ON A PROBLEM OF CHEN AND FANG RELATED TO INFINITE ADDITIVE COMPLEMENTS

SÁNDOR Z. KISS AND CSABA SÁNDOR

Abstract. Two infinite sets A and B of nonnegative integers are called additive complements if their sumset contains every nonnegative integer. In 1964, Danzer constructed infinite additive complements A and B with $A(x)B(x) = (1 + o(1))x$ as $x \to \infty$, where $A(x)$ and $B(x)$ denote the counting function of the sets A and B, respectively. In this paper we solve a problem of Chen and Fang by extending the construction of Danzer.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers and let A and B be infinite sets of nonnegative integers. We define their sum by $A+B = \{a+b : a \in A, b \in B\}$. We say A and B are infinite additive complements if their sum contains all nonnegative integers i.e., $A + B = \mathbb{N}$. Let $A(x)$ be the number of elements of A up to x i.e.,

$$A(x) = \sum_{\substack{a \in A \\ a \le x}} 1.$$

Since A and B are infinite additive complements, every nonnegative integer x can be written in the form $a + b = x$, where $a \in A$, $b \in B$. Then clearly [7] we have $A(x)B(x) \ge x + 1$, which implies that

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \ge \liminf_{x\to\infty} \frac{A(x)B(x)}{x} \ge 1.$$

According to a conjecture of H. Hanani [3], the above result can be sharpened in the following way.

Conjecture 1.1 (Hanani, 1957)**.** *If A and B are infinite additive complements, then*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 1.$$

Later, Danzer [2] disproved the above conjecture of Hanani.

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Theorem 1.2 (Danzer, 1964)**.** *There exist infinite additive complements A and B such that*

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1.$$

Let $A_1, \ldots, A_r$ be infinite sets of nonnegative integers. We define their sum by $A_1 + A_2 + \ldots + A_r = \{a_1 + a_2 + \ldots + a_r : a_i \in A_i, 1 \le i \le r\}$. Chen and Fang extended the notion of additive complements to more than two sets in the following way [1]. The infinite sets $A_1, \ldots, A_r$ of nonnegative integers are said to form infinite additive complements if their sum contains all nonnegative integers. Again, it is easy to see that $A_1(x) \cdots A_r(x) \ge (A_1 + \ldots + A_r)(x) = x + 1$, thus

$$\liminf_{x\to\infty} \frac{A_1(x) \cdots A_r(x)}{x} \ge 1.$$

Furthermore, they posed the following problem.

Problem 1.3. *For each integer $r \ge 3$ find additive complements $A_1, \ldots, A_r$ such that*

$$\lim_{x\to\infty} \frac{A_1(x) \cdots A_r(x)}{x} = 1.$$

In this paper we solve this problem. Note that our construction is the extension of Danzer's result to $r > 2$.

Theorem 1.4. *For each integer $h \ge 2$ there exist infinite sets of nonnegative integers $A_1, \ldots, A_h$ with the following properties:*

(1) $A_1 + \ldots + A_h = \mathbb{N}$,
(2) $A_1(x) \cdots A_h(x) = (1 + o(1))x$ *as* $x \to \infty$.

Let $R_{A+B}(n)$ be the number of representations of the integer n in the form $a + b = n$, where $a \in A$, $b \in B$. W. Narkiewicz [4] proved the following theorem.

Theorem 1.5 (Narkiewicz, 1960)**.** *If $R_{A+B}(n) \ge C$ for every sufficiently large integer n, where C is a constant and*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \le C,$$

then

$$\lim_{x\to\infty} \frac{A(2x)}{x} = 1,$$

or

$$\lim_{x\to\infty} \frac{B(2x)}{x} = 1.$$

Additive complements A, B are called exact if $A(x)B(x) = (1+o(1))x$ as $x \to \infty$. For any $h \geq 2$ integer let us define the system of sets $\mathcal{A}_h$ by

$$\mathcal{A}_h = \{A \subset \mathbb{N} : \text{there exist } A_2, \dots, A_h \subset \mathbb{N},$$
$$A + A_2 + \dots + A_h = \mathbb{N}, A(x) \cdot A_2(x) \cdots A_h(x) = (1+o(1))x \text{ as } x \to \infty\}.$$

Theorem 1.4 implies that $\mathcal{A}_h \neq \emptyset$ for every $h \geq 2$. We prove that the $\mathcal{A}_h$'s form an infinite chain.

