Generalized asymptotic Sidon basis

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Abstract

Let $h, k \geq 2$ be integers. We say a set A of positive integers is an asymptotic basis of order k if every large enough positive integer can be represented as the sum of k terms from A. A set of positive integers A is called a $B_h[g]$ set if every positive integer can be represented as the sum of h terms from A in at most g different ways. In this paper we prove the existence of $B_h[1]$ sets which are asymptotic bases of order $2h+1$ by using probabilistic methods.

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1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. Let $h, k \geq 2$ be integers. Let $A \subset \mathbb{N}$ be an infinite set and let $R_{h,A}(n)$ denote the number of solutions of the equation

$$a_1 + a_2 + \cdots + a_h = n, \quad a_1 \in A, \ldots, a_h \in A, \quad a_1 \leq a_2 \leq \ldots \leq a_h, \tag{1}$$

where $n \in \mathbb{N}$. A set of positive integers A is called a $B_h[g]$ set if for every $n \in \mathbb{N}$, the number of representations of n as the sum of h terms in the form (1) is at most g, that is $R_{h,A}(n) \leq g$. We denote the fact that A is a $B_h[g]$ set by $A \in B_h[g]$. We say a set $A \subset \mathbb{N}$ is an asymptotic basis of order k if there exists a positive integer n_0 such that $R_{k,A}(n) > 0$ for $n > n_0$. In [4] and [5], P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set (or $B_2[1]$ set) which is an asymptotic basis of order 3. It is easy to see that a Sidon set cannot be an asymptotic basis of order 2. J. M. Deshouillers and A. Plagne in [3] constructed a Sidon set which is an asymptotic basis of order at most 7. In [7] it was proved the existence of Sidon sets which are asymptotic bases of order

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5 by using probabilistic methods. In [1] and [9] this result was improved on by proving the existence of a Sidon set which is an asymptotic basis of order 4. It was also proved [1] that there exists a $B_2[2]$ set which is an asymptotic basis of order 3. In this paper we will prove a general theorem that concerns $B_h[1]$-sets for any $h \geq 2$ instead of just $B_2[g]$ sets. Namely, we prove the existence of an asymptotic basis of order $2h+1$ which is also a $B_h[1]$ set.

Theorem 1. *For every integer $h \geq 2$ there exists a $B_h[1]$ set which is an asymptotic basis of order $2h+1$.*

Before we prove the above theorem, we propose some open problems for further research. In general, for $k, h \geq 2$ integers, one can investigate the existence of an asymptotic basis of order k which is a $B_h[1]$ set.

Problem 1. *Determine the smallest value of $k = k(h)$ for which there exists an asymptotic basis of order k which is a $B_h[1]$ set.*

It is easy to see that there does not exist a $B_h[1]$ set which is an asymptotic basis of order k with $k < h$ because it does not have enough elements. On the other hand, if $k = h$, the generalization of the well known conjecture of Erdős and Turán asserts that there does not exist a $B_h[g]$ set which is an asymptotic basis of order h. This conjecture is still unsolved even for $h = 2$. In this paper, we prove the case $k = 2h+1$ by using probabilistic methods. More advanced probabilistic tools may help to handle the case $k = 2h$. To prove the existence of a $B_h[1]$ set which is an asymptotic basis of order $2h-1$ seems hopeless.

It is natural to ask whether there exist a $B_h[g]$ set which is an asymptotic basis of order $h+1$ for some $g = g(h)$. For $3 \leq h < k$, it was proved [8] the existence of a $B_h[g]$ set which is an asymptotic basis of order k. In [8], the order of magnitude of $g = g(h)$ was not controlled. This motivates the study of the following problem.

Problem 2. *Determine the smallest value of $g = g(h)$ for which there exists an asymptotic basis of order $h+1$ which is a $B_h[g]$ set.*

In the following section we give a short survey of the probabilistic method we will use.

2 Probabilistic tools

To prove Theorem 1 we use the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in the book of Halberstam and Roth [6]. In this paper we denote the probability measure by $\mathbb{P}$, and the expectation of a random variable Y by $\mathbb{E}(Y)$. Let Ω denote the set of the strictly increasing sequences of positive integers.

