# CHARACTERIZATION OF QUASI-ARITHMETIC MEANS WITHOUT REGULARITY CONDITION 

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#### Abstract

In this paper we show that bisymmetry, which is an algebraic property, has a regularity improving feature. More precisely, we prove that every bisymmetric, partially strictly monotonic, reflexive and symmetric function $F: I^{2} \rightarrow I$ is continuous. As a consequence, we obtain a finer characterization of quasiarithmetic means than the classical results of Aczél [1, Kolmogoroff [18, Nagumo [20] and de Finetti (11.


## 1. Introduction

Our main goal is to prove a somewhat surprising result, namely, a purely algebraic property (bisymmetry) has a regularity improving feature. In other word, a better analytic behaviour of a map is resulted by a non-analytic object.

The story of bisymmetry is an old one. It is developed, together with its role in the characterization of quasi-arithmetic means, by János Aczél in 1948 (see [1]).

Quasi-arithmetic means are central objects in theory of functional equations, in particular in the theory of means (see e.g. [6], 10], [12], [13], [14], [17], [21], [22], [23], [24], [25]).

Aczél's motivation were the works of Kolmogoroff 18], Nagumo [20] and de Finetti [11]. They considered quasi-arithmetic means as a sequence of maps, where the $n$th member of this sequence is the $n$ variable quasi-arithmetic mean. They characterize this sequence by

[^0]means of reflexivity, continuity, increasing property, symmetry and associativity in the sense of Kolmogorofflu

Aczél dealt only with the two-variable case, and he changed associativity in the sense of Kolmogoroff into bisymmetry. In his proof (see [1] or [3, proof of Theorem 1 on page 290]) continuity is used essentially but a little bit furtively. This was our motivation for a sophisticated examination of the proof in question. It turned out at last that continuity is superfluous in the characterization of two-variable quasi-arithmetic means, which is somehow a striking outcome of the present investigation. In the second section we will see that the situation is completely different if we assume associativity, which is a close relative of bisymmetry.

Concerning the structure of our work, we summarize the related concepts and results in the next section. The third one is devoted to our main theorem and its consequences together with their proofs. In the last section we pose some open questions connecting to our current work.

## 2. Preliminary results and notations

We keep the following notations throughout the text. Let $I \subseteq \mathbb{R}$ be a proper interval (i.e. the cardinality of $I$ is at least 2 , in notation $|I| \geq 2)$ and $F: I \times I \rightarrow \mathbb{R}$ be a map.

The map $F$ is said to be
(i) reflexive, if $F(x, x)=x$ for every $x \in I$;
(ii) partially strictly monotone / partially strictly increasing / partially monotone / partially increasing, if the functions $x \mapsto F\left(x, y_{0}\right)$, $y \mapsto F\left(x_{0}, y\right)$ are strictly monotone / strictly increasing / monotone / increasing for every fixed $x_{0} \in I$ and $y_{0} \in I$, respectively;
(iii) symmetric, if $F(x, y)=F(y, x)$ for every $x, y \in I$;
(iv) bisymmetric, if

$$
\begin{equation*}
F(F(x, y), F(u, v))=F(F(x, u), F(y, v)) \tag{BS}
\end{equation*}
$$

for every $x, y, u, v \in I$;
(v) associative, if

$$
\begin{equation*}
F(F(x, y), z))=F(x, F(y, z)) \tag{AS}
\end{equation*}
$$

[^1]for every $x, y, z \in I$.
The map $F$ is said to be a mean on the interval $I$ if its value is always between the minimum and the maximum of the variables, that is to say,
\[

$$
\begin{equation*}
\min \{x, y\} \leq F(x, y) \leq \max \{x, y\} \tag{1}
\end{equation*}
$$

\]

for every $x, y \in I$. If the previous inequalities are strict, whenever $x$ is different from $y$, then $F$ is called a strict mean on $I$.

