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FROZEN PERCOLATION ON THE BINARY TREE IS NONENDOGENOUS

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In frozen percolation, i.i.d. uniformly distributed activation times are assigned to the edges of a graph. At its assigned time, an edge opens provided neither of its endvertices is part of an infinite open cluster; in the opposite case, it freezes. Aldous (2000) showed that such a process can be constructed on the infinite 3-regular tree and asked whether the event that a given edge freezes is a measurable function of the activation times assigned to all edges. We give a negative answer to this question, or, using an equivalent formulation and terminology introduced by Aldous and Bandyopadhyay (2005), we show that the recursive tree process associated with frozen percolation on the oriented binary tree is nonendogenous. An essential role in our proofs is played by a frozen percolation process on a continuous-time binary Galton Watson tree that has nice scale invariant properties.

1. Introduction.

1.1. *Frozen percolation on the 3-regular tree.* Let (T, E) be a regular tree where each vertex has degree 3, and let $\mathcal{U} = (U_e)_{e\in E}$ be an i.i.d. collection of uniformly distributed $[0, 1]$-valued random variables, indexed by the edges of the tree. We write $E_t := \{e \in E : U_e \le t\}$ $(t \in [0, 1])$. Aldous [Ald00] has proved the following theorem.

THEOREM 1 (Frozen percolation on the 3-regular tree). *It is possible to couple $\mathcal{U}$ to a random subset $F \subset E$ with the following properties:*

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(i) *$e \notin F$ if and only if no endvertex of e is part of an infinite cluster of $E_{U_e} \backslash (F \cup \{e\})$.*
(ii) *The law of $(\mathcal{U}, F)$ is invariant under automorphisms of the tree.*

At time $t \in [0,1]$, we call edges in $E_t \backslash F$ *open*, edges in $E_t \cap F$ *frozen*, and all other edges *closed*. Then property (i) can be described in word as follows. Initially all edges are closed. At its *activation time* U_e, the edge e opens provided neither of its endvertices is at that moment part of an infinite open cluster; in the opposite case, it freezes.

It is not known if properties (i) and (ii) uniquely determine the joint law of $(\mathcal{U}, F)$. However, it is possible to obtain an object that is unique in law by adding one natural additional property. To formulate this, we view T as an oriented graph $(T, \vec{E})$ where $\vec{E} := \big\{(v,w),(w,v) : \{v,w\} \in E\big\}$ contains two oriented edges for every unoriented edge in E. A *ray* is an infinite sequence of oriented edges $(v_n, w_n)_{n\geq 0}$ such that $v_n = w_{n-1}$ and $w_n \neq v_{n-1}$ $(n \geq 1)$. We let

$$\begin{aligned}\vec{E}_{(v,w)} := \big\{(v',w') : \exists \text{ a ray } (v_n,w_n)_{n\geq 0} \text{ and } m \geq 0 \\ \text{s.t. } (v_0,w_0) = (v,w) \text{ and } (v_m,w_m) = (v',w')\big\}\end{aligned} \tag{1.1}$$

denote the union of all rays that start with (v,w), and we let $E_{(v,w)} := \big\{\{v',w'\} : (v',w') \in \vec{E}_{(v,w)}\big\}$ denote the associated set of unoriented edges. For each subset S of T, we let

$$\partial S := \big\{(v,w) \in \vec{E} : v \in S,\ w \in T \backslash S\big\} \tag{1.2}$$

denote the collection of oriented edges pointing out of S, and we let $E_S := \big\{\{v,w\} \in E : v \in S \text{ and } w \in S\big\}$ denote the set of edges induced by S. We say that S is a *subtree* if its induced subgraph (S, E_S) is a tree.

Let $\mathcal{U} = (U_e)_{e \in E}$ be as before and let $\vec{E}_t := \{(v,w) \in \vec{E} : U_{\{v,w\}} \leq t\}$ $(t \in [0,1])$. The existence part of the following theorem was proved in [Ald00], but the uniqueness part is new.

THEOREM 2 (Frozen percolation on the oriented 3-regular tree). *It is possible to couple $\mathcal{U}$ to a random subset $\vec{F} \subset \vec{E}$ with the following properties:*

(i) *$(v,w) \in \vec{F}$ if and only if there exists a ray $(v_n,w_n)_{n\geq 0}$ with $(v_0,w_0) = (v,w)$ and $(v_n,w_n) \in \vec{E}_{U_{\{v,w\}}} \backslash \vec{F}$ for all $n \geq 1$.*
(ii) *The law of $(\mathcal{U}, \vec{F})$ is invariant under automorphisms of the tree.*
(iii) *Let $\mathcal{U}_{(v,w)} := (U_e)_{e \in E_{(v,w)}}$ and $\vec{F}_{(v,w)} := \vec{F} \cap \vec{E}_{(v,w)}$. Then, for each finite subtree $S \subset T$, the random variables $(\mathcal{U}_{(v,w)}, \vec{F}_{(v,w)})_{(v,w) \in \partial S}$ are independent of each other and of $(\mathcal{U}_e)_{e \in E_S}$.*

These properies uniquely determine the joint law of $(\mathcal{U},\vec{F})$. *Moreover, setting* $F := \big\{\{v,w\} \in E : (v,w) \in \vec{F}$ *or* $(w,v) \in \vec{F}\big\}$ *defines a pair* $(\mathcal{U},F)$ *with properties (i) and (ii) of Theorem 1.*

In this paper, our main interest is not in uniqueness in law but rather in almost sure uniqueness. In [Ald00, Section 5.7], Aldous asked whether the set F of frozen edges is measurable w.r.t. the σ-field generated by $\mathcal{U}$, and cautiously conjectured that this might indeed be the case. In [AB05, Thm 55], an apparent proof of this conjecture by Bandyopadhyay was announced that appeared on the arXiv [Ban04] but turned out to contain an error. In the last posted update of [Ban04] from 2006, Bandyopadhyay reported on numerical simulations (similar to those shown in Figure 3 below) that suggested nonuniqueness, and from this moment on this seems to have been the generally held belief. We finally settle the issue by proving this.

THEOREM 3 (Frozen percolation is not almost sure unique). *Let* $(\mathcal{U},F)$ *be the pair defined in Theorem 2 and let* F' *be a copy of* F, *conditionally independent of* F *given* $\mathcal{U}$. *Then* $F \neq F'$ *a.s. In particular, the random variable* F *is not measurable w.r.t. the* σ*-field generated by* $\mathcal{U}$.

The proofs of Theorems 2 and 3 will be given in Subsection 3.7.

1.2. *Frozen percolation on the oriented binary tree.* For a given oriented edge $(v,w) \in \vec{E}$ of the 3-regular tree, the set $\vec{E}_{(v,w)}$ of oriented edges that lie on rays starting with (v,w) can naturally be labeled with the space $\mathbb{T}$ of all finite words $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 0)$ made up from the alphabet $\{1,2\}$. We call $|\mathbf{i}| := n$ the length of the word $\mathbf{i}$ and denote the word of length zero by $\varnothing$, which we distinguish notationally from the empty set $\emptyset$. The concatenation of two words $\mathbf{i} = i_1 \cdots i_n$ and $\mathbf{j} = j_1 \cdots j_m$ is denoted by $\mathbf{ij} := i_1 \cdots i_n j_1 \cdots j_m$.

Apart from using $\mathbb{T}$ to label oriented edges as above, we can also interpret $\mathbb{T}$ as labeling the vertices of a binary tree with root $\varnothing$, in which each vertex $\mathbf{i}$ has two descendants $\mathbf{i}1, \mathbf{i}2$ and each vertex $\mathbf{i} = i_1 \cdots i_n$ $(n \geq 1)$ except the root has a unique predecessor $\overleftarrow{\mathbf{i}} := i_1 \cdots i_{n-1}$. By definition, a *ray* starting at $\mathbf{i}$ is a sequence $(\mathbf{i}_n)_{n\geq 0}$ such that $\mathbf{i}_0 = \mathbf{i}$ and $\overleftarrow{\mathbf{i}}_n = \mathbf{i}_{n-1}$ $(n \geq 1)$. For any $A \subset \mathbb{T}$ and $\mathbf{i} \in \mathbb{T}$, we write $\mathbf{i} \xrightarrow{A} \infty$ if there exists a ray $(\mathbf{i}_n)_{n\geq 0}$ with $\mathbf{i}_0 = \mathbf{i}$ and $\mathbf{i}_n \in A$ $(n \geq 0)$.

We write $\mathbf{i} \prec \mathbf{j}$ if $\mathbf{j} = \mathbf{ik}$ for some $\mathbf{k} \in \mathbb{T}$. By definition, a *rooted subtree* of $\mathbb{T}$ is a set $\mathbb{U} \subset \mathbb{T}$ with the property that $\mathbf{i} \prec \mathbf{j} \in \mathbb{U}$ implies $\mathbf{i} \in \mathbb{U}$. For each nonempty rooted subtree $\mathbb{U}$ of $\mathbb{T}$, we let $\partial\mathbb{U} := \{\mathbf{i} \in \mathbb{T}\backslash\mathbb{U} : \overleftarrow{\mathbf{i}} \in \mathbb{U}\}$ denote the *boundary* of $\mathbb{U}$ relative to $\mathbb{T}$, and we use the convention that $\partial\mathbb{U} = \{\varnothing\}$ if $\mathbb{U} = \emptyset$.

Let $\tau = (\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an i.i.d. collection of uniformly distributed $[0,1]$-valued random variables. In the picture where elements of $\mathbb{T}$ label oriented edges in $\vec{E}_{(v,w)}$, this corresponds to the collection of activation times $(U_e)_{e\in E_{(v,w)}}$. Using the same picture, let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be a collection of real random variables, which correspond to the first time when there is an infinite open ray of edges starting with a given oriented edge, with $X_{\mathbf{i}} := \infty$ if this never happens. Note that $X_{\mathbf{i}}$ takes values in $I := [0,1]\cup\{\infty\}$. By properties (ii) and (iii) of Theorem 2, for each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$,

(1.3) the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. and independent of $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

Using also property (i), it is easy to see that the random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ satisfy the inductive relation (compare [AB05, formula (65)])

$$X_{\mathbf{i}} = \gamma[\tau_{\mathbf{i}}](X_{\mathbf{i}1}, X_{\mathbf{i}2}) \qquad (\mathbf{i}\in\mathbb{T}), \tag{1.4}$$

where $\gamma : [0,1]\times I^2 \to I$ is defined as

$$\gamma[t](x,y) := \begin{cases} x\wedge y & \text{if } x\wedge y > t, \\ \infty & \text{otherwise.} \end{cases} \tag{1.5}$$

Generalising from the set-up of Theorem 2, we will more generally be interested in collections of random variables $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ such that $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. uniformly distributed on $[0,1]$ and (1.3) and (1.4) hold. As will be explained in the next subsection, in the terminology of [AB05], such a collection forms a *Recursive Tree Process* (RTP). The theory of RTPs provides us with a convenient general framework to reformulate and prove Theorems 2 and 3.

1.3. *Recursive Tree Processes.* Roughly speaking, a *Recursive Tree Process* (RTP) is a stationary Markov chain in which time has a tree-like structure and flows in the direction of the root. The state at each node of the tree is a function of the states of its descendants and i.i.d. randomness attached to the nodes. Following [AB05], we call an RTP *endogenous* if the state at the root is measurable w.r.t. the σ-field generated by the i.i.d. randomness attached to the nodes. It has been shown in [AB05, Thm 11] that endogeny is equivalent to *bivariate uniqueness*. We first explain these concepts in a general setting and then specialise to frozen percolation.