Theorem 1.6. *We have $\mathcal{A}_2 \supseteq \mathcal{A}_3 \supseteq \dots$*

It follows from Theorem 1.6 that if $A \in \mathcal{A}_2$, then $A(x) = x^{o(1)}$ or $A(x) = x^{1+o(1)}$ as $x \to \infty$. Then for any $h \geq 2$, $A \in \mathcal{A}_h$ implies that $A(x) = x^{o(1)}$ or $A(x) = x^{1+o(1)}$ as $x \to \infty$. If the sets $A_1, \dots, A_h \subset \mathbb{N}$ satisfy $A_1 + \dots + A_h = \mathbb{N}$ and $A_1(x) \cdots A_h(x) = (1+o(1))x$ as $x \to \infty$, then $A_i(x) = x^{1+o(1)}$ or $A_i(x) = x^{o(1)}$ for every $1 \leq i \leq h$ while $x \to \infty$. As a corollary, one can get from Theorem 1.4 that

Corollary 1.7. *Let $A_1, \dots, A_h$ be infinite sets of nonnegative integers such that $A_1 + \dots + A_h = \mathbb{N}$ and*

$$A_1(x) \cdots A_h(x) = (1+o(1))x$$

as $x \to \infty$. Then there exists an index i such that $A_i(x) = x^{1+o(1)}$ and $A_j(x) = x^{o(1)}$ for every $1 \leq j \leq h$ with $j \neq i$ as $x \to \infty$.

We pose the following problems for further research.

Problem 1.8. *Does $\mathcal{A}_h \neq \mathcal{A}_{h+1}$ hold for every $h \geq 2$?*

Problem 1.9. *Assume that $A_1 + \dots + A_h = \mathbb{N}$ and $A_1(x) \cdots A_h(x) = (1+o(1))x$ hold as $x \to \infty$. Does there exist a permutation $i_1, \dots, i_h$ of the indices $1, \dots, h$ such that $A_{i_j}(x) = (A_{i_{j-1}}(x))^{o(1)}$ for every $2 \leq j \leq h$ as $x \to \infty$?*

The statement in Problem 1.9 holds for $h = 2$.

The exact complemets have been investigated by many authors in the last few decades. In particular, they studied what kind of sets A of nonnegative integers with $A(x) = x^{o(1)}$ as $x \to \infty$ have exact additive complement. It was proved in [2] that the sequence $a_n = (n!)^2 + 1$ has an exact complement. In [5] Ruzsa showed that the set of the powers of an integer $a \geq 3$ has an exact complement. Furthermore, in [6] he proved that the set of powers of 2 has an exact complement. Moreover, he also proved in [6] that $A = \{a_1, a_2, \dots\}$ with $1 \leq a_1 < a_2 < \dots$ has an exact complement if $\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = \infty$. In view of these results, it is natural to ask

Problem 1.10. *Is it true that if $A \in \mathcal{A}_2$, $A(x) = x^{o(1)}$ as $x \to \infty$, then $A(x) = O(\log x)$?*

2. Proof of Theorem 1.4

For any nonnegative integers $a < b$, let us define $[a, b] = \{x \in \mathbb{N} : a \le x \le b\}$. The following lemma plays the key role in the proof of Theorem 1.4.

Lemma 2.1. *Assume that $A_1, \dots, A_h \subset \mathbb{N}$ are infinite subsets with the following properties*

(1) $A_1 + \ldots + A_h = \mathbb{N}$,

(2) *there exists a monotone increasing arithmetic function $f_h(n) \ge 0$ with*

$$\lim_{n\to\infty} f_h(n) = \infty$$

such that the equation $a_1 + \ldots + a_h = n$, $a_i \in A_i$ has a solution with $a_i \ge f_h(n)$,

(3) $A_1(x) \cdots A_h(x) = (1 + o(1))x$ *as* $x \to \infty$.