Observation 1. If a map $F$ is reflexive and partially strictly increasing, then it is a strict mean on $I$.

Proof. Let $x, y \in I$ be arbitrary. We can assume without the loss of generality that $x<y$. Then, using the assumptions, we can write

$$
x=F(x, x)<F(x, y)<F(y, y)=y .
$$

The following fundamental result is due to Aczél [1], which also can be found in [3, Theorem 1. page 287].

Theorem 1. A function $F: I^{2} \rightarrow I$ is continuous, reflexive, partially strictly monotonic, symmetric and bisymmetric mapping if and only if there is a continuous, strictly increasing function $f:[0,1] \rightarrow I$ that satisfies

$$
\begin{equation*}
F(x, y)=f\left(\frac{f^{-1}(x)+f^{-1}(y)}{2}\right), \quad x, y \in I \tag{2}
\end{equation*}
$$

The following observation indicates that monotonicity of F in Theorem 1 can be substituted by increasing property. Hence, in the sequel we would focus on partially increasing mappings.
Observation 2. Let $F: I^{2} \rightarrow I$ be a reflexive, partially strictly monotone, symmetric mapping. Then $F$ is partially strictly increasing.

Proof. Clearly, if $F$ is partially strictly monotone and symmetric on the interval $I$, then it is either strictly increasing in each of its variables or strictly decreasing in each of its variables. Reflexivity implies that $F$ can only be increasing in each of its variables.

The function $F$ of the form (2) is called quasi-arithmetic mean as it is a conjugate of the arithmetic mean by an order preserving bijection $f$.

Apart from quasi-arithmetic means, bisymmetry is strongly connected to associativity. The following theorem due to Aczél is wellknown (see e.g. [2, Theorem 1. page 107]).

Theorem 2. A function $F: I^{2} \rightarrow I$ is an associative, partially strictly increasing and continuous mapping if and only if there is a proper interval $J \subset \mathbb{R}$ and a continuous and strictly monotonic function $f: J \rightarrow I$ that satisfies

$$
\begin{equation*}
F(x, y)=f\left(f^{-1}(x)+f^{-1}(y)\right), \quad x, y \in I \tag{3}
\end{equation*}
$$

It is important to note that the original theorem assumes cancellative $\overbrace{}^{2}$ property instead of partially strictly increasing property, however, every partially strictly increasing two-place operation is automatically cancellative, and every continuous cancellative operation on an interval is partially strictly monotone. A simpler and constructive proof of Theorem 2, given by Craigen and Páles, can be found in 9]. The $n$-variable case proved by Couceiro and Marichal in [8].

It is clear that every map is of the form (3) is bisymmetric, as well.
This theorem immediately implies the following result.
Corollary 3. There is no reflexive, associative, partially strictly increasing, symmetric and continuous mapping on $I^{2}$, where $I$ is an arbitrary interval.

Proof. According to Theorem 2 the assumptions without reflexivity implies that the map in question can be written in the form of (3), which is clearly not reflexive.

We show similar implication in more general settings, where the concepts (reflexivity, associativity, etc.) on a totally ordered set $X$ can be defined in a similar way as on an interval $I$.

Proposition 4. Let $X$ be a strictly totally ordered set with $|X| \geq$ 2. Then there is no reflexive, associative, partially strictly increasing mapping on $X^{2}$.

Proof. Let us assume that $F: X^{2} \rightarrow X$ is reflexive, associative and partially strictly increasing map. Then for arbitrary $a, b \in X, a<b$ we have $a<F(a, b)<b$. Using associativity and reflexivity we can write

$$
F(a, F(a, b))=F(F(a, a), b)=F(a, b),
$$

which contradicts to the partially strictly increasing property of $F$.
On the other hand, it is well-known that if we weaken the conditions such as the partial functions are increasing (not strictly), then uncountably many such functions do exist even if they are reflexive.

[^2]The general description of reflexive, associative, partially increasing functions are not known. However, assuming the existence of a neutral element, we have the following characterization.