Slightly generalising our previous notation, let $\mathbb{T}$ denote the space of all finite words $\mathbf{i} = i_1\cdots i_n$ $(n\geq 0)$ made up from the alphabet $\{1,\ldots,d\}$, where $d\geq 1$ is some fixed integer. All previous notation involving the binary tree generalizes in a straightforward manner to the d-ary tree $\mathbb{T}$. Let I and Ω be Polish spaces, let $\gamma : \Omega \times I^d \to I$ be a measurable function, and let $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be i.i.d. Ω-valued random variables with common law $\mathbf{p}$. Let ν be a probability law on I that solves the *Recursive Distributional Equation* (RDE)

$$X_\varnothing \stackrel{\mathrm{d}}{=} \gamma[\omega_\varnothing](X_1, \ldots, X_d), \tag{1.6}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, $X_\varnothing$ has law ν, and $X_1, \ldots, X_d$ are copies of $X_\varnothing$, independent of each other and of $\omega_\varnothing$. A simple argument based on Kolmogorov's extension theorem (see [MSS20, Lemma 8]) tells us that the i.i.d. random variables $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ can be coupled to I-valued random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ in such a way that:

(i) For each finite rooted subtree $\mathbb{U} \subset \mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are i.i.d. with common law ν and independent of $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}}$.

(ii) $X_{\mathbf{i}} = \gamma[\omega_{\mathbf{i}}](X_{\mathbf{i}1}, \ldots, X_{\mathbf{i}d}) \qquad (\mathbf{i} \in \mathbb{T})$.

Moreover, these conditions uniquely determine the joint law of $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. We call the latter the *Recursive Tree Process* (RTP) corresponding to the maps γ and solution ν of the RDE (1.6). By definition, the RTP $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is *endogenous* if the random variable $X_\varnothing$ is measurable w.r.t. the σ-field generated by the random variables $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. It has been shown in [AB05, Thm 11] that this is equivalent to *bivariate uniqueness*, as we now explain.

Let $\mathcal{P}(I)$ denote the space of all probability measures on I. We can define a map $T : \mathcal{P}(I) \to \mathcal{P}(I)$ by

$$T(\mu) := \text{ the law of } \gamma[\omega_\varnothing](X_1, \ldots, X_d), \tag{1.7}$$

where $X_1, \ldots, X_d$ are i.i.d. with law μ and independent of $\omega_\varnothing$. In particular, solutions to the RDE (1.6) correspond to fixed points of T. Similarly, we can define a *bivariate map* $T^{(2)} : \mathcal{P}(I^2) \to \mathcal{P}(I^2)$ by

$$T^{(2)}(\mu^{(2)}) := \text{ the law of } \big(\gamma[\omega_\varnothing](X_1, \ldots, X_d), \gamma[\omega_\varnothing](X_1^*, \ldots, X_d^*)\big), \tag{1.8}$$

where $(X_1, X_1^*), \ldots, (X_d, X_d^*)$ are i.i.d. with common law $\mu^{(2)}$ and independent of $\omega_\varnothing$. A trivial way to construct a fixed point of $T^{(2)}$ is to set

$$\overline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X_\varnothing) \in \cdot\,\big] \tag{1.9}$$

where the law ν of $X_\varnothing$ is a fixed point of T. We will refer to $\overline{\nu}^{(2)}$ as the *trivial* fixed point or as the *diagonal* fixed point of $T^{(2)}$ with marginal distribution ν. A more interesting way to construct a fixed point of $T^{(2)}$ goes as follows. Let $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to the map γ and a fixed point ν

of T, and let $(X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be a copy of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, conditionally independent given $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. It follows from [MSS18, Lemma 2 and Prop 4] that

$$\underline{\nu}^{(2)} := \mathbb{P}\big[(X_\varnothing, X'_\varnothing) \in \cdot\,\big] \tag{1.10}$$

is also a fixed point of $T^{(2)}$. Let us denote by $(T^{(2)})^n$ the n-th iterate of the bivariate map $T^{(2)}$. By [MSS18, Lemma 2 and Prop. 3], one has

$$(T^{(2)})^n(\nu\otimes\nu) \underset{n\to\infty}{\Longrightarrow} \underline{\nu}^{(2)}. \tag{1.11}$$

The following theorem links endogeny to bivariate uniqueness. The essential idea goes back to [AB05, Thm 11]. In its present form, it follows from [MSS18, Thms 1 and 5 and Lemma 14]. Below, $\mathcal{P}(I^2)_\nu$ denotes the space of all probability measures on I^2 whose one-dimensional marginals are given by ν. Note that condition (ii) below and formula (1.11) suggest a method to numerically investigate whether an RTP is endogenous, compare Figure 3 below.

THEOREM 4 (Endogeny and bivariate uniqueness). *Let $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to a map γ and a solution ν of the corresponding RDE (1.6). Then the following statements are equivalent:*

(i) *The RTP $(\omega_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is endogenous.*
(ii) $\underline{\nu}^{(2)} = \overline{\nu}^{(2)}$.
(iii) *The bivariate map $T^{(2)}$ has a unique fixed point in $\mathcal{P}(I^2)_\nu$.*
(iv) $(T^{(2)})^n(\mu^{(2)}) \underset{n\to\infty}{\Longrightarrow} \overline{\nu}^{(2)}$ *for all* $\mu^{(2)} \in \mathcal{P}(I^2)_\nu$.

Note that since we know that $\underline{\nu}^{(2)}$ and $\overline{\nu}^{(2)}$ are fixed points, the implications (iv)⇒(iii)⇒(ii) are trivial. The implication (ii)⇒(i) follows from our characterisation of $\underline{\nu}^{(2)}$ in (1.10), so the essential claim is that (i) implies (iv).

1.4. *Nonendogeny.* Specialising from the general set-up of the previous subsection, we set $d := 2$ and as our i.i.d. randomness $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ we use an i.i.d. collection $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ of uniformly distributed $[0,1]$-valued random variables. We set $I := [0,1]\cup\{\infty\}$, and choose for $\gamma : \Omega\times I^2 \to I$ the map defined in (1.5). Using these objects, we define a map $T : \mathcal{P}(I) \to \mathcal{P}(I)$ as in (1.7). The associated RDE $T(\mu) = \mu$ then takes the form (compare (1.6))

$$X_\varnothing \overset{\mathrm{d}}{=} \gamma[\tau_\varnothing](X_1, X_2), \tag{1.12}$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution, $X_\varnothing$ has law μ, and X_1, X_2 are copies of $X_\varnothing$, independent of each other and of $\tau_\varnothing$. Solutions to the RDE

(1.12) are not unique. We will describe all solutions of (1.12) in Lemma 33 and Proposition 37 below.

Let $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an RTP corresponding to the map γ in (1.5) and an arbitrary solution to the RDE (1.12). We set

$$\mathbb{T}^t := \big\{\mathbf{i}\in\mathbb{T} : \tau_{\mathbf{i}} \leq t\big\} \quad \text{and} \quad \mathbb{F} := \big\{\mathbf{i}\in\mathbb{T} : \tau_{\mathbf{i}} \geq X_{\mathbf{i}1}\wedge X_{\mathbf{i}2}\big\}, \tag{1.13}$$

and define I-valued random variables $(X^{\uparrow}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ by

$$X^{\uparrow}_{\mathbf{i}} := \inf\big\{t\in[0,1] : \mathbf{i} \xrightarrow{\mathbb{T}^t\backslash\mathbb{F}} \infty\big\}, \tag{1.14}$$

with $X^{\uparrow}_{\mathbf{i}} := \infty$ if the set on the right-hand side is empty. In line with our interpretation where elements of $\mathbb{T}$ represent oriented edges in $\vec{E}_{(v,w)}$ (with (v,w) fixed), we say that at time $t\in[0,1]$, points in $\mathbb{T}^t\backslash\mathbb{F}$ are *open*, points in $\mathbb{T}^t\cap\mathbb{F}$ are *frozen*, and all other points in $\mathbb{T}$ are *closed*. We call $\tau_{\mathbf{i}}$ the *activation time* of $\mathbf{i}$ and refer to $X_{\mathbf{i}}$ and $X^{\uparrow}_{\mathbf{i}}$ as its *burning time* and *percolation time*, respectively. Note that since the subset of $[0,1]$ on the right-hand side of (1.14) is a.s. closed (in the topological sense), $\mathbf{i}$ percolates at time t if and only if $X^{\uparrow}_{\mathbf{i}} \leq t$. Formula (1.13) says that initially, all points $\mathbf{i}\in\mathbb{T}$ are closed. At its activation time $\tau_{\mathbf{i}}$, the point $\mathbf{i}$ freezes if at that moment one of its descendants is burnt, and opens otherwise.

It follows from the inductive relation (1.4) that $X_{\mathbf{i}} > \tau_{\mathbf{i}}$ a.s., i.e., a point $\mathbf{i}$ can only burn after its activation time. Comparing the definition of $\mathbb{F}$ in (1.13) with the definition of the map γ in (1.5), we observe that if $\mathbf{i}$ burns at some time $X_{\mathbf{i}}\in[0,1]$, then $\mathbf{i}$ must be open at that time. Moreover, by (1.5), if $\mathbf{i}$ burns at some time $X_{\mathbf{i}}\in[0,1]$, then there must be a ray starting at $\mathbf{i}$ of points that burn at the same time as $\mathbf{i}$. In particular, such a ray must be open, which proves that

$$X^{\uparrow}_{\mathbf{i}} \leq X_{\mathbf{i}} \quad \text{a.s.} \quad (\mathbf{i}\in\mathbb{T}). \tag{1.15}$$

We will prove Theorem 2 by showing that the opposite inequality holds a.s. if and only if $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is the RTP corresponding to one particular solution of the RDE (1.12). This solution is described by the following lemma, which we cite from [Ald00, Lemma 3]. Note that (1.16) below implies that $\nu(\{\infty\}) = \frac{1}{2}$.

LEMMA 5 (Special solution to the RDE). *Let ν denote the probability measure on I defined by*

$$\nu\big((t,1]\cup\{\infty\}\big) := 1\wedge\frac{1}{2t} \qquad \big(t\in[0,1]\big). \tag{1.16}$$

Then ν solves the RDE (1.12).

We will deduce Theorem 2 from the following, more precise theorem. Aldous proved the "if" part of the statement below in [Ald00], but the "only if" part is new. Theorem 6 is proved in Subsection 3.5.

THEOREM 6 (Frozen percolation on the oriented binary tree). *Consider an RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map γ in (1.5) and an arbitrary solution μ to the RDE (1.12). Let $(X^{\uparrow}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be defined as in (1.14). Then one has $X^{\uparrow}_{\varnothing} = X_{\varnothing}$ a.s. if and only if $\mu = \nu$, the measure defined in (1.16).*

Using the language of RTPs, we can formulate our main result as follows. Theorem 3 will follow from the theorem below in a straightforward manner using methods from [Ald00]. Theorem 7 is proved in Subsection 3.2.

THEOREM 7 (Frozen percolation on the binary tree is nonendogenous). *The RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map γ from (1.5) and the law ν from (1.16) is nonendogenous.*

Because of Theorem 4 (iii), to prove Theorem 7, it suffices to show that the bivariate RDE has, apart from the trivial solution $\overline{\nu}^{(2)}$, at least one more solution in the space $\mathcal{P}(I^2)_\nu$. We will explicitly identify such a solution in formula (3.86) below, so in principle the proof of Theorem 7 can be completed by checking directly that the measure in (3.86) solves the bivariate RDE and has the right marginals.

We will follow a somewhat less direct approach, which will yield the additional information that the measure in (3.86) is indeed $\underline{\nu}^{(2)}$. Perhaps more importantly, our approach explains why our formula (3.86) has the special form that it has. First, we change the problem, replacing frozen percolation on the oriented binary tree by frozen percolation on a continuum tree that we will call the *Marked Binary Branching Tree* (MBBT), and replacing the inductive relation (1.4) by (1.29). The advantage of the MBBT is that it enjoys a nice scaling property, detailed in Proposition 9 and Lemma 11 below, that will significantly simplify our analysis. Once we have found the nontrivial solution $\underline{\rho}^{(2)}$ to the bivariate RDE for frozen percolation on the MBBT, a simple trick, explained in Subsection 3.5, also allows us to identify the measure $\underline{\nu}^{(2)}$ that we were originally interested in.

1.5. *The Marked Binary Branching Tree.* Roughly speaking, the *marked binary branching tree* is the family tree of a continuous-time, rate one binary branching process, equipped with a marked Poisson point process of intensity one and uniformly distributed $[0,1]$-valued marks. We now introduce this object more formally.