Lemma 1. *Let*

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

be real numbers satisfying

$$0 \leq \alpha_n \leq 1 \quad (n = 1, 2, \ldots).$$

Then there exists a probability space (Ω, X, $\mathbb{P}$) with the following two properties:

(i) For every natural number n, the event $E^{(n)} = \{A\colon\ A \in \Omega,\ n \in A\}$ is measurable, and $\mathbb{P}(E^{(n)}) = \alpha_n$.

(ii) The events $E^{(1)}$, $E^{(2)}, \dots$ are independent.

See Theorem 13 in [6], p. 142. We denote the characteristic function of the event $E^{(n)}$ by $\varrho(A, n)$:

$$\varrho(A,n) = \begin{cases} 1, \text{ if } n \in A \\ 0, \text{ if } n \notin A. \end{cases}$$

Furthermore, for some $A = \{a_1, a_2, \dots\} \in \Omega$ we denote the number of solutions of

$$a_{i_1} + a_{i_2} + \ldots + a_{i_h} = n$$

with

$$a_{i_1} \in A, a_{i_2} \in A, \dots, a_{i_h} \in A,\ \ 1 \le a_{i_1} < a_{i_2} < \cdots < a_{i_h} < n$$

by $r_h(n)$. Then

$$r_{h,A}(n) = r_h(n) = \sum_{\substack{(a_1,a_2,\dots,a_h)\in\mathbb{N}^h \\ 1\le a_1<\ldots<a_h<n \\ a_1+a_2+\ldots+a_h=n}} \varrho(A,a_1)\varrho(A,a_2)\ldots\varrho(A,a_h). \tag{2}$$

Let $R_h^*(n)$ denote the number of those representations of n in the form (1) in which there are at least two equal terms. Thus we have

$$R_{h,A}(n) = r_h(n) + R_h^*(n). \tag{3}$$

In the proof of Theorem 1 we use the following lemma:

Lemma 2. *(Borel-Cantelli) Let $X_1, X_2, \dots$ be a sequence of events in a probability space. If*

$$\sum_{j=1}^{+\infty} \mathbb{P}(X_j) < \infty,$$

then with probability 1, at most a finite number of the events X_j can occur.

See [6], p. 135.

3 Proof of Theorem 1

Let h be fixed and let $\alpha = \frac{2}{4h+1}$. Define the sequence α_n in Lemma 1 by

$$\alpha_n = \frac{1}{n^{1-\alpha}},$$

so that $\mathbb{P}(\{A\colon\ A \in \Omega,\ n \in A\}) = \frac{1}{n^{1-\alpha}}$. The proof of Theorem 1 consists of three parts. In the first part we prove similarly as in [8] that with probability 1, A is an asymptotic basis of order $2h+1$. In particular, we show that $R_{2h+1,A}(n)$ tends to infinity as n goes to infinity. In the second part we show that by deleting finitely many elements from A

we can obtain a $B_h[1]$ set. Finally, we show that the above deletion does not destroy the asymptotic basis property.

By (3), to prove that A is an asymptotic basis of order $2h+1$ it is enough to show $r_{2h+1,A}(n) > 0$ for every n large enough. To do this, we apply the following lemma with $k = 2h+1$.

Lemma 3. *Let $k \geq 2$ be a fixed integer and let $\mathbb{P}(\{A\colon\ A \in \Omega,\ n \in A\}) = \frac{1}{n^{1-\alpha}}$ where $\alpha > \frac{1}{k}$. Then with probability 1, $r_{k,A}(n) > cn^{k\alpha-1}$ for every sufficiently large n, where $c = c(\alpha, k)$ is a positive constant.*