Theorem 5. [16, Theorem 2.2.] A function $F: I^{2} \rightarrow I$ is reflexive, associative, partially increasing, and has a neutral element $e \in I$ (i.e. $F(x, e)=F(e, x)=x$ for every $x \in I)$ if and only if there exists a monotone decreasing function $g: I \rightarrow I$ with $g(e)=e$ such that

$$
F(x, y)= \begin{cases}\min (x, y), & \text { if } y<g(x), \text { or }  \tag{4}\\ & y=g(x) \text { and } x<g^{2}(x) \\ \max (x, y), & \text { if } y>g(x), \text { or } \\ & y=g(x) \text { and } x>g^{2}(x), \\ \min (x, y) \text { or } \max (x, y), & \text { if } y=g(x) \text { and } x=g^{2}(x) .\end{cases}
$$

Moreover, $F(x, y)=F(y, x)$ except perhaps the set of points $(x, y) \in I^{2}$ satisfying $y=g(x)$ and $x=g^{2}(y)$.
Remark 1. Clearly, such an $F$ in Theorem is symmetric if $F(x, y)=$ $F(y, x)$ when $g(x)=y$, and it is continuous, if $F$ is the minimum or the maximum on $I^{2}$.

The following lemma is a folklore, the interested reader is referred to [7, Lemma 22.].
Lemma 6. Let $X$ be a set with cardinality $|X| \geq 2$ and let $F: X^{2} \rightarrow X$ be a map. If $F$ is bisymmetric and has a neutral element, then it is associative and symmetric.

Thus we get the following result as a corollary.
Corollary 7. Let $F: I^{2} \rightarrow I$ be a reflexive, bisymmetric, partially increasing function that has a neutral element $e \in I$. Then $F$ is symmetric, satisfies (4) with $F(x, y)=F(y, x)$ if $g(x)=y$.

If $F$ is continuous, then $F$ is the minimum or the maximum on $I^{2}$.
Thus, for such a bisymmetric family of functions the continuity assumption is essential. One can find uncountably many different discontinuous functions satisfying (4).

It is also worthy to note that the projections $F_{1}(x, y)=x$ and $F_{2}(x, y)=y$ are reflexive, bisymmetric, partially increasing functions and continuous but not symmetric (and have no neutral element).

## 3. Main results

Theorem 8. Let us assume that $a, b \in \mathbb{R}, a<b$ and $F:[a, b]^{2} \rightarrow[a, b]$ is a reflexive, partially strictly increasing, symmetric and bisymmetric
map. Then there is a continuous function $f:[0,1] \rightarrow[a, b]$ such that

$$
\begin{equation*}
F(x, y)=f\left(\frac{f^{-1}(x)+f^{-1}(y)}{2}\right), \quad x, y \in[a, b] \tag{5}
\end{equation*}
$$

Proof. At first we imitate Aczél's algorithmic construction of function $f$ (see Aczél and Dhombres [3] on the pages 287 - 290). Thus let $f:[0,1] \rightarrow[a, b]$ be a function, defined recursively on the set of dyadic numbers diad $[0,1]$. We introduce the following notations for the sake of simplicity:

$$
\mathcal{D}:=\operatorname{diad}[0,1], \quad x \circ y:=F(x, y), \quad x, y \in[a, b] .
$$

Let

$$
\begin{gathered}
f(0)=a, f(1)=b \\
f\left(\frac{1}{2}\right)=a \circ b, \quad f\left(\frac{1}{4}\right)=a \circ(a \circ b), \quad f\left(\frac{3}{4}\right)=a \circ(b \circ b)
\end{gathered}
$$

and so forth. The function $f$ is defined by satisfying the following identity

$$
\begin{equation*}
f\left(\frac{d_{1}+d_{2}}{2}\right)=f\left(d_{1}\right) \circ f\left(d_{2}\right) \tag{6}
\end{equation*}
$$

for every $d_{1}, d_{2} \in \mathcal{D}$. In [3] it was shown that the function $f$ is welldefined and strictly increasing on $\mathcal{D} 3^{3}$

In the sequel we show that $f(\mathcal{D})$ is dense in $[a, b]$, which implies that $f$ is continuous.