Let $(A_h)_{h\geq 0}$ be a continuous-time branching process, started with a single particle, where each particle splits into two new particles with rate one. We view A_h as an evolving set. In particular, the cardinality $|A_h|$ is a Markov process in $\mathbb{N}$ that jumps from a to $a+1$ with rate a, and $A_0 = \{x_0\}$ is a set containing a single element x_0. In the next subsection, we will make a more explicit choice for the labels of elements of A_h. We choose $(A_h)_{h\geq 0}$ to be right-continuous and let $(A_{h-})_{h\geq 0}$ denote its left-continuous modification.

For each pair of times $g, h \geq 0$ and individuals $x \in A_g$, $y \in A_h$ that are alive at these times, let $\tau(x, y)$ denote the last time in $[0, g \wedge h]$ that a common ancestor existed of x and y, and let

$$d\big((x,g),(y,h)\big) := g + h - 2\tau(x,y) \tag{1.17}$$

denote their genetic distance. Then the random set

$$\mathcal{T} := \big\{(x,h) : x \in A_{h-},\ h \geq 0\big\} \tag{1.18}$$

equipped with the metric (1.17) is a random continuum tree. In pictures, we draw x horizontally and h vertically, and from now on, we refer to h as the *height*, rather than time, of a point $(x, h) = z \in \mathcal{T}$. We call $\varnothing := (x_0, 0)$ the *root* of $\mathcal{T}$.

Conditional on $\mathcal{T}$, we let Π_0 be a Poisson point set on $\mathcal{T}$ whose intensity measure is the length measure on $\mathcal{T}$, and conditional on $\mathcal{T}$ and Π_0, we let $(\tau_z)_{z\in\Pi_0}$ be i.i.d. uniformly distributed $[0, 1]$-valued marks. We think of $z = (x, h) \in \Pi_0$ as a hole on $\mathcal{T}$ that disappears (i.e., gets filled in) at time τ_z. We observe that

$$\Pi = \big\{(z, \tau_z) : z \in \Pi_0\big\} \tag{1.19}$$

is a Poisson set of intensity one on $\mathcal{T} \times [0, 1]$ and that Π_0 as well as the marks $(\tau_z)_{z\in\Pi_0}$ can be read off from Π. For lack of better name, we call the pair $(\mathcal{T}, \Pi)$ the *Marked Binary Branching Tree* (MBBT). See Figure 1 for an illustration.

We set

$$\Pi_t := \big\{z \in \Pi_0 : \tau_z > t\big\} \quad (t \in [0, 1]). \tag{1.20}$$

Intuitively, Π_t is the set of holes on $\mathcal{T}$ which are still present at time t. For any set $A \subset \mathcal{T}$ and point $z \in \mathcal{T}$, we write $\varnothing \xrightarrow{\mathcal{T}\backslash A} z$ if $\varnothing$ and z are connected in $\mathcal{T}\backslash A$ and we write $z \xrightarrow{\mathcal{T}\backslash A} \infty$ if there exists an infinite, continuous, upward path through $\mathcal{T}\backslash A$. We start with a simple observation. Note that below, in contrast with our earlier notation E_t, points in Π_t play the role of points that can *not* be passed at time t. The proof of the following lemma can be found in Subsection 3.6.

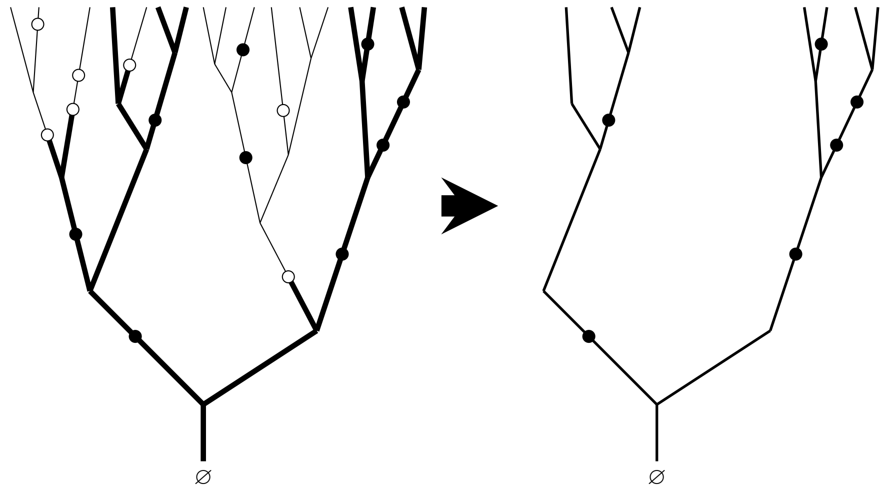

FIG 1. *Scaling of the marked binary branching tree. On the left: at time t, points in Π_t are still closed and marked with white circles, while points in $\Pi_0 \backslash \Pi_t$ have already opened and are marked with black circles. On the right: removing the loose ends from the open cluster at the root yields a stretched version of the original marked binary branching tree.*

LEMMA 8 (Oriented percolation on the marked binary branching tree). *One has*

$$\mathbb{P}\big[\varnothing \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty\big] = t \quad (0 \le t \le 1). \tag{1.21}$$

Indeed, if we cut $\mathcal{T}$ at points of Π_t, then the remaining connected component of the root is the family tree of a branching process where particles split into two with rate one and die with rate $1-t$. It is an elementary exercise in branching theory to show that the survival probability of such a branching process is t. The fact that the survival probability is a linear function of t reflects a scaling property of the marked binary branching tree that will be important in our analysis. Below, we view $(\mathcal{T}, \Pi)$ as a marked metric space, i.e., we consider two marked trees to be equal if one can be mapped onto the other by an isometry that preserves the marks. The following result is proved in Subsection 3.6.

PROPOSITION 9 (Scaling). *Let $(\mathcal{T}, \Pi)$ be the marked binary branching tree. Fix $0 < t < 1$ and define*

$$\mathcal{T}' := \big\{z \in \mathcal{T} : \varnothing \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} z \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty\big\}, \quad \Pi' := \big\{(z, \tau_z) \in \Pi : z \in \mathcal{T}'\big\}. \tag{1.22}$$

Then the probability that $\mathcal{T}' \neq \emptyset$ is t and conditional on this event, the pair $(\mathcal{T}', \Pi')$, viewed as a marked metric space, is equally distributed with the stretched marked binary branching tree $(\mathcal{T}'', \Pi'')$ defined as

$$\mathcal{T}'' := \big\{(x, t^{-1}h) : (x,h) \in \mathcal{T}\big\},\ \Pi'' := \big\{(x, t^{-1}h, t\tau_{(x,h)}) : (x,h,\tau_{(x,h)}) \in \Pi\big\}. \tag{1.23}$$

In words, this says that if we cut off all parts of $\mathcal{T}$ that lie above points $z \in \Pi_t$, then remove the loose ends of the tree, and condition on the event that the remaining tree is nonempty, then we end up with the family tree of a branching process where particles split into two with rate t, equipped with a marked Poisson point set with intensity t and i.i.d. uniformly distributed on $[0,t]$-valued marks. See Figure 1 for an illustration.

1.6. *Frozen percolation on the MBBT.* In the previous subsection, we have been deliberately vague about the labeling of elements of the evolving set-valued branching process $(A_h)_{h\geq 0}$. We now make an explicit choice, which naturally leads to an RTP that is closely related to, but different from the one introduced in Subsection 1.4.

We will construct the branching process $(A_h)_{h\geq 0}$ in such a way that $A_0 = \{\varnothing\}$ and $A_h \subset \mathbb{T}$ for all $h \geq 0$. (Note that by a slight abuse of notation, $\varnothing$ now denotes both the root of the discrete tree $\mathbb{T}$ and of the continuum tree $\mathcal{T}$, the latter being defined as $\varnothing = (\varnothing, 0)$.) Each element $\mathbf{i} \in A_h$ branches with rate one into two new elements labeled $\mathbf{i}1$ and $\mathbf{i}2$. In addition, we arrange things in such a way that each element $\mathbf{i} \in A_h$ is with rate one replaced by a new element labeled $\mathbf{i}1$. The idea of this is to encode the Poisson point set Π_0 from the MBBT in terms of the labels of elements of A_h, in such a way that Π_0 is given by the collection of points $(\mathbf{i}, h)$ for which $\mathbf{i}$ is at time h replaced by $\mathbf{i}1$.

We will give an explicit construction of the MBBT based on three collections of i.i.d. random variables:

(i) $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. uniformly distributed on $[0,1]$,
(ii) $(\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. uniformly distributed on $\{1,2\}$,
(iii) $(\ell_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. exponentially distributed with mean $1/2$.

We interpret $\ell_{\mathbf{i}}$ as the lifetime of the individual $\mathbf{i}$ and let

$$b_{i_1\cdots i_n} := \sum_{k=0}^{n-1} \ell_{i_1\cdots i_k} \quad \text{and} \quad d_{i_1\cdots i_n} := \sum_{k=0}^{n} \ell_{i_1\cdots i_k} \tag{1.24}$$

with $b_\varnothing := 0$ and $d_\varnothing := \ell_\varnothing$ denote the birth and death times of $\mathbf{i}$. The random variable κ_i indicates what happens with the individual $\mathbf{i}$ at the end of its lifetime. If $\kappa_{\mathbf{i}} = 1$, then it is replaced by a single new individual with label $\mathbf{i}1$, and if $\kappa_{\mathbf{i}} = 2$, then it is replaced by two new individuals with labels $\mathbf{i}1$ and $\mathbf{i}2$. In line with this, we let $\mathbb{S}$ denote the random subtree of $\mathbb{T}$ defined by

$$\mathbb{S} := \big\{ i_1 \cdots i_n \in \mathbb{T} : i_m \leq \kappa_{i_1 \cdots i_{m-1}} \ \forall 1 \leq m \leq n \big\}, \tag{1.25}$$

which is the collection of all individuals that will ever be born. Recall that $\partial\mathbb{U}$ denotes the boundary of a rooted subtree $\mathbb{U} \subset \mathbb{T}$ relative to $\mathbb{T}$. Likewise, for any rooted subtree $\mathbb{U} \subset \mathbb{S}$ we let $\nabla\mathbb{U} := \partial\mathbb{U} \cap \mathbb{S}$ denote the *boundary* of $\mathbb{U}$ relative to $\mathbb{S}$.

For $h \geq 0$, we let

$$\begin{aligned} \mathbb{T}_h &:= \big\{\mathbf{i} \in \mathbb{T} : d_{\mathbf{i}} \leq h\big\}, \qquad & \partial\mathbb{T}_h &= \big\{\mathbf{i} \in \mathbb{T} : b_{\mathbf{i}} \leq h < d_{\mathbf{i}}\big\}, \\ \mathbb{S}_h &:= \mathbb{T}_h \cap \mathbb{S}, & \nabla\mathbb{S}_h &= \partial\mathbb{T}_h \cap \mathbb{S} \end{aligned} \tag{1.26}$$

denote the sets of individuals that have died by time h and those that are alive at time h, respectively. Note that the former are a.s. finite rooted subtrees of $\mathbb{T}$ and $\mathbb{S}$, respectively, and the latter are their boundaries. Then

$$(\nabla\mathbb{S}_h)_{h\geq 0} = (A_h)_{h\geq 0} \tag{1.27}$$

gives an explicit construction of the branching process $(A_h)_{h\geq 0}$ we have earlier described in words. Defining $\mathcal{T}$ as in (1.18) and setting

$$\Pi := \big\{(\mathbf{i}, d_{\mathbf{i}}, \tau_{\mathbf{i}}) : \mathbf{i} \in \mathbb{S},\ \kappa_{\mathbf{i}} = 1\big\} \tag{1.28}$$

yields an explicit construction of the MBBT $(\mathcal{T}, \Pi)$ based on i.i.d. randomness.