The proof of Lemma 3 can be found in [8]. It is clear from (3) that

$$\mathbb{P}(\mathcal{E}) = 1, \tag{4}$$

where $\mathcal{E}$ denotes the event

$$\mathcal{E} = \{A : A \in \Omega, \exists n_0(A) = n_0 \ such\ that\ R_{2h+1,A}(n) \geq cn^{\frac{1}{4h+1}}\ for\ n > n_0\},$$

where c is a suitable positive constant. In the next step we prove that removing finitely elements from A we get a $B_h[1]$ set with probability 1. To do this, it is enough to show that with probability 1, $R_{h,A}(n) \leq 1$ for every n large enough. Note that in a representation of n as the sum of h terms there can be equal summands. To handle this situation we consider the terms of a representation $a_1 + \ldots + a_h = n$ as a vector $(a_1, \ldots, a_h) \in \mathbb{N}^h$. We denote the set which elements are the coordinates of the vector $\bar{x}$ as $Set(\bar{x})$. Of course, if two or more coordinates of $\bar{x}$ are equal, this value appears only once in $Set(\bar{x})$. We say that two vectors $\bar{x}$ and $\bar{y}$ are disjoint if $Set(\bar{x})$ and $Set(\bar{y})$ are disjoint sets. We define $r^*_{l,A}(n)$ as the maximum number of pairwise disjoint representations of n as sum of l (not necessarily distinct) elements of A, i.e., the maximum number of pairwise disjoint vectors of $R_l(n)$ with their coordinates in A. We say that A is a $B^*_l[g]$ sequence if $r^*_{l,A}(n) \leq g$ for every n.

Lemma 4. *Suppose $\mathbb{P}(\{A{:}A \in \Omega,\ n \in A\}) = \frac{1}{n^{1-\alpha}}$, where $\alpha = \frac{2}{4h+1}$.*

(i) For every $2 \leq k \leq h$, almost always there exists a finite set A_k such that
$r^*_{k,A\setminus A_k}(n) \leq 1$.

(ii) For every $h+1 \leq k \leq 2h$, almost always there exists a finite set A_k such that
$r^*_{k,A\setminus A_k}(n) \leq 4h+1$.

Proof. We need the following proposition (see Lemma 3.7 in [2]).

Proposition 1. *For a sequence $A \in \Omega$, for every $k \geq 2$ and $n \geq 1$,*

$$\mathbb{P}(r^*_{k,A}(n) \geq s) \leq C_{k,\alpha,s}\ n^{(k\alpha-1)s}$$

, where $C_{k,\alpha,s}$ depends only on k, α and s.

We apply Proposition 1 with $s = 2$. Then we have

$$\mathbb{P}(r^*_{k,A}(n) \geq 2) \leq C_{k,\alpha}\ n^{2(k\alpha-1)} = C_{k,\alpha}\ n^{-\frac{8h-4k+2}{4h+1}}.$$

Since $2 \leq k \leq h$, we have

$$\mathbb{P}(r^*_{k,A}(n) \geq 2) \leq C_{k,\alpha}\ n^{-\frac{4h+2}{4h+1}}$$

then by the Borel-Cantelli lemma we get that almost always there exists an n_k such that $r^*_{k,A}(n) \leq 1$ for $n \geq n_k$. It follows that

$$r^*_{k,A\setminus A_k}(n) \leq 1,$$

where $A_k = A \cap [0, n_k]$.
Assume that $h < k \leq 2h$. We apply Proposition 1 with $s = 4h + 2$. Then we have

$$\mathbb{P}(r^*_{k,A}(n) \geq 4h+2) \leq C_{k,h,\alpha}\ n^{(4h+2)(k\alpha-1)} = C_{k,\alpha}\ n^{-(2h+1)\frac{8h-4k+2}{4h+1}}.$$

Since $h < k \leq 2h$, we have

$$\mathbb{P}(r^*_{k,A}(n) \geq 4h+2) \leq C_{k,\alpha}\ n^{-\frac{4h+2}{4h+1}}$$

then by the Borel-Cantelli lemma we get that almost always there exists an n_k such that $r^*_{k,A}(n) \leq 4h+1$ for $n \geq n_k$. It follows that

$$r^*_{k,A\setminus A_k}(n) \leq 4h+1,$$

where $A_k = A \cap [0, n_k]$. □

It follows from (4) and Lemma 4 that there exists a set A and for every $2 \leq k \leq h$ finite sets $A_k \subset A$ such that

$$R_{2h+1,A}(n) \geq cn^{\frac{1}{4h+1}} \tag{5}$$

for $n \geq n_0$ and for every $2 \leq k \leq h$,

$$r^*_{k,A\setminus A_k}(n) \leq 1, \tag{6}$$

and for every $h < k \leq 2h$,

$$r^*_{k,A\setminus A_k}(n) \leq 4h+1. \tag{7}$$

Set $B = A \setminus \cup_{k=1}^{2h} A_k$. In the next step we show that B is both a $B_h[1]$ set and a $B_{2h}[g]$ set for some g. We apply the following proposition (see Remark 3.10 in [2]).