Suppose for the contrary that $f(\mathcal{D})$ is not dense in $[a, b]$. In the following steps we show that this is impossible.

First step: We prove that $\overline{f(\mathcal{D})}$ (the closure of $f(\mathcal{D})$ ) is uncountable. For this, we construct an extension of $f$ on the whole unit interval.

Let $x \in[0,1] \backslash \mathcal{D}$. Then there is a strictly monotone increasing sequence $\left\{d_{n}\right\}_{n \in \mathbb{N}}$ in $\mathcal{D}$ such that $d_{n} \rightarrow x$ as $n \rightarrow \infty$. Because $f$ is strictly increasing on $\mathcal{D}$ (see [3) and $[a, b]$ is compact, we have that $f\left(d_{n}\right)$ is convergent. Let us define $f(x)$ as the limit of this sequence, that is

$$
f(x):=\lim _{n \rightarrow \infty} f\left(d_{n}\right) .
$$

If $y \in[0,1] \backslash \mathcal{D}$ and $x \neq y$, then we can assume that $x<y$. Then, there exists a strictly monotone increasing dyadic sequence $\left\{s_{n}\right\}_{n \in \mathbb{N}}$ such that $s_{n} \rightarrow y$. If $j \in \mathbb{N}$ is large enough, we have that

$$
d_{i}<s_{j}, \quad i \in \mathbb{N},
$$

[^3]which entails
$$
f\left(d_{i}\right)<f\left(s_{j}\right)<f(y)
$$

Hence,

$$
f(x)=\lim _{n \rightarrow \infty} f\left(d_{n}\right) \leq f\left(s_{j}\right)<f(y)
$$

In other words, as $f$ is strictly increasing on $\mathcal{D}$, we have that its extension is strictly increasing on the whole unit interval. So, it is injective, which implies that $\overline{f(\mathcal{D})}$ is uncountable.

Second step: We prove that $\overline{f(\mathcal{D})}$ has uncountably many two-sided accumulation points, which can be defined as follows.

Let $H \subset \mathbb{R}$ be a set. A point $\alpha$ of $H$ is said to be

- isolated, if there exists an $\varepsilon>0$ such that

$$
] \alpha-\varepsilon, \alpha+\varepsilon[\cap H=\emptyset
$$

- two-sided accumulation point, if for every $\varepsilon>0$, we have

$$
] \alpha-\varepsilon, \alpha[\cap H \neq \emptyset \quad \text { and } \quad] \alpha, \alpha+\varepsilon[\cap H \neq \emptyset
$$

- one-sided accumulation point, if it is neither isolated, nor twosided accumulation point.
The set $\overline{f(\mathcal{D})}$ has at most countably many isolated points, otherwise there would be uncountably many disjoint, proper intervals in the compact interval $[a, b]$. From the same reason, $\overline{f(\mathcal{D})}$ has at most countably many half-sided accumulation points.

Since $\overline{f(\mathcal{D})}$ is uncountable, we have that it has at least uncountably many two-sided accumulation points.