Instead of giving a description of oriented frozen percolation on $(\mathcal{T}, \Pi)$ similar to Theorem 6, we immediately jump to the corresponding RTP for the percolation times. Letting $Y_{\mathbf{i}}$ denote the first time when there is an infinite upwards open path in frozen percolation on $(\mathcal{T}, \Pi)$ starting from the point $(\mathbf{i}, b_{\mathbf{i}})$, it is not hard to see that $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}}$ must satisfy the inductive relation

$$Y_{\mathbf{i}} = \chi[\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}](Y_{\mathbf{i}1}, Y_{\mathbf{i}2}), \tag{1.29}$$

where $\chi : [0,1] \times \{1,2\} \times I^2 \to I$ is the function

$$\chi[\tau, \kappa](x, y) := \begin{cases} x & \text{if } \kappa = 1,\ x > \tau, \\ \infty & \text{if } \kappa = 1,\ x \leq \tau, \\ x \wedge y & \text{if } \kappa = 2. \end{cases} \tag{1.30}$$

Note that in (1.29), $Y_{\mathbf{i}}$ is a priori only defined for $\mathbf{i} \in \mathbb{S}$. The definition of $\mathbb{S}$ in (1.25) is such, however, that in cases when $\mathbf{i} \in \mathbb{S}$ but $\mathbf{i}2 \notin \mathbb{S}$, the value of $Y_{\mathbf{i}2}$ is irrelevant for the outcome of the function χ. The subtree $\mathbb{S}$ plays an important role in the theory of continuous-time RTPs, see [MSS20, Sect. 1.4].

Like in the case of the oriented binary tree (as discussed in Subsection 1.4) it is possible to go the other way, i.e., starting from a solution to the RDE corresponding to the map χ, one can construct an RTP $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ where now $Y_{\mathbf{i}}$ is defined for all $\mathbf{i} \in \mathbb{T}$, and then restrict to $\mathbb{S}$ to construct oriented frozen percolation on the MBBT. In the present setting, it turns out that the "right" solution to the corresponding RDE is given by the following lemma. Since we will later (in Subsection 3.5 below) see that frozen percolation on the MBBT and on the oriented binary tree can be mapped into each other, we will at this moment not explain why in the present setting, Lemma 10 describes the "right" solution to the RDE.

LEMMA 10 (Special solution to the RDE). *Let ρ denote the probability measure on I defined by*

$$\rho\big([0,t]\big) := \tfrac{1}{2}t \qquad \big(t \in [0,1]\big), \qquad \rho(\{\infty\}) := \tfrac{1}{2}. \tag{1.31}$$

Then ρ solves the RDE

$$Y_\varnothing \stackrel{\mathrm{d}}{=} \chi[\tau_\varnothing, \kappa_\varnothing](Y_1, Y_2), \tag{1.32}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, $Y_\varnothing$ has law ρ, and Y_1, Y_2 are copies of $Y_\varnothing$, independent of each other and of $\tau_\varnothing, \kappa_\varnothing$.

PROOF. Let Y_1, Y_2 be i.i.d. with law ρ, let τ and κ be independent r.v.'s that are uniformly distributed on $[0,1]$ and $\{1,2\}$, respectively, and define $Y_\varnothing := \chi[\tau,\kappa](Y_1,Y_2)$, where χ is defined in (1.30). We claim that $Y_\varnothing$ has law ρ. Indeed, for each $t \in [0,1]$, we have

$$\begin{aligned}
\mathbb{P}[Y_\varnothing \le t] &= \tfrac{1}{2}\int_0^1 \mathbb{P}[\chi[s,1](Y_1,Y_2) \le t]\,\mathrm{d}s + \tfrac{1}{2}\mathbb{P}[Y_1 \wedge Y_2 \le t] \\
&= \tfrac{1}{2}\int_0^1 \mathbb{P}[s \le Y_1 \le t]\,\mathrm{d}s + \tfrac{1}{2}\big(1 - \mathbb{P}[Y_1 > t]^2\big) \\
&= \tfrac{1}{2}\int_0^t \big(\tfrac{1}{2}t - \tfrac{1}{2}s\big)\,\mathrm{d}s + \tfrac{1}{2}\big(1 - (1 - \tfrac{1}{2}t)^2\big) = \tfrac{1}{2}t.
\end{aligned} \tag{1.33}$$

$\square$

We will prove that the RTP $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}},Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map χ from (1.30) and law ρ from (1.31) is nonendogenous. We apply Theorem 4. We will explicitly identify the special solutions $\underline{\rho}^{(2)}$ and $\overline{\rho}^{(2)}$ to the bivariate RDE and show that they are not equal.

It is clear from the definitions of $\underline{\rho}^{(2)}$ and $\overline{\rho}^{(2)}$ in (1.9) and (1.10) that both measures are symmetric under a permutation of the two coordinates and that their one-dimensional marginals equal ρ. The main advantage of working with the MBBT is that as a result of the scaling property described in Proposition 9, the measures $\underline{\rho}^{(2)}$ and $\overline{\rho}^{(2)}$ are also scale invariant. We let $\mathcal{P}_*(I^2)_\rho$ denote the space of symmetric measures $\mu^{(2)}$ on I^2 whose one-dimensional marginals are given by ρ and that are moreover scale invariant in the sense that

$$\mu^{(2)}\big([0,tr]\times[0,ts]\big)=t\mu^{(2)}\big([0,r]\times[0,s]\big)\qquad\big(r,s,t\in[0,1]\big). \tag{1.34}$$

The following lemma is proved in Subsection 3.6.

Lemma 11 (Scale invariance). *One has* $\underline{\rho}^{(2)},\overline{\rho}^{(2)}\in\mathcal{P}_*(I^2)_\rho$.

By Theorem 4, to show that the RTP $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}},Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is nonendogenous, it suffices to show that apart from the trivial fixed point $\overline{\rho}^{(2)}$, the bivariate map $T^{(2)}$ has at least one other fixed point in $\mathcal{P}(I^2)_\rho$. The following theorem identifies all fixed points in $\mathcal{P}_*(I^2)_\rho$. Since there are precisely two of them, by Lemma 11 we conclude that the nontrivial fixed point is $\underline{\rho}^{(2)}$. The following theorem is proved in Subsection 2.1.

Theorem 12 (Nonendogeny). *The bivariate map $T^{(2)}$ associated with the map χ from (1.30) has precisely two fixed points $\rho^{(2)}_1,\rho^{(2)}_2$ in $\mathcal{P}_*(I^2)_\rho$. For each $c\geq 0$, let $f_c:[0,1]\to[0,1]$ denote the continuous function given by the unique solution to the Cauchy problem*

$$\tfrac{\partial}{\partial r}f_c(r)=\frac{cr}{f_c(r)-r/2},\quad 0\leq r<1,\qquad f_c(0)=\tfrac12. \tag{1.35}$$

The equation

$$f_c(1)^2-\tfrac12 f_c(1)=2c \tag{1.36}$$

is solved for precisely two values of c in $[0,\infty)$. Denoting these by c_1 and c_2 with $c_1<c_2$, we have $c_1=0$ and $c_2\in(0,1/4)$. The measures $\rho^{(2)}_i$ $(i=1,2)$ are uniquely characterised by

$$\rho^{(2)}_i\big(\{[0,r]\times I\}\cup\{I\times[0,s]\}\big)=(s\vee r)f_{c_i}\Big(\frac{r\wedge s}{r\vee s}\Big),\quad\big((r,s)\in[0,1]^2\backslash\{(0,0)\}\big). \tag{1.37}$$

One has $\rho_1^{(2)} = \overline{\rho}^{(2)}$, the trivial fixed point defined as in (1.9).

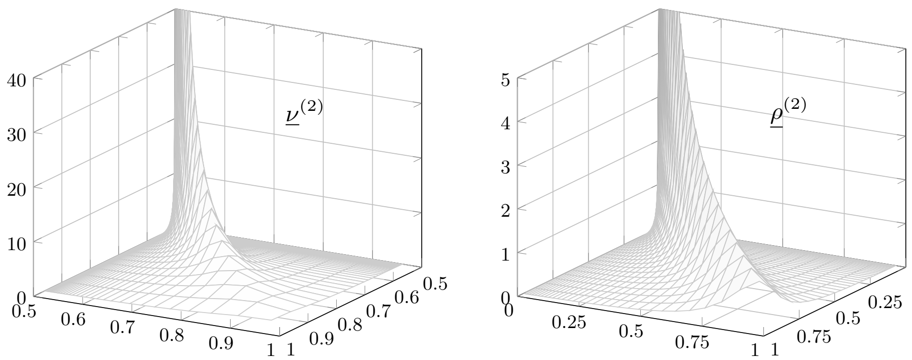

FIG 2. *The nontrivial solutions $\underline{\nu}^{(2)}$ and $\underline{\rho}^{(2)}$ of the bivariate RDE for frozen percolation on the oriented binary tree and the MBBT, respectively. Plotted are the densities of the restrictions of the measures to $[\frac{1}{2}, 1]^2$ and $[0, 1]^2$, respectively.*

Numerically, we find $c_2 \approx 0.01770838$. The function f_{c_2} is increasing and convex with $f_{c_2}(0) = \frac{1}{2}$ and $f_{c_2}(1) \approx 0.5629165415$. Lemma 11 allows us to identify $\rho_2^{(2)}$ as $\underline{\rho}^{(2)}$, the nontrivial fixed point defined in (1.10). As a result of Theorem 12, we also have an explicit expression for the nontrivial solution $\underline{\nu}^{(2)}$ to the bivariate RDE for frozen percolation on the oriented binary tree, see formula (3.86) below. Numerical data for $\underline{\nu}^{(2)}$ and $\underline{\rho}^{(2)}$ are plotted in Figure 2.

1.7. *Discussion.*

Frozen percolation on finite graphs. Let $G = (V, E)$ be a finite graph. Let $(U_e)_{e \in E}$ be i.i.d. uniformly distributed on $[0, 1]$ and let $(\Lambda_v)_{v \in V}$ be an independent i.i.d. collection of exponentially distributed random variables with mean λ^{-1}. Now consider a process where edges and vertices can be in two possible states: edges are either *closed* or *open*, and vertices are either *available* or *frozen*. Initially, all edges are closed and all vertices are available. The evolution is as follows:

(i) At time U_e, the edge e becomes open, provided neither of its endvertices is frozen.
(ii) At time Λ_v, all vertices of the open component containing v become frozen.

We call such a process *frozen percolation* on the finite graph G, and by a certain analogy with forest fire models, we call λ the *lightning rate*.

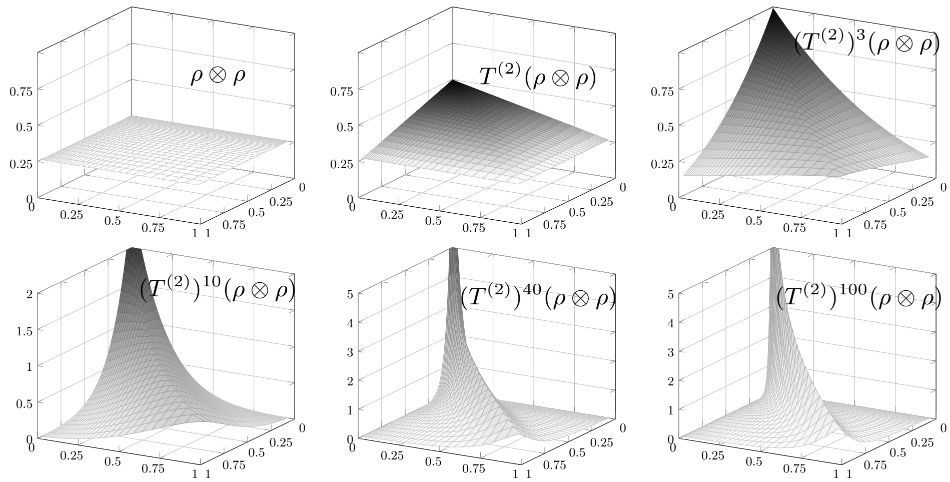

FIG 3. *Iterating the bivariate map $T^{(2)}$ on the product measure $\rho \otimes \rho$ produces a series of measures that by (1.11) converge to the nontrivial fixed point $\underline{\rho}^{(2)}$. Plotted is the density of the restriction of $(T^{(2)})^n(\rho \otimes \rho)$ to the unit square for $n = 0, 1, 3, 10, 40$, and 100. The last plot is already very close to the theoretical limit.*

One is typically interested in the limit when G is large. Let us therefore consider a sequence $G_n = (V_n, E_n)$ of finite graphs with $|V_n| = n$ vertices and with lightning rates λ_n, and make two assumptions:

(A1) The graphs G_n converge to a weak local limit G in the sense of Benjamini and Schramm.
(A2) $n^{-1} \ll \lambda_n \ll 1$ as $n \to \infty$.