For $m \in \mathbb{N}$, let $g(m)$ be an integral-valued strictly increasing function such that $g(f_h(n)) \ge n^2$ for every $n \in \mathbb{N}$. Put for shortness

$$\Phi_n = g(n + 1)! + h(g(n + 1) - 1)!,$$

$$\Delta_n = n - \lceil \sqrt{n} \rceil,$$

and for $n \ge 6$ let

$$M_n = [g(n)! - 2\Delta_n, \Phi_n].$$

Furthermore, for $1 \le i \le h$, let $B_i = \{0\} \cup \{g(a)! + a : a \in A_i\}$ and define the sets of integers

$$B_{h+1} = \{a : 0 \le a \le \Phi_5 - 1\} \cup \bigcup_{n\ge 6} \{\alpha \in M_n : \Delta_n \mid \alpha\}.$$

Then

(i) $B_1 + \ldots + B_{h+1} = \mathbb{N}$,

(ii) *there exists a monotone increasing arithmetic function $f_{h+1}(n) \ge 0$ with*

$$\lim_{n\to\infty} f_{h+1}(n) = \infty$$

such that the equation $b_1 + \ldots + b_{h+1} = n$, $b_i \in B_i$ has a solution with $b_i \ge f_{h+1}(n)$,

(iii) $B_1(x) \cdots B_{h+1}(x) = (1 + o(1))x$ *as* $x \to \infty$.

2.1. **Proof of the lemma.** Now we prove that for any $N \geq 6$,

$$B_1 + \ldots + B_h + \{\alpha \in M_N : \Delta_N \mid \alpha\} \supseteq [\Phi_{N-1} - 2\Delta_N + N, \Phi_N].$$

Consider an element from the interval on the right hand side i.e., let y be

$$\Phi_{N-1} - 2\Delta_N + N \leq y \leq \Phi_N.$$

It is clear that there exists an $\lceil \sqrt{N} \rceil \leq m \leq N-1$ with $y \equiv m \pmod{\Delta_N}$. By (2), there exist $a_1, \ldots, a_h$ integers with $a_i \in A_i$ such that $m = a_1 + \ldots + a_h$ and $a_i \geq f_h(m)$. Since $f_h(m)$ is a monotone increasing function and $g(m)$ is a strictly increasing function, we have

$$g(a_i) \geq g(f_h(m)) \geq g(f_h(\lceil \sqrt{N} \rceil)) \geq (\lceil \sqrt{N} \rceil)^2 \geq N$$

and so $g(a_i)! \equiv 0 \pmod{\Delta_N}$. Let $b_i = g(a_i)! + a_i$. Then $b_i \in B_i$ for every $1 \leq i \leq h$. It follows that

$$\sum_{i=1}^{h} b_i = \sum_{i=1}^{h} (g(a_i)! + a_i) \equiv \sum_{i=1}^{h} a_i \equiv m \equiv y \pmod{\Delta_N},$$

which implies that $\frac{y-(b_1+\ldots+b_h)}{\Delta_N}$ is an integer and clearly

$$y = b_1 + \ldots + b_h + \frac{y - (b_1 + \ldots + b_h)}{\Delta_N} \cdot \Delta_N.$$

In view of these facts, it is enough to show that

$$g(N)! - 2\Delta_N \leq y - (b_1 + \ldots + b_h) \leq \Phi_N.$$

Since $g(n)$ is a strictly increasing function, we have

$$0 \leq b_i = g(a_i)! + a_i \leq g(m)! + m \leq g(N-1)! + N - 1 < (g(N)-1)! + N$$

and so

$$0 \leq \sum_{i=1}^{h} b_i < h((g(N)-1)! + N).$$

It follows that

$$y - (b_1 + \ldots + b_h) \geq y - h((g(N)-1)! + N)$$

$$\geq g(N)! - 2(N - \lceil \sqrt{N} \rceil) + h((g(N)-1)! + N) - h(g(N)-1)! + N)$$

$$= g(N)! - 2\Delta_N$$

and

$$y - (b_1 + \ldots + b_h) \leq y \leq \Phi_N.$$

Thus for $N \geq 6$, we have

$$B_1 + \ldots + B_{h+1} \supseteq [\Phi_{N-1} - 2\Delta_N + N, \Phi_N] \supseteq [\Phi_{N-1}, \Phi_N].$$

This implies that

$$B_1 + \ldots + B_{h+1} \supseteq \bigcup_{N \ge 6} [\Phi_{N-1}, \Phi_N] = [\Phi_5, +\infty).$$

Moreover, for $1 \le i \le h$, $0 \in B_i$ and $B_{h+1} \supseteq [0, \Phi_5 - 1]$. Therefore,

$$[0, \Phi_5 - 1] \subseteq B_1 + \ldots + B_{h+1}$$

and so $B_1 + \ldots + B_{h+1} = \mathbb{N}$, which proves (i).