Third step: Suppose that $f(\mathcal{D})$ is not dense in $[a, b]$. Then there is a point $z \in] 0,1[$ such that

$$
\begin{equation*}
\lim _{d_{i} \rightarrow z-} f\left(d_{i}\right)<\lim _{D_{i} \rightarrow z+} f\left(D_{i}\right), \tag{7}
\end{equation*}
$$

where $\left\{d_{i}\right\}_{i \in \mathbb{N}}$ and $\left\{D_{i}\right\}_{i \in \mathbb{N}}$ are arbitrary dyadic sequences from $\mathcal{D}$ tending to $z$ from the left and from the right, respectively. Let us define $X, Y \in[a, b]$ in the following way:

$$
X:=\lim _{d_{i} \rightarrow z-} f\left(d_{i}\right), \quad Y:=\lim _{D_{i} \rightarrow z+} f\left(D_{i}\right)
$$

The values $X$ and $Y$ are independent from the choices of the sequences $\left\{d_{i}\right\}_{i \in \mathbb{N}}$ and $\left\{D_{i}\right\}_{i \in \mathbb{N}}$, respectively (see [3] on page 289).

Fourth step: We prove that for arbitrary $\alpha \neq \beta$ two-sided accumulation points we have

$$
] \alpha \circ X, \alpha \circ Y[\cap] \beta \circ X, \beta \circ Y[=\emptyset,
$$

where $X$ and $Y$ were defined in the previous step.

Let $\alpha, \beta \in[a, b], \alpha<\beta$ be two-sided accumulation points of $\overline{f(\mathcal{D})}$. Then, there are $d_{\alpha}, d_{\beta} \in \mathcal{D}$ such that

$$
\begin{equation*}
\alpha<f\left(d_{\alpha}\right)<f\left(d_{\beta}\right)<\beta \tag{8}
\end{equation*}
$$

Moreover, there exist dyadic sequences $\left\{d_{n}\right\}_{n \in \mathbb{N}}$, and $\left\{D_{n}\right\}_{n \in \mathbb{N}}$ in the interval $[0,1]$ such that

$$
d_{1}<d_{2}<\cdots<z<\cdots<D_{2}<D_{1}
$$

and

$$
f\left(d_{n}\right) \rightarrow X \quad \text { and } \quad f\left(D_{n}\right) \rightarrow Y
$$

as $n \rightarrow \infty$. It also follows from the definitions of the sequences and from the strictly increasing property of $f$ on $\mathcal{D}$ that

$$
\begin{equation*}
f\left(d_{n}\right)<X<Y<f\left(D_{m}\right) \tag{9}
\end{equation*}
$$

for every $n, m \in \mathbb{N}$.
If $n$ and $m$ are large enough, then

$$
\frac{D_{m}+d_{\alpha}}{2}<\frac{d_{n}+d_{\beta}}{2}
$$

Applying (6) and the strictly increasing property of $f$ on $\mathcal{D}$, we can write

$$
\begin{gathered}
f\left(d_{n}\right) \circ f\left(d_{\alpha}\right)<f\left(D_{m}\right) \circ f\left(d_{\alpha}\right)=f\left(\frac{D_{m}+d_{\alpha}}{2}\right)< \\
f\left(\frac{d_{n}+d_{\beta}}{2}\right)=f\left(d_{n}\right) \circ f\left(d_{\beta}\right)<f\left(D_{m}\right) \circ f\left(d_{\beta}\right) .
\end{gathered}
$$

It follows then

$$
] f\left(d_{n}\right) \circ f\left(d_{\alpha}\right), f\left(D_{m}\right) \circ f\left(d_{\alpha}\right)[\cap] f\left(d_{n}\right) \circ f\left(d_{\beta}\right), f\left(D_{m}\right) \circ f\left(d_{\beta}\right)[=\emptyset
$$

By the chain of inequalities (9), we have that

$$
\begin{aligned}
& ] X \circ f\left(d_{\alpha}\right), Y \circ f\left(d_{\alpha}\right)[\subset] f\left(d_{n}\right) \circ f\left(d_{\alpha}\right), f\left(D_{m}\right) \circ f\left(d_{\alpha}\right)[, \\
& ] X \circ f\left(d_{\beta}\right), Y \circ f\left(d_{\beta}\right)[\subset] f\left(d_{n}\right) \circ f\left(d_{\beta}\right), f\left(D_{m}\right) \circ f\left(d_{\beta}\right)[,
\end{aligned}
$$

which implies that

$$
] X \circ f\left(d_{\alpha}\right), Y \circ f\left(d_{\alpha}\right)[\cap] X \circ f\left(d_{\beta}\right), Y \circ f\left(d_{\beta}\right)[=\emptyset
$$