We recall that a sequence of graphs converge to a weak local limit if the neighbourhood of a typical (uniformly chosen) vertex converges in law to a (possibly random) rooted graph; see [BS01] or [Hof17b, Section 1.4]. Assumption (A2) guarantees that in the limit, small open clusters with size of order one never freeze, but giant components that occupy a positive fraction of all vertices freeze immediately.

We can think of this as a model for polymerisation, where open components represent polymers that grow through merger with neighbours. Polymers that grow too large become part of the "gel" and are unable to grow any further. In the model we have just described, this is guaranteed by the lightning process, which has certain mathematical advantages. However, one can also think about alternative models where polymers are, e.g., prevented from growing when they reach a certain deterministic size.

If p_c is the critical value for percolation in the local limit graph G from assumption (A1), then up to time p_c, open clusters grow as in normal percolation. Since beyond this time, large clusters are prevented from growing further, one can expect the model to exhibit *self-organised criticality* (SOC) in the sense of [Bak96, Jen98], i.e., in the whole time regime beyond time p_c we can expect phenomena that are usually associated with the behaviour of large systems at their critical point. Statements of this form have indeed been proved. With the model described above in mind, we will give a short overview of the literature and mention some open problems.

Frozen percolation on the complete graph. Although historically not the oldest, frozen percolation on the complete graph is one of the most natural models to consider. Since in this case, the degree of each vertex is n, it is more natural to take the $(U_e)_{e\in E}$ to be uniformly distributed on $[0,n]$ instead of $[0,1]$. The complete graph does not have a weak local limit, but one can take the local limit of the combined object consisting of the complete graph and the edge activation times U_e. The resulting limiting object is called the Poisson Weighted Infinite Tree (PWIT) [AS04, Sect 4.2].

Following a suggestion in [Ald00, Sect. 5.5], one of us has studied frozen percolation on the complete graph. In [Rat09], it was shown that the fraction of clusters of sizes $k\in\mathbb{N}$ at time t converges to a solution of Smoluchowski's equations with multiplicative kernel, an infinite system of differential equation that serves as a deterministic model of polymerisation, and that is known to exhibit self-organised criticality (SOC).

The closely related forest fire model of [RT09] is further studied in [CFT15, CRY18, Cra18]. In [CRY18] it is shown that the asymptotic distribution of a typical cluster is that of a critical multi-type Galton-Watson tree after gelation.

Aldous [Ald00, Sect. 5.5] in fact suggested to study the variant of the mean field frozen percolation model where clusters are frozen when their size exceeds a deterministic threshold $1\ll\alpha(n)\ll n$. This model is studied in in [MN14]. Their Theorem 1.3 states that at any time $t\geq 1$, the limiting distribution of a typical non-frozen cluster is that of a critical Galton-Watson tree with Poisson offspring distribution, again establishing SOC. As an open problem, we mention:

PROBLEM 1. *Construct frozen percolation on the PWIT and show that it is the local weak limit of the models in [Rat09, MN14].*

Coagulation equations. The relation of frozen percolation on the complete graph to Smoluchowski's coagulation equations has already been mentioned. A remark of Stockmayer [Sto43] on these equations inspired Aldous' work for the 3-regular tree. In [Ald00, Section 1.1] Aldous compares the post-gel behaviour of Smoluchowski's coagulation equations to the self-similar behaviour of his model. In [Ald99] Aldous surveys the connections between variants of Smoluchowski's coagulation equations and various stochastic models of coagulation.

The configuration model [Hof17a] is a well-studied random graph whose weak local limit is well-known. In particular, one can choose the parameters of the configuration model so that this limit is the 3-regular or more generally any d-regular graph. The configuration model has a dynamical construction where to vertices there are assigned "half-edges" or "arms" that are then randomly linked. In [MN15] a variant of this model is treated where components freeze once their size exceeds a fixed threshold. They link the model to a variant of Smoluchowski's equations and it is shown that after gelation, the asymptotic distribution of a typical non-frozen cluster is that of a critical Galton-Watson tree.

The mathematical connection between more general stochastic models of coalescence where clusters with a size above a certain threshold are frozen and Smoluchowski's equation with more general kernels is established in [FL09].

Frozen percolation the 3-regular tree. Aldous' work on frozen percolation on the 3-regular tree is the first example of a dynamically constructed random graph model that exhibits SOC. In [Ald00, Prop 11 and Thm 14] it is proved that at any time $t \in [\frac{1}{2}, 1]$, a typical finite cluster is distributed as a critical percolation cluster on the binary tree, and infinite clusters are distributed as the incipient infinite cluster. As an open problem, we mention:

PROBLEM 2. *Show that frozen percolation on the 3-regular tree is the weak local limit of frozen percolation on a suitable sequence of finite graphs.*

When proving convergence, it is very useful to have a unique characterization of the limit. A unique characterization of frozen percolation on the 3-regular tree is provided by our Theorem 2. We do not know if condition (iii) is in fact needed for uniqueness. Likewise, the following question is still open:

QUESTION 3. *Do conditions (i) and (ii) of Theorem 1 uniquely determine the law of* $(\mathcal{U}, F)$*?*

We note that using Theorem 3, it is not hard to show that (i) alone is not sufficient for distributional uniqueness.

Let us note here that a variant of Aldous' frozen percolation model on the binary tree where clusters with size greater than a large number N are frozen was introduced in [BKN12]. The law of the cluster of the origin at time $t \in [0,1]$ in the frozen percolation model with freezing threshold N locally converges to the corresponding law in the frozen percolation model of Aldous [Ald00] as $N \to \infty$ (see [BKN12, Theorem 1]).

Nonendogeny. In line with Problem 2, we expect that for a suitable sequence of finite graphs whose weak local limit is the 3-regular tree, if we couple two frozen percolation processes on these graphs by using the same edge activation times but independent lightning processes, then the weak local limit should be the process $(\mathcal{U}, F, F')$ from Theorem 3. In particular, the local limit of such processes should be a.s. different because of nonendogeny.

Even though the basic question of endogeny has now been settled for the binary tree, more detailed questions remain open. In Section 3.3, we classify all solutions to RDE (1.32). This leads to the question:

QUESTION 4. *For which solutions of the RDE (1.32) is the corresponding RTP nonendogenous?*

Even for the RTP in Theorem 7, one would like to understand better what is going on.

QUESTION 5. *By Theorem 7, the σ-field generated by $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is larger than the σ-field generated by $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Give an explicit characterisation of the extra randomness needed to construct $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$.*

In this context, we mention that in [Ban06, Thm 1.2], it is proved that the tail σ-algebra of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is trivial. Proposition 1.1 of [Ban06] states that generally, endogeny of a RTP implies its tail-triviality, however our main result exemplifies that the converse implication does not necessarily hold.

Related to our previous question is the following problem. Let $X'_\varnothing$ denote the first time when there is an infinite path of open or frozen edges starting at the root. Then clearly $X'_\varnothing \leq X_\varnothing$ a.s. If the answer to the following question is positive, then this is all that can be said with certainty about $X_\varnothing$ based on $(\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$.

QUESTION 6. *Let $\xi := \mathbb{P}\big[X_\varnothing \in \cdot \,\big|\, (\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big]$. Is it true that the support of ξ is a.s. equal to $[X'_\varnothing, \infty)$?*

In Theorem 12, we have shown that the bivariate RDE has precisely two scale invariant fixed points. We believe that there exist fixed points that are not scale invariant. To see why, recall that we suggested that $\underline{\nu}^{(2)}$ should describe the local limit of two finite frozen percolation processes that use the same edge activation times but independent lightning processes. We believe that the local limit of two processes that use the same lightning process up to some time $\frac{1}{2} < s < 1$ and independent lightning processes thereafter should be described by a fixed point of $T^{(2)}$ that is neither $\underline{\nu}^{(2)}$ nor $\overline{\nu}^{(2)}$.

It has been shown in [MSS20, Thm 1] that for each initial state, the differential equation

$$\frac{\partial}{\partial h}\mu_h^{(2)} = T^{(2)}(\mu_h^{(2)}) - \mu_h^{(2)} \qquad (h \geq 0) \tag{1.38}$$

has a unique solution.

Problem 7. *For frozen percolation on the oriented binary tree, find all fixed points of (1.38) and their domains of attraction.*

In [MSS20, Prop 12] Problem 7 is solved for a different RTP, which is also nonendogenous. In that example, $\underline{\nu}^{(2)}$ and $\overline{\nu}^{(2)}$ turned out to be the only fixed points, where the trivial fixed point $\overline{\nu}^{(2)}$ is unstable and the nontrivial fixed point $\underline{\nu}^{(2)}$ attracts all other initial states. One wonders if the situation for frozen percolation is similar. In general, we ask:

Question 8. *For a general RTP, can one prove nonendogeny by proving that the trivial fixed point $\overline{\nu}^{(2)}$ is unstable?*

Our proof of Theorem 7 is based on an explicit formula for $\underline{\nu}^{(2)}$. Ultimately, one would like to be able to prove nonendogeny without having to solve the bivariate RDE.

Frozen percolation on regular trees. In Problem 1, we have already mentioned frozen percolation on the PWIT. Aldous [Ald00, Sect. 5.4] observed that his construction can be carried out on any d-regular tree, and even gave a formula for the distribution of freezing times on d-regular trees. This leads to:

Question 9. *Are frozen percolation on the PWIT or on general d-regular trees endogenous?*

We conjecture the answer to this question to be negative, but this does not follow from the methods of this paper. Our main results are for the MBBT and essentially rely on the nice scaling property of the latter that simplifies our formulas. The fact that we are also able to treat the oriented binary tree and consequently the unoriented 3-regular tree depends on a trick that uses in an essential way that the MBBT is a binary tree.

Nevertheless, we hope that our methods will be useful in answering Question 9. The reason for this optimism is that the MBBT can be seen as the near-critical scaling limit of percolation on a wide class of oriented trees, such as oriented d-ary trees or the PWIT.

Indeed, since edges with $U_e \leq p_{\rm c}$ belong to finite clusters when they open, from the point of view of frozen percolation it does not matter when they open. In view of this, let us focus only on those edges whose activation times lie between $p_{\rm c}$ and $p_{\rm c} + \varepsilon$ for some small $\varepsilon > 0$. If we condition on the event that there is an infinite path starting at the root along edges with activation times $U_e \leq p_{\rm c} + \varepsilon$, and cut off all parts of the tree that do not lie on such an infinite path, then the scaling limit as $\varepsilon \to 0$ of our tree $\mathcal{T}$, and the locations marked with the (scaled) activation times of edges with $p_{\rm c} < U_e < p_{\rm c} + \varepsilon$ converge to the marked Poisson process Π on $\mathcal{T}$.

In view of this, we expect that on a general class of oriented trees, frozen percolation is nonendogenous and the nontrivial fixed point $\underline{\nu}^{(2)}$ of the bivariate RDE will in a small neighbourhood of the critical point look similar to the nontrivial fixed point from Theorem 12.

Frozen percolation on integer lattices. One can try to "naively" define frozen percolation on any infinite graph as in property (i) of Theorem 1, by specifying that clusters stop growing as soon as they reach infinite size. It is an observation of Benjamini and Schramm that such a process cannot be defined on the planar square lattice (for a sketch of a proof, see [BT01, Section 3]). The following question is open:

QUESTION 10. *For which $d \geq 3$ does there exists a frozen percolation process on the nearest-neighbour lattice $\mathbb{Z}^d$ that satisfies property (i) of Theorem 1?*

There exists an extensive literature for finite versions of frozen percolation on the planar lattice. A model where clusters with diameter greater than a large number N are frozen was introduced in [BLN12]. The behaviour of this model is rather different from the the analogous model of [BKN12] on the binary tree that we have discussed after Problem 2, because in planar diameter-frozen percolation all frozen clusters freeze in the critical time window around the Bernoulli percolation threshold p_c, the frozen clusters look similar to critical percolation clusters, and moreover macroscopic non-frozen clusters asymptotically have full density as $N \to \infty$, c.f. [BLN12, Kis15]. In [BN17] it is shown that the particular mechanism to freeze clusters (the "boundary rules") matters strongly, i.e., if we modify the diameter-frozen site percolation model on the triangular lattice in a way that the outer boundary of frozen connected components can become occupied (and later freeze) then frozen clusters in the terminal configuration have non-vanishing density as $N \to \infty$.