If $\Phi_{N-1} \le n \le \Phi_N$, then there exists a representation $n = b_1 + \ldots + b_{h+1}$, where $b_i = g(a_i)! + a_i \ge a_i \ge f_h(\lceil \sqrt{N} \rceil)$ and $b_{h+1} \ge g(N)! - 2\Delta_N \ge N! - 2\Delta_N$, which proves (ii) with a suitable function $f_{h+1}(n)$.

To prove (iii) we assume that $\Phi_{N-1} \le x \le \Phi_N$. Since $g(N)$ is strictly increasing, $g(N + 2h) \ge g(N + 1) + h$. This implies that

$$x \le \Phi_N = (g(N+1) + h)(g(N+1) - 1)!$$

$$\le g(N+2h)(g(N+1) - 1)! < g(N+2h)! + N + 2h$$

and

$$x \ge \Phi_{N-1} > g(N)! + h(N-1)! \ge g(N-1)! + N - 1.$$

Therefore, we have $A_i(N) \le B_i(x) \le A_i(N + 2h)$ for every $1 \le i \le h$. Thus we have, $B_i(x) = A_i(N) + O(1) = (1 + o(1))A_i(N)$ as $x \to \infty$ for every $1 \le i \le h$. Now, we have

$$B_1(x) \cdots B_h(x) = (1 + o(1))A_1(N) \cdots A_h(N) = (1 + o(1))N$$

as $x \to \infty$. It remains to prove that $B_{h+1}(x) = \frac{x}{N}(1 + o(1))$ as $x \to \infty$. It follows from the definition of B_{h+1} that for $x \ge \Phi_5$ we have

$$B_{h+1}(x) = \Phi_5 + \sum_{n=6}^{N-1} \left(\frac{\Phi_n}{\Delta_n} - \frac{g(n)!}{\Delta_n} + 3 \right) + \left\lfloor \frac{x}{\Delta_N} - \frac{g(N)!}{\Delta_N} + 3 \right\rfloor$$

$$= O(N) + \sum_{n=6}^{N-1} \left(\frac{\Phi_n}{\Delta_n} - \frac{g(n)!}{\Delta_n} \right) + \left(\frac{x}{\Delta_N} - \frac{g(N)!}{\Delta_N} \right).$$

By $x \ge \Phi_{N-1} \ge N!$, we have $O(N) = o\left(\frac{x}{N}\right)$ as $x \to \infty$. It follows from (2) in Lemma 2.1 that $n \ge f_h(n)$. Then by the definition of $g(n)$, we have

$$g(n) \ge g(f_h(n)) \ge n^2.$$

Applying this observation, a straightforward computation shows that

$$\frac{\Phi_n}{\Delta_n} - \frac{g(n)!}{\Delta_n} = \left(1 + O\left(\frac{1}{n^2}\right)\right) \cdot \frac{\Phi_n}{\Delta_n} = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \cdot \frac{g(n+1)!}{n+1}.$$

Hence,

$$\sum_{n=6}^{N-1} \frac{\Phi_n}{\Delta_n} - \frac{g(n)!}{\Delta_n} = \sum_{n=6}^{N-1} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \cdot \frac{g(n+1)!}{n+1}.$$

In the next step, we show that

$$\sum_{n=6}^{N-1}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\cdot\frac{g(n+1)!}{n+1}=(1+o(1))\cdot\frac{g(N)!}{N}$$

as $N\to\infty$. Since $g(m)$ is strictly increasing,

$$\frac{g(N+1)!}{g(N)!}\geq\frac{(g(N)+1)!}{g(N)!}=g(N)+1\geq N+1\geq\frac{N+1}{N},$$

which implies that $\frac{g(N)!}{N}$ is monotone increasing. By $g(m)\geq m^2$, we have

$$g(N-1)!\leq\frac{1}{N^2}g(N)!.$$

On the other hand,

$$\frac{g(N-1)!}{N-1}\leq\frac{g(N)!/N^2}{N-1}=O\left(\frac{g(N)!}{N^3}\right).$$