Even more so, applying (8), we have

$$
] X \circ \alpha, Y \circ \alpha[\cap] X \circ \beta, Y \circ \beta[=\emptyset
$$

It means, that $[a, b]$ contains uncountably many accumulation points and hence uncountably many disjoint, proper intervals, which is impossible. So, $f(\mathcal{D})$ is necessarily dense in $[a, b]$.

Now, we are going to prove, that the density of $f(\mathcal{D})$ implies the continuity of $f$. Let $x \in[0,1] \backslash \mathcal{D}$ and let $\left\{d_{n}\right\}_{n \in \mathbb{N}},\left\{D_{n}\right\}_{n \in \mathbb{N}}$ be dyadic sequences from $\mathcal{D}$ such that $d_{n}$ tends to $x$ strictly monotone increasingly from the left, and $D_{n}$ tends to $x$ strictly monotone decreasingly from the right. Let us consider the values

$$
L:=\lim _{n \rightarrow \infty} f\left(d_{n}\right), \quad R:=\lim _{n \rightarrow \infty} f\left(D_{n}\right) .
$$

Because of the strictly increasing property of $f$ it is clear that $L \leq R$. If $L<R$, then

$$
f(\mathcal{D}) \cap] L, R[=\emptyset,
$$

which contradicts to the fact that $f(\mathcal{D})$ is dense in $[a, b]$. Consequently, we obtain that $f$ is continuous and strictly monotone increasing on $[0,1]$, which entails that $f$ fulfils (6) on $[0,1]$. Thus, we get the desired form (5) of $F$ on the whole interval $[a, b]$.

Theorem 9. Let $F: I^{2} \rightarrow I$ be a reflexive, partially strictly increasing, symmetric and bisymmetric mapping. Then $F$ is continuous.

Proof. If $I$ is compact then we have the statement from Theorem 8 , Otherwise, one can approximate $I$ by a sequence of compact subintervals of $I$ ?

As an immediate consequence of the previous theorem we get a more natural new characterization theorem of quasi-arithmetic means.

Corollary 10. A function $F: I^{2} \rightarrow I$ is a quasi-arithmetic mean if and only if it is reflexive, partially strictly increasing, symmetric and bisymmetric.

Proof. If $F$ is of the form (2), then it is trivially reflexive, partially strictly increasing, symmetric and bisymmetric.

The opposite direction comes from Theorem 9 ,

## 4. Further directions

Problems connecting to the bivariate case. The following Theorem can be found in [3, p. 296].
Theorem 11. Let $F: I^{2} \rightarrow I$ be a partially strictly monotonic and bisymmetric continuous mapping. Then
(1) there are constants $A, B, C \in \mathbb{R}, A B \neq 0$, and a continuous, strictly monotonic function $f: J \rightarrow I$ that satisfies

$$
\begin{equation*}
F(x, y)=f\left(A f^{-1}(x)+B f^{-1}(y)+C\right), \quad x, y \in I \tag{10}
\end{equation*}
$$

[^4](2) $F$ is reflexive if and only if there is a continuous, strictly monotonic $f: J \rightarrow I$ that satisfies
\[

$$
\begin{equation*}
F(x, y)=f\left(r f^{-1}(x)+(1-r) f^{-1}(y)\right), \quad x, y \in I, r \in \mathbb{R} \backslash\{0,1\} \tag{11}
\end{equation*}
$$

\] where $J \subset \mathbb{R}$ is a proper interval.

The proof of Theorem 11 is based on the reflexive, symmetric case of Theorem 1 and it relies on the fact that some functions are continuous such as $g_{a}(z)=F(F(a, z), F(z, a))$. This naturally motivates the following open problems.