The percolation on the planar lattice where clusters with volume (cardinality) greater than a large number N are frozen was introduced in [BN17], the main result being that if we restrict the process to a large box with side-length n, then the probability that the origin freezes depends on the relation between N and n in an oscillatory fashion. Thus the behaviour of the volume-frozen process is substantially different from that of the diameter-frozen process. In [BKN18] it is shown that in the volume-frozen model many frozen clusters surrounding the origin appear successively, each new cluster having a diameter much smaller than the previous one. In [BKN18] it is also proved that in the full planar case ($n = \infty$) with high probability (as $N \to \infty$), the origin does not belong to a frozen cluster in the final configuration. In [BN18] it is proved that if the freezing mechanism in a box of size n is governed by independent lightnings hitting the vertices then the density of frozen sites depends on the relation between the lightning rate and n in an oscillatory fashion.

Self-destructive percolation and forest fire model on infinite graphs. The "naive" definition of the forest fire model on an infinite graph $G = (V, E)$ (dating back to [DS92]) is as follows: vacant sites become occupied at rate 1 and infinite occupied clusters become vacant instantaneously. Similarly to the case of the frozen percolation model, it is a highly non-trivial question whether such a process exists.

The model of *self-destructive percolation* was introduced by [BB04] in order to address this question on the planar lattice: given some $p > p_c$, let us switch all of the sites which are in an infinite occupied component into vacant state (destruction) and then turn any vacant site occupied with probability δ (enhancement). Denote by $\delta(p)$ the smallest enhancement needed for the appearance of an infinite cluster in the enhanced configuration. Theorem 4.1 of [BB04] states that if $\lim_{p \searrow p_c} \delta(p) > 0$ then the forest fire process cannot be defined on the planar lattice. Theorem 1 of [KMS15] states that indeed $\lim_{p \searrow p_c} \delta(p) > 0$ on the planar lattice. However, we have $\lim_{p \searrow p_c} \delta(p) = 0$ on non-amenable graphs [AST14] and $\mathbb{Z}^d$ for high enough d [ADKS15]. Also note that in mean field percolation models we have $\lim_{p \searrow p_c} \frac{\delta(p)}{p - p_c} = 1$. This asymptotic relation becomes an exact equality of the lengths of growth and recovery time intervals if one considers self-destructive (frozen) percolation on the MBBT, moreover the general solutions to the RDE (1.32) (c.f. Section 3.3) and the associated RTP's (c.f. Section 3.4) also exhibit time intervals of (supercritical) growth and (subcritical) recovery, which are of equal length.

Currently it is an open question whether it is possible to define a forest fire process on the nearest-neighbour lattice $\mathbb{Z}^d$, $d \geq 3$. In [BT01] a variant of the forest fire model (with site-dependent occupation rates) is constructed on the half-line. The construction of the variant of the forest fire model with a positive rate of lightning per vertex on $\mathbb{Z}^d$ is given in [Dur06a, Dur06b]: if a lightning hits a vertex v, then all of the sites in the occupied cluster of v become vacant instantaneously. In [Gra14, Gra16] a variant of the forest fire model on the half-plane is defined where components that touch the boundary (or become infinite) are destroyed. It is shown that before (and including) the critical time, the effect of the destruction mechanism is only felt locally near the boundary of the half-plane, whereas after the critical time, it is felt globally on the entire half-plane.

Outline. The rest of the paper is devoted to proofs. We prove Theorem 12 in Section 2 and the remaining results in Section 3. The paper concludes with a small appendix on skeletal branching processes, which are related to the scaling property of the MBBT described in Proposition 9.

Even though Theorem 7, which is proved in Subsection 3.2, is our main result, considerable extra effort is needed to prove additional results, in particular, uniqueness of the nontrivial fixed point in Theorem 12 and its subsequent identification as $\underline{\rho}^{(2)}$ with the help of Lemma 11, as well as Theorem 2, which depends on the classification of general solutions to the RDE (1.32) in Subsection 3.3.

2. The bivariate RDE.

2.1. *Main line of the proof.* In this section, we prove Theorem 12. The main steps of the proof are summarised in the following lemmas. We first need a convenient way to parametrise elements of the space $\mathcal{P}_*(I^2)_\rho$.

LEMMA 13 (Parametrisation of the space of interest). *For each $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$, there exists a unique continuous function $f : [0,1] \to \mathbb{R}$ such that*

$$\rho^{(2)}\big(\{[0,r]\times I\}\cup\{I\times[0,s]\}\big) = (s\vee r) f\left(\frac{r\wedge s}{r\vee s}\right), \quad \big((r,s)\in[0,1]^2\backslash\{(0,0)\}\big), \tag{2.1}$$

and such a function f uniquely characterizes $\rho^{(2)}$. In particular, the trivial fixed point $\overline{\rho}^{(2)}$ corresponds to $\overline{f}(r) = \frac{1}{2}$, $(r \in [0,1])$.

There are a priori many ways of parametrising elements of $\mathcal{P}_*(I^2)_\rho$. The parametrisation in terms of the function f from (2.1) turns out to lead to a particularly simple form of the bivariate RDE.

LEMMA 14 (Bivariate RDE). *An element $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$ is a fixed point of the bivariate map $T^{(2)}$ associated with the map χ from (1.30) if and only if the function $f : [0,1] \to \mathbb{R}$ from (2.1) is continuously differentiable on $[0,1)$ and satisfies*

$$\text{(i) } \frac{\partial}{\partial r} f(r) = \frac{cr}{f(r) - r/2} \qquad \big(r \in [0,1)\big), \qquad \text{(ii) } f(0) = \tfrac{1}{2}, \qquad \text{(iii) } f(1)^2 - \tfrac{1}{2} f(1) = 2c, \tag{2.2}$$

for some $c \geq 0$.

In particular, the trivial fixed point $\overline{f}(r) = \frac{1}{2}$ solves (2.2) with $c = \overline{c} := 0$. The following lemma shows that there is exactly one other, nontrivial solution.

LEMMA 15 (Nontrivial solution of (2.2)). *For each $c \geq 0$, there exists a unique solution f_c to (2.2) (i) and (ii). There exists a unique $c_2 > 0$ such that the function f_{c_2} also satisfies (2.2) (iii). Moreover, we have $c_2 \in (0, \frac{1}{4})$.*

In Lemma 13, we have shown that a probability law $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$ is uniquely characterised by the corresponding function f from (2.1), but we have not given sufficient conditions for a function $f : [0,1] \to \mathbb{R}$ to correspond to an element of $\mathcal{P}_*(I^2)_\rho$. In view of this, to complete the proof of Theorem 12, we need one more lemma.

LEMMA 16 (Nontrivial solution of the bivariate RDE). *The function f_{c_2} from Lemma 15 defines through (2.1) a probability measure $\rho_2^{(2)} \in \mathcal{P}_*(I^2)_\rho$. The restriction of $\rho_2^{(2)}$ to $[0,1]^2$ has a density w.r.t. the Lebesgue measure. In particular, $\rho_2^{(2)}$ puts no mass on the diagonal $\big\{(r,r) : r \in [0,1]\big\}$.*

PROOF OF THEOREM 12. By Lemmas 13, 14, 15, and 16, the bivariate map $T^{(2)}$ has, apart from the trivial fixed point $\overline{\rho}^{(2)}$, precisely one more fixed point in $\mathcal{P}_*(I^2)_\rho$, which is given as in (1.37) in terms of the function f_{c_2}. $\square$

We will prove Lemmas 13, 14, 15 and 16 in Sections 2.2, 2.3, 2.4 and 2.5, respectively.

2.2. *Parametrisation of scale invariant measures.* In this subsection we prove Lemma 13. We also prepare for the proof of Lemma 16 by giving sufficient conditions for a function $f:[0,1]\to\mathbb{R}$ to define a measure $\rho^{(2)}\in\mathcal{P}_*(I^2)_\rho$ through (2.1).

LEMMA 17 (Encoding $\rho^{(2)}$ as a bivariate function). *Any $\rho^{(2)}\in\mathcal{P}(I^2)_\rho$ is uniquely characterised by the continuous function $F:[0,1]^2\to[0,1]$ defined as*

$$F(r,s):=\rho^{(2)}\big(\{[0,r]\times I\}\cup\{I\times[0,s]\}\big),\qquad \big(r,s\in(0,1]\big). \tag{2.3}$$

Moreover, $\rho^{(2)}\in\mathcal{P}_*(I^2)_\rho$ if and only if F is symetric in the sense that $F(r,s)=F(s,r)$ and

$$F(tr,ts)=tF(r,s)\qquad \big(r,s,t\in[0,1]\big). \tag{2.4}$$

PROOF. Since both marginals of $\rho^{(2)}$ are equal to ρ, formula (2.3) is equivalent to

$$\begin{aligned} &\text{(i) } \rho^{(2)}\big(\{\infty\}\times\{\infty\}\big)=1-F(1,1),\\ &\text{(ii) } \rho^{(2)}\big([0,r]\times\{\infty\}\big)=F(r,1)-\tfrac{1}{2},\\ &\text{(iii) } \rho^{(2)}\big(\{\infty\}\times[0,s]\big)=F(1,s)-\tfrac{1}{2},\\ &\text{(iv) } \rho^{(2)}\big([0,r]\times[0,s]\big)=\tfrac{1}{2}r+\tfrac{1}{2}s-F(r,s). \end{aligned} \tag{2.5}$$

Since these functions uniquely determine the restrictions of $\rho^{(2)}$ to $\{(\infty,\infty)\}$, $[0,1]\times\{\infty\}$, $\{\infty\}\times[0,1]$, and $[0,1]^2$, the function F determines $\rho^{(2)}$ uniquely. Moreover, we see from (2.5) (iv) that $\rho^{(2)}$ is scale invariant in the sense of (1.34) if and only if (2.4 holds. Since the marginals of $\rho^{(2)}$ are equal to ρ, and ρ has no atoms in $[0,1]$, we see from (2.5) (iv) that F is a continuous function. $\square$

If a closed subset C of $\mathbb{R}^d$ is the closure of its interior, then we say that a function is n times continuously differentiable on C is all partial derivatives up to n-th order exist on the interior of C and can be extended to continuous functions on C.

LEMMA 18 (Sufficient conditions on F corresponding to $\rho^{(2)}\in\mathcal{P}(I^2)_\rho$). *Let $\Delta:=\big\{(r,s)\in[0,1]^2: 0\le r\le s\big\}$ and let $F:\Delta\to[0,\infty)$ be a twice continuously differentiable function such that:*

(i) $F(1,1)\le 1$, (ii) $F(0,s)=\frac{1}{2}s$, (iii) $r\mapsto F(r,1)$ *is nondecreasing,*

(iv) $\frac{\partial}{\partial r}F(r,s)\big|_{r=s}=\frac{\partial}{\partial s}F(r,s)\big|_{r=s}$, (v) $g(r,s):=-\frac{\partial^2}{\partial r\partial s}F(r,s)\ge 0.$

Extend F and g to $[0,1]^2$ by setting $F(s,r) := F(r,s)$ and $g(s,r) := g(r,s)$ for $\big((r,s) \in \Delta\big)$. Then there exists a unique probability measure $\rho^{(2)} \in \mathcal{P}(I^2)_\rho$ such that (2.3) holds, and the restriction of $\rho^{(2)}$ to $[0,1]^2$ has density g with respect to the Lebesgue measure.