By using the above observations, we have

$$\begin{aligned}\sum_{n=6}^{N-1}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\cdot\frac{g(n+1)!}{n+1}&=\sum_{n=7}^{N-1}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\cdot\frac{g(n)!}{n}+\frac{g(N)!}{N}(1+o(1))\\&=\sum_{n=7}^{N-1}O\left(\frac{g(N-1)!}{N-1}\right)+(1+o(1))\frac{g(N)!}{N}\\&=O\left(N\frac{g(N)!}{N^3}\right)+\frac{g(N)!}{N}(1+o(1))=\frac{g(N)!}{N}(1+o(1))\end{aligned}$$

as $x\to\infty$. It is clear that

$$\begin{aligned}\frac{x}{\Delta_N}-\frac{g(N)!}{\Delta_N}&=\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\left(\frac{x-g(N)!}{N}\right)\\&=(1+o(1))\frac{x-g(N)!}{N}\end{aligned}$$

as $x\to\infty$. Then it follows that

$$B_{h+1}(x)=o\left(\frac{x}{N}\right)+(1+o(1))\frac{g(N)!}{N}+\frac{x-g(N)!}{N}(1+o(1))=(1+o(1))\frac{x}{N}$$

as $x\to\infty$, which proves (iii). The proof of Lemma 2.1 is completed.

2.2. **Proof of Theorem 1.4.** Now, we prove Theorem 1.4 by induction on h. We show that there exist infinite sets $A_1,\dots,A_h\subset\mathbb{N}$ with the following properties:

(1) $A_1+\ldots+A_h=\mathbb{N}$,
(2) there exists a monotone increasing arithmetic function $f_h(n)\geq 0$ with

$$\lim_{n\to\infty}f_h(n)=\infty$$

such that the equation $a_1+\ldots+a_h=n$, $a_i\in A_i$ has a solution with $a_i\geq f_h(n)$,
(3) $A_1(x)\cdots A_h(x)=(1+o(1))x$ as $x\to\infty$.

For $h = 1$ consider the set of natural numbers and the function $f_1(n) = n$, which gives the result. Assume that the statement of Theorem 1.4 holds for h. For $h + 1$ the result follows from Lemma 2.1. (Actually, for $h = 2$ our construction is the same as the construction of Danzer [2]). The proof of Theorem 1.4 is completed.

3. Proof of Theorem 1.6

Let $h \geq 2$. We will prove that $\mathcal{A}_{h+1} \subseteq \mathcal{A}_h$. Let $A \in \mathcal{A}_{h+1}$. Then there exist $A_2, \ldots, A_{h+1} \subseteq \mathbb{N}$ such that $A + A_2 + \ldots + A_{h+1} = \mathbb{N}$ and $A(x)A_2(x)\cdots A_{h+1}(x) = (1 + o(1))x$ as $x \to \infty$. Let $A_h^* = A_h + A_{h+1}$. It is clear that $A_h^*(x) \leq A_h(x) \cdot A_{h+1}(x)$. Then we have

$$A + A_2 + \ldots + A_{h-1} + A_h^* = \mathbb{N}$$

and so $A(x)A_2(x)\cdots A_{h-1}(x)A_h^*(x) \geq x + 1$. On the other hand,

$$A(x)A_2(x)\cdots A_{h-1}(x)A_h^*(x) \leq A(x)A_2(x)\cdots A_{h+1}(x) = (1 + o(1))x$$

as $x \to \infty$, thus we have

$$A(x)A_2(x)\cdots A_{h-1}(x)A_h^*(x) = (1 + o(1))x$$

as $x \to \infty$, which implies that $A \in \mathcal{A}_h$. The proof of Theorem 1.6 is completed.

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Institute of Mathematics, Budapest University of Technology and Economics, Egry József utca 1, 1111 Budapest, Hungary
Email address: ksandor@math.bme.hu

Institute of Mathematics, Budapest University of Technology and Economics, Egry József utca 1, 1111 Budapest, Hungary
Email address: csandor@math.bme.hu