Question 1. Is that true that every partially strictly monotonic and bisymmetric function $F: I^{2} \rightarrow I$ is automatically continuous, that satisfies (10)?

Question 2. Is that true that every reflexive, partially strictly monotonic and bisymmetric function $F: I^{2} \rightarrow I$ is automatically continuous, that satisfies (11)?

Problems connecting to the $n$-arity case. Analogously we can extend every property of $F$ introduced in Section 2 to $n$-ary functions. Hence we can talk about reflexivity, partially (strict) monotonicity, continuity of $G: I^{n} \rightarrow I$.

Symmetry is defined as

$$
G\left(x_{1}, \ldots, x_{n}\right)=G\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)
$$

holds for all $x_{1}, \ldots, x_{n}$ and $\sigma \in S_{n}$, where $S_{n}$ is the permutation group of $n$ elements.

Bisymmety is defined as

$$
\begin{aligned}
& G\left(G\left(x_{1,1}, \ldots, x_{1, n}\right), G\left(x_{2,1}, \ldots, x_{2, n}\right), \ldots, G\left(x_{n, 1}, \ldots, x_{n, n}\right)\right)= \\
& G\left(G\left(x_{1,1}, \ldots, x_{n, 1}\right), G\left(x_{1,2}, \ldots, x_{n, 2}\right), \ldots, G\left(x_{1, n}, \ldots, x_{n, n}\right)\right)
\end{aligned}
$$

for all $x_{i, j} \in[a, b]$. This property has a significant role in economics, especially, in the theory of aggregation functions (see e.g. [4, [5], [15], [19]).

The following general questions arise naturally as $n$-ary analogue.
Question 3. Let $G: I^{n} \rightarrow I$ be a partially strictly increasing and bisymmetric function. Is that true that $G$ is continuous.

This problem seems too general at this moment. Therefore we formalize the following direct analogue of our main result.

Question 4. Let $G: I^{n} \rightarrow I$ be a reflexive, partially strictly increasing, symmetric and bisymmetric function. Is that true that there is a proper
interval $J \subset \mathbb{R}$ and a continuous function $f: J \rightarrow I$, such that

$$
G\left(x_{1}, \ldots, x_{n}\right)=f\left(\frac{f^{-1}\left(x_{1}\right)+\cdots+f^{-1}\left(x_{n}\right)}{n}\right), \quad x_{1}, \ldots, x_{n} \in I
$$

hence $G$ is continuous?

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[^1]:    ${ }^{1}$ Associativity in the sense of Kolmogoroff (alternative names associativity with repetitions or decomposability see [15) is different from the classical associativity. A sequence of functions $\left\{F_{n}\right\}_{n \in \mathbb{N}}$, where $F_{n}: I^{n} \rightarrow \mathbb{R}$ is an $n$-place map, is associative in the sense of Kolmogoroff if $F_{n}\left(x_{1}, \ldots, x_{k}, x_{k+1}, \ldots, x_{n}\right)=$ $F_{n}\left(F_{k}\left(x_{1}, \ldots, x_{k}\right), \ldots, F_{k}\left(x_{1}, \ldots, x_{k}\right), x_{k+1}, \ldots, x_{n}\right)$ for every $k \in\{1, \ldots, n\}, n \in \mathbb{N}$ and $x_{1}, \ldots, x_{n} \in I$.

[^2]:    ${ }^{2} \mathrm{~A} \operatorname{map} F: I^{2} \rightarrow I$ is said to be cancellative if either $F(x, a)=F(y, a)$ or $F(a, x)=F(a, y)$ implies $x=y$ for every $x, y, a \in I$.

[^3]:    ${ }^{3}$ We note that there was also shown in [3] that if $F$ is continuous, then so is $f$. We do not assume continuity here.

[^4]:    ${ }^{4}$ For further details see [3] the end of the proof of Theorem 1. on page 290-291].