PROOF. Uniqueness follows from Lemma 17. By (2.5), condition (i) guarantees that the mass at (∞,∞) is nonnegative, while conditions (ii) and (iii) guarantee that the restrictions of $\rho^{(2)}$ to $[0,1] \times \{\infty\}$ and $\{\infty\} \times [0,1]$ are nonnegative measures.

To complete the proof, we will show that conditions (ii), (iv) and (v) imply that (2.5) (iv) defines a measure on $[0,1]^2$ with density g. Equivalently, we must show that

$$D(r,s) := \int_0^r \mathrm{d}r' \int_0^s \mathrm{d}s'\, g(r',s') - \tfrac{1}{2}r - \tfrac{1}{2}s + F(r,s) \qquad \big((r,s) \in \Delta\big) \tag{2.6}$$

is identically zero. Conditions (ii), (iv) and (v) imply that

$$D(0,s) = 0, \quad \tfrac{\partial}{\partial r} D(r,s)\big|_{r=s} = \tfrac{\partial}{\partial s} D(r,s)\big|_{r=s}, \quad \text{and} \quad \tfrac{\partial^2}{\partial r \partial s} D(r,s) = 0 \tag{2.7}$$

$((r,s) \in \Delta)$ The third equality implies that $D(r,s) = u(r) + v(s)$ for some differentiable functions u and v, but then $D(0,s) \equiv 0$ implies that $u(0) + v(s) \equiv 0$, thus v is constant and therefore $u'(r) = \frac{\partial}{\partial r} D(r,s)\big|_{r=s} = \frac{\partial}{\partial s} D(r,s)\big|_{r=s} = v'(r) = 0$, so u is also a constant, so $D(r,s) = 0$ for any $0 \leq r \leq s \leq 1$. □

PROOF OF LEMMA 13. Given $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$, let F be as in (2.3) and let $f : [0,1] \to \mathbb{R}$ be the continuous function defined by

$$f(r) := F(r,1) = \rho^{(2)}\left(\{[0,r] \times I\} \cup \{I \times [0,1]\}\right), \quad 0 \leq r \leq 1. \tag{2.8}$$

Then $F(r,s) = sf(r/s)$ $(s \neq 0)$ by (2.4) and hence (2.1) follows by symmetry. The fact that f uniquely characterizes the measure $\rho^{(2)}$ follows from (2.1) and Lemma 17. The trivial fixed point $\overline{\rho}^{(2)}$ of $T^{(2)}$ is the distribution of (Y,Y), where $Y \sim \rho$. In this case $\overline{f}(r) = \mathbb{P}(\{Y \leq r\} \cup \{Y \leq 1\}) = \frac{1}{2}$ for any $r \in [0,1]$. □

LEMMA 19 (Sufficient conditions on f corresponding to $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$). *Let $f : [0,1] \to \mathbb{R}$ be a twice continuously differentiable function such that*

$$\text{(i) } f(1) \leq 1, \quad \text{(ii) } f(0) = \tfrac{1}{2}, \quad \text{(iii) } r \mapsto f(r) \textit{ is nondecreasing,}$$
$$\text{(iv) } 2f'(1) = f(1), \qquad \text{(v) } f''(r) \geq 0 \quad (r \in [0,1]).$$

Then there exists a unique probability measure $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$ such that (2.1) holds, and the restriction of $\rho^{(2)}$ to $[0,1]^2$ has a density with respect to the Lebesgue measure.

PROOF. For $0 \le r \le s$, define $F(r,s) := sf(r/s)$ if $s \neq 0$ and $:= 0$ otherwise, and $F(s,r) := F(r,s)$. Then (2.1) is equivalent to (2.5) so uniqueness follows from Lemma 17. Since F is symmetric and satisfies (2.4), the same lemma shows that if $\rho^{(2)}$ exists, then $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$.

To get existence, we apply Lemma 18. We claim that conditions (i)–(v) of that lemma follow from the corresponding conditions of the present lemma. This is trivial for conditions (i)–(iii). Condition (iv) of Lemma 18 yields

$$f'\big(\frac{r}{s}\big) = f\big(\frac{r}{s}\big) - \frac{r}{s} f'\big(\frac{r}{s}\big) \qquad (r = s), \tag{2.9}$$

which corresponds to the present condition (iv). Finally, condition (v) of Lemma 18 requires that

$$-\tfrac{\partial^2}{\partial r \partial s} s f\big(\frac{r}{s}\big) = -\tfrac{\partial}{\partial s} f'\big(\frac{r}{s}\big) = \frac{r}{s^2} f''\big(\frac{r}{s}\big) \ge 0, \tag{2.10}$$

which corresponds to the present condition (v). $\square$

2.3. *Bivariate RDE and controlled ODE .* In this subsection we prove Lemma 14, i.e., we equivalently reformulate the bivariate fixed point property $T^{(2)}\rho^{(2)} = \rho^{(2)}$ for a scale invariant measure $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$ as the controlled ODE problem (2.2) for the function f defined in (2.8). We start by deriving an integral expression for the map $T^{(2)}$. Equation (2.11) below is an adaptation of equations (11) and (12) of [Ban04] to the MBBT.

LEMMA 20 (Bivariate map). *Let $\rho^{(2)} \in \mathcal{P}(I^2)_\rho$, let $T^{(2)}$ denote the bivariate map defined as in (1.8) for the map χ in (1.30), taking uniformly distributed $\tau_\varnothing, \kappa_\varnothing$ as its input. Let F be defined in terms of $\rho^{(2)}$ as in (2.3) and let $\widetilde{F}$ be defined similarly in terms of $T^{(2)}(\rho^{(2)})$. Then*

$$\begin{aligned}\widetilde{F}(r,s) = F(r,s) &- \frac{F(r,s)^2}{2} + \frac{r^2}{8} + \frac{s^2}{8} \\ &+ \tfrac{1}{2}\int_0^{r\wedge s} \big(F(r,s) - F(t,s) - F(r,t) + F(t,t)\big)\,\mathrm{d}t, \quad r,s \in (0,1].\end{aligned} \tag{2.11}$$

PROOF. Let $\tau_\varnothing$ and $\kappa_\varnothing$ denote independent random variables, where $\tau_\varnothing \sim \mathrm{Uni}[0,1]$ and $\kappa_\varnothing$ is uniformly distributed on $\{1,2\}$. Let (Y_1, Y_1^*) and (Y_2, Y_2^*)

denote I^2-valued random variables with distribution $\rho^{(2)}$, independent from each other and of $\tau_\varnothing$, $\kappa_\varnothing$. Let us define

$$Y_\varnothing := \chi[\tau_\varnothing,\kappa_\varnothing](Y_1,Y_2), \qquad Y^*_\varnothing := \chi[\tau_\varnothing,\kappa_\varnothing](Y^*_1,Y^*_2), \tag{2.12}$$

where χ is defined in (1.30). Then the distribution of $(Y_\varnothing, Y^*_\varnothing)$ is $T^{(2)}(\rho^{(2)})$. It follows that

$$\widetilde{F}(r,s) = \tfrac{1}{2}\mathbb{P}[\,Y_\varnothing \le r \ \text{ or } \ Y^*_\varnothing \le s \,|\, \kappa_\varnothing = 1] + \tfrac{1}{2}\mathbb{P}[\,Y_\varnothing \le r \ \text{ or } \ Y^*_\varnothing \le s \,|\, \kappa_\varnothing = 2]. \tag{2.13}$$

Here

$$\begin{aligned}
&\mathbb{P}[\,Y_\varnothing \le r \ \text{ or } \ Y^*_\varnothing \le s \,|\, \kappa_\varnothing = 1] \\
&\quad\stackrel{(1.30)}{=} \int_0^1 \mathbb{P}[\,Y_1 \in (t, t\vee r] \ \text{ or } \ Y^*_1 \in (t, t\vee s]\,)\,\mathrm{d}t \\
&\quad= \int_0^1 \mathbb{P}[\,Y_1 \in (t, t\vee r]\,]\,\mathrm{d}t + \int_0^1 \mathbb{P}[\,Y^*_1 \in (t, t\vee s]\,]\,\mathrm{d}t \\
&\qquad - \int_0^1 \mathbb{P}[\,Y_1 \in (t, t\vee r],\, Y^*_1 \in (t, t\vee s]\,]\,\mathrm{d}t \\
&\quad\stackrel{(1.31)}{=} \int_0^r \tfrac{1}{2}(r-t)\,\mathrm{d}t + \int_0^s \tfrac{1}{2}(s-t)\,\mathrm{d}t - \int_0^{r\wedge s} \mathbb{P}[\,Y_1 \in (t,r],\, Y^*_1 \in (t,s]\,]\,\mathrm{d}t \\
&\quad\stackrel{(*)}{=} \frac{r^2}{4} + \frac{s^2}{4} - \int_0^{r\wedge s} \big(F(t,s) + F(r,t) - F(t,t) - F(r,s)\big)\,\mathrm{d}t,
\end{aligned} \tag{2.14}$$

where in $(*)$ we used (2.3) and inclusion-exclusion. Moreover

$$\begin{aligned}
&\mathbb{P}[\,Y_\varnothing \le r \ \text{ or } \ Y^*_\varnothing \le s \,|\, \kappa = 2] \stackrel{(1.30)}{=} 1 - \mathbb{P}[\,Y_1 \wedge Y_2 > r,\, Y^*_1 \wedge Y^*_2 > s\,] \\
&\quad = 1 - \mathbb{P}[\,Y_1 > r,\, Y^*_1 > s\,]^2 = 1 - (1 - F(r,s))^2 = 2F(r,s) - F(r,s)^2.
\end{aligned} \tag{2.15}$$

Now (2.11) follows as a combination of (2.13), (2.14) and (2.15). □

LEMMA 21 (Scale invariant bivariate fixed point). *$\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$ satisfies $T^{(2)}(\rho^{(2)}) = \rho^{(2)}$ if and only if the function f in (2.1) satisfies*

$$f(r)^2 = \frac{1}{4} + rf(r) - \int_0^r f(u)\,\mathrm{d}u + \left(\frac{1}{4} + \frac{f(1)}{2} - \int_0^1 f(s)\,\mathrm{d}s\right) r^2, \ \ r \in [0,1]. \tag{2.16}$$

PROOF. If $\rho^{(2)} \in \mathcal{P}(I^2)_\rho$ then $\rho^{(2)}$ is symmetric, so Lemmas 17 and 20 imply that $T^{(2)}(\rho^{(2)}) = \rho^{(2)}$ holds if and only if for any $0 < r \le s \le 1$

$$F(r,s)^2 = \frac{r^2}{4} + \frac{s^2}{4} + \int_0^r \big(F(r,s) - F(t,s) - F(t,r) + F(t,t)\big)\,\mathrm{d}t. \tag{2.17}$$

If $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$, then the function F from (2.3) can be expressed in the function f from (2.1) as

$$F(r,s) = sf\left(\frac{r}{s}\right), \qquad 0 < r \le s \le 1. \tag{2.18}$$

Plugging this into (2.17), dividing both sides by s^2 and using the substitution $u = t/s$ in the integral we obtain

(2.19)
$$f\left(\frac{r}{s}\right)^2 = \frac{(r/s)^2}{4} + \frac{1}{4} + \int_0^{r/s} \left(f\left(\frac{r}{s}\right) - f(u) - \frac{r}{s} f\left(\frac{u}{r/s}\right) + uf(1) \right) \mathrm{d}u,$$

which holds for all $0 < r \le s \le 1$ if and only if

(2.20)
$$f(r)^2 = \frac{1}{4} + \frac{r^2}{4} + \int_0^r \left(f(r) - f(u) - rf\left(\frac{u}{r}\right) + uf(1) \right) \mathrm{d}u, \quad 0 < r \le 1.$$

Evaluating the integrals, using the substitution $s = u/r$, we arrive at (2.16), which also holds for $r = 0$ since f is continuous. □

REMARK 22. For any $\rho^{(2)} \in \mathcal{P}_*(I^2)_\rho$, setting $s = 1$ in (2.1) yields (2.8), which shows that f is nondecreasing. Since the marginals of $\rho^{(2)}$ are ρ, we have $f(0) = \frac{1}{2}$. If $f(1) = \frac{1}{2}$, then we must have $f(r) = \frac{1}{2}$, $r \in [0,1]$. In this case (2.16) holds. This is the $\overline{f}$ associated to the (scale invariant) diagonal fixed point $\overline{\rho}^{(2)}$ of $T^{(2)}$.

LEMMA 23 (Controlled ODE). *Let $f : [0,1] \to [0,\infty)$ be continuous and nondecreasing with $f(1) > \frac{1}{2}$. Then f satisfies (2.16) if and only if f is continuously differentiable and solves*

$$\text{(i)}\ f(0) = \frac{1}{2}, \quad \text{(ii)}\ f'(r) = \frac{cr}{f(r) - r/2}, \quad r \in [0,1], \tag{2.21}$$

where

$$c = \frac{1}{4} + \frac{f(1)}{2} - \int_0^1 f(s)\,\mathrm{d}s > 0. \tag{2.22}$$

PROOF. Plugging in $r = 0$ into (2.16) we obtain $f(0) = \frac{1}{2}$. Using this, we have

(2.23)
$$\text{(i)}\ f(r)^2 = \frac{1}{4} + \int_0^r 2f(u)\mathrm{d}f(u), \quad \text{(ii)}\ rf(r) - \int_0^r f(u)\,\mathrm{d}u = \int_0^r u\,\mathrm{d}f(u),$$

where both integrals in (2.23) are Stieltjes. Inserting this into (2.16) yields

$$\int_0^r (2f(u)-u)\,\mathrm{d}f(u) = cr^2, \quad 0 \le r \le 1, \tag{2.24}$$

with c as in (2.21). Since f is nondecreasing with $f(0)=\frac{1}{2}$, we observe that $2f(u)-u \ge 2f(0)-u > 0$ for all $u \in [0,1)$. Combining this with the assumption $f(1) > \frac{1}{2}$ we get $m := \min_{0\le u\le 1}(2f(u)-u) > 0$, since $u \mapsto 2f(u)-u$ is a continuous function on the compact interval $[0,1]$. Since the right-hand side of (2.24) is has Lipschitz constant $2c$, we conclude that f is $2c/m$-Lipschitz-continuous on $[0,1]$. Thus, by the Radon-Nykodim theorem, there exists a Lebesgue-a.s. unique measurable function $f^\circ : [0,1] \to [0,2c/m]$ such that $\int_0^r f^\circ(u)\,\mathrm{d}u = f(r)-\frac{1}{2}$ for all $0 \le r \le 1$.

By (2.24) we have $\int_0^r (2f(u)-u)\,f^\circ(u)\,\mathrm{d}u = cr^2$ for any $0 \le r \le 1$, thus $(2f(r)-r)f^\circ(r) = 2cr$ for Lebesgue-almost all $r \in [0,1]$, from which it follows that the Radon-Nykodim derivative f° can be chosen to be the continuous function $f^\circ(r) = \frac{2cr}{2f(r)-r}$, therefore f is continuously differentiable, $f' = f^\circ$ and (2.21) holds.

Since f is nondecreasing, (2.21) implies $c \ge 0$. Solving (2.21) with $c=0$ yields $f(1)=\frac{1}{2}$, contradicting $f(1) > \frac{1}{2}$, so we conclude that $c > 0$.

Assume, conversely, that f solves (2.21) and (2.22). Then (2.21) (ii) implies (2.24) and (2.21) (i) yields (2.23) (i). Combining this with (2.23) (ii) and (2.22), we see that f solves (2.16). $\square$

LEMMA 24 (Well-defined ODE). *For each $c \in [0,\infty)$, there exists a unique continuous function $f_c : [0,1] \to \mathbb{R}$ that solves (2.2) (i) and (ii).*

PROOF. Solutions to (2.2) (i) and (ii) exist and are unique up to the first time τ when $f(r) = \frac{1}{2}r$. Since solutions are nondecreasing with $f(0)=\frac{1}{2}$, we have $\tau \ge 1$. If $\tau = 1$ then $f(1) = \frac{1}{2}$ which corresponds to the case $f(r) = \frac{1}{2}$ $(r \in [0,1])$, so f is in any case continuous on $[0,1]$. $\square$

LEMMA 25 (Integral equation for f_c). *The function f_c from Lemma 24 satisfies*

$$f_c(r)^2 = \tfrac{1}{4} + rf_c(r) - \int_0^r f_c(u)\,\mathrm{d}u + cr^2, \quad 0 \le r \le 1, \quad c \in [0,\infty). \tag{2.25}$$

PROOF. (2.25) holds for $r=0$ since $f_c(0)=\frac{1}{2}$ and the derivatives of the two sides of (2.25) are equal for all $0 \le r \le 1$ by (2.21). $\square$

PROOF OF LEMMA 14. We note that the function $\overline{f}(r) = \frac{1}{2}$ $(r \in [0,1])$ solves (2.21) for $r \in [0,1)$ and $c = 0$. In view of this, Lemma 21, Remark 22, and Lemma 23 show that $T^{(2)}(\rho^{(2)}) = \rho^{(2)}$ if and only if the function f from (2.1) satisfies (2.2) (i) and (ii) with $c = \frac{1}{4} + \frac{1}{2}f(1) - \int_0^1 f(s)\mathrm{d}s \geq 0$. To see that this latter condition is equivalent to (2.2) (iii), we insert $r = 1$ into (2.25) which yields $\frac{1}{4} + \frac{f_c(1)}{2} - \int_0^1 f_c(s)\,\mathrm{d}s = f_c(1)^2 - \frac{f_c(1)}{2} - c$. $\square$

2.4. *Finding the nontrivial control parameter .* The goal of this subsection is to prove Lemma 15. By Lemma 24, the ODE (2.2) (i) with the left boundary condition (2.2) (ii) has a unique solution f_c for all $c \geq 0$. We need to prove the existence and uniqueness of a control parameter $c_2 > 0$ for which f_{c_2} also solves the right boundary condition (2.2) (iii). In Lemma 26 we solve the ODE and obtain an implicit equation for $f_c(1)$. In Lemma 28 we use this to rewrite (2.2) (iii) as $h(c) = 1$ for some explicit function h (see (2.33)). In Lemma 29 we show that there is a unique $c_2 > 0$ such that $h(c_2) = 1$ holds, and $c_2 \in (0, \frac{1}{4})$.

Given any $c \in (0,\infty)$, let us define $g_+(c), g_-(c), A_+(c), A_-(c)$ by

$$g_\pm(c) := \frac{1}{4}\left(1 \pm \sqrt{1+16c}\right), \qquad A_\pm(c) := -\frac{1}{2} \pm \frac{1}{2\sqrt{1+16c}}. \tag{2.26}$$

LEMMA 26 (Solution of ODE for f_c). *For any $c > 0$, the function f_c from Lemma 24 is given by $f_c(r) = r g_c(r)$ ($r \in (0,1]$), where $g_c(r)$ is the unique element of $(g_+(c), +\infty)$ that satisfies*

$$\tfrac{1}{2}\left(g_c(r) - g_+(c)\right)^{A_+(c)} \left(g_c(r) - g_-(c)\right)^{A_-(c)} = r. \tag{2.27}$$

PROOF. If we define $g_c(r) := f_c(r)/r$ for any $r \in (0,1]$, then we can use (2.2) (i) to show that the function $r \mapsto g_c(r)$ solves the ODE

$$\frac{g_c(r) - 1/2}{c - g_c(r)(g_c(r) - 1/2)} g_c'(r) = \frac{1}{r}, \qquad r \in (0,1]. \tag{2.28}$$

We first find the general solution of this ODE by integrating both sides of (2.28). In order to calculate the indefinite integral of the l.h.s., we perform the substitution $g = g_c(r)$ and apply the partial fraction decomposition

$$\frac{1/2 - g}{g^2 - g/2 - c} \overset{(2.26)}{=} \frac{A_+(c)}{g - g_+(c)} + \frac{A_-(c)}{g - g_-(c)}. \tag{2.29}$$

Integrating and then exponentiating both sides of (2.28), we obtain that the general solution of (2.28) satisfies the implicit equation $R(g_c(r)) = r$ for any $r \in (0,1]$, where

$$R(g) := \alpha^* \left(g - g_+(c)\right)^{A_+(c)} \left(g - g_-(c)\right)^{A_-(c)}, \quad g \in (g_+(c), +\infty) \tag{2.30}$$

for some positive constant α^*. Note that the function $g \mapsto R(g)$ is strictly decreasing (since both $A_+(c)$ and $A_-(c)$ are negative) and that it satisfies $\lim_{g\to g_+(c)} R(g) = +\infty$ as well as $\lim_{g\to\infty} R(g) = 0$. Therefore, the equation $R(g) = r$ has a unique solution g for any $r \in (0, 1]$. In order to identify the value of α^*, we observe that (2.2) (ii) is equivalent to $\lim_{r\to 0_+} g_c(r)r = \frac{1}{2}$, which is in turn equivalent to

$$\lim_{g\to\infty} g\alpha^* \left(g - g_+(c)\right)^{A_+(c)} \left(g - g_-(c)\right)^{A_-(c)} = \tfrac{1}{2}. \tag{2.31}$$

Noting that $A_+(c) + A_-(c) = -1$ (c.f. (2.26)), we obtain $\alpha^* = \frac{1}{2}$ using (2.31). □

LEMMA 27 (f_c is increasing and concave). *For any $c > 0$, the function f_c from Lemma 24 is twice continuously differentiable with $f_c(0) = \frac{1}{2}$, $f_c'(r) \geq 0$, and $f_c''(r) > 0$ $\big(r \in (0, 1]\big)$.*

PROOF. The facts that $f_c(0) = \frac{1}{2}$ and $f_c'(r) \geq 0$ are immediate from (2.2) (i) and (ii). To see that f_c is twice continuously differentiable with $f_c''(r) \geq 0$, we observe that by (2.2) (i),

$$f_c''(r) = \frac{\partial}{\partial r} \frac{c}{r^{-1} f_c(r) - \frac{1}{2}} = \frac{\partial}{\partial r} \frac{c}{g_c(r) - \frac{1}{2}}, \tag{2.32}$$

where g_c is the function in Lemma 26. Since the function in (2.30) is strictly decreasing, $g_c(r)$ is strictly decreasing, and hence the right-hand side of (2.32) is strictly positive for $r > 0$. □

Let us define

$$h(c) := \tfrac{1}{4} \left(\frac{\sqrt{1 + 32c} - \sqrt{1 + 16c}}{\sqrt{1 + 32c} + \sqrt{1 + 16c}} \right)^{\frac{1}{\sqrt{1+16c}}} \frac{1}{c}, \quad c \in (0, \infty). \tag{2.33}$$

LEMMA 28 (Right boundary condition). *Let $c \in (0, +\infty)$. The following conditions are equivalent:*

$$2c = f_c(1)^2 - \tfrac{1}{2} f_c(1) \tag{2.34}$$

$$f_c(1) = \tfrac{1}{4}\left(1 + \sqrt{1 + 32c}\right) \tag{2.35}$$

$$h(c) = 1 \tag{2.36}$$

PROOF. The positive solution of the quadratic equation (2.34) is (2.35). By Lemma 26, $f_c(1)$ is the unique element of $(g_+(c), +\infty)$ that satisfies

$$\tfrac{1}{2} \left(f_c(1) - g_+(c)\right)^{A_+(c)} \left(f_c(1) - g_-(c)\right)^{A_-(c)} = 1. \tag{2.37}$$

Now by (2.37) and $A_+(c) + A_-(c) = -1$, (2.35) is equivalent to

$$2\left(\sqrt{1+32c} - \sqrt{1+16c}\right)^{A_+(c)} \left(\sqrt{1+32c} + \sqrt{1+16c}\right)^{A_-(c)} = 1. \tag{2.38}$$

Finally, the equivalence of the condition (2.38) and (2.36) (c.f. (2.33)) follows using elementary algebra. □

LEMMA 29 (Existence and uniqueness of the positive root). *There exists exactly one* $c_2 \in (0, +\infty)$ *such that* $h(c_2) = 1$. *Moