq 4$, the result is proved in Theorem 5.2. When $n\geq 6$ and all a_i are odd, Proposition 5.3 gives the result. Finally in the cases when $n=5$ and all a_i odd, Proposition 6.6 verifies the claim. This completes the proof of Theorem 1.2. □

7. Appendix: the Jones polynomial for pretzel knots

In this section we provide a convenient formula for the Jones polynomial of pretzel knots with odd coefficients. Recall that the Jones polynomial $V_K(t)$ is defined by the skein relation

$$t^{-1}V_{L_+}(t)-tV_{L_-}(t)=(t^{\frac{1}{2}}-t^{-\frac{1}{2}})V_{L_0}(t)$$

and normalization $V_U(t)=1$ on the unknot U.

Suppose that $P=P(a_1,\ldots,a_n)$ is an n-strand pretzel knot with a_i odd. Let $s=t^{\frac{1}{2}}$, $k\in\mathbb{Z}$ be an integer and $v_i\in\{0,1\}$. We define functions $P_{v_i,k}(s)$ as follows. For $v_i=0$ take

$$P_{0,k}(s)=-s^{-2k}.$$

If $v_i=1$ and $k>0$, take

$$P_{1,k}(s)=\sum_{j=1}^{k}(-1)^j\cdot s^{1-2j};$$

and if $v_i=1$ and $k<0$, take

$$P_{1,k}(s)=\sum_{j=1}^{-k}(-1)^j\cdot s^{-1+2j}.$$

For a fixed vector $v\in\{0,1\}^n$ multiply the terms $P_{v_i,a_i}(s)$ corresponding to the twisting numbers $a_1,\ldots,a_n$ of the given pretzel knot, and multiply the result with the Jones polynomial of the $d(v)$-component unlink, where $d(v)=|(n-1)-\sum_{i=1}^n v_i|$, resulting in

$$Q_{v,a_1,\ldots,a_n}(s)=(-s-s^{-1})^{d(v)}\cdot P_{v_1,a_1}(s)\cdot P_{v_2,a_2}(s)\cdots P_{v_n,a_n}(s).$$

Finally, add these terms and get $W_P(s)=\sum_{v\in\{0,1\}^n}Q_{v,a_1,\ldots,a_n}(s)$. The verification of the fact that we get the Jones polynomial follows the same route as the description of the Jones polynomial through spanning tree expansion, as given in [13].

Proposition 7.1. *With the substitution $t=s^2$ the function $W_P(s)$ provides the Jones polynomial $V_P(t)$ of the n-strand pretzel knot P with all odd coefficients.* □

Remark 7.2. *This formula can also be used to prove the formula of Lemma 6.4.*

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