Proposition 2. *For integers $s \geq 2$ and $g \geq 1$,*

$$B^*_s[g] \cap B_{s-1}[\ell] \subseteq B_s[g(s(\ell-1)+1)].$$

By using the definition of B, the fact that $B^*_2[1] = B_2[1]$ and (6), (7) it follows that

$$B \in B_2[1] \cap B^*_3[1] \cap \ldots \cap B^*_h[1] \cap B^*_{h+1}[4h+1] \cap \ldots \cap B^*_{2h}[4h+1].$$

Applying Proposition 2 with $g = \ell = 1$ we get by induction that for every $2 \leq s \leq h$ if $B \in B^*_s[1] \cap B_{s-1}[1]$ then $B \in B_s[1]$, thus B is a $B_h[1]$ set. Applying Proposition

2 with $g=4h+1$, $\ell=1$ we get that $B\in B_{h+1}[4h+1]$. Using Proposition 2 again with $g=4h+1$, $\ell=4h+1$ we get that if $B\in B^*_{h+2}[4h+1]\cap B_{h+1}[4h+1]$ then $B\in B_{h+2}[(4h+1)(4h(h+2)+1)]$. Continuing this process, we obtain that for $1<k\le h$, we have $B\in B_k[1]$ and for $h+1\le k\le 2h$, $B\in B_k[g_k]$ for some positive integer g_k.

Let $\cup_{k=1}^{2h}A_k=\{d_1,\dots,d_w\}$ $(d_1<\dots<d_w)$. Now we show that $A\in B_{2h}[G]$ where

$$G=2^w\cdot\max_{1<k\le 2h}g_k.$$

We prove by contradiction. Assume that there exists a positive integer n with $R_{2h,A}(n)>2^w\cdot\max_{1<k\le 2h}g_k$. Then there exist indices $1\le i_1<i_2<\dots<i_j\le w$ such that the number of representations of n in the form $n=d_{i_1}+\dots+d_{i_j}+c_{j+1}+\dots+c_{2h}$, where $c_{j+1},\dots,c_{2h}\in B$ is more than $\max_{1<k\le 2h}g_k$. It follows that

$$R_{2h-j,B}(n-(d_{i_1}+\dots+d_{i_j}))>\max_{1<k\le 2h}g_k\ge g_{2h-j}$$

which is a contradiction since $B\in B_{2h-j}[g_{2h-j}]$.

Finally, we prove similarly as in [7] that B is an asymptotic basis of order $2h+1$, i.e., the deletion of $\cup_{k=1}^{2h}A_k$ from A does not destroy its asymptotic basis property. We prove by contradiction. Assume that there exist infinitely many positive integers M which cannot be represented as the sum of $2h+1$ numbers from B. Choose such an M large enough. In view of (5), we have $R_{2h+1,A}(M)>cM^{\frac{1}{4h+1}}$. It follows from our assumption that every representation of M as the sum of $2h+1$ numbers from A contains at least one element from $A\setminus B=\cup_{k=1}^{2h}A_k$. Then by the pigeon hole principle there exists a $y\in\cup_{k=1}^{2h}A_k$ which is in at least $\frac{R_{2h+1,A}(M)}{w}$ representations of M. As $A\in B_{2h}[G]$, it follows that with probability 1,

$$\frac{cM^{\frac{1}{4h+1}}}{w}<\frac{R_{2h+1,A}(M)}{w}\le R_{2h,A}(M-y)\le G,$$

which is a contradiction if M is large enough. We conclude that B is an asymptotic basis of order $2h+1$, and a $B_h[1]$ set. This completes the proof of Theorem 1.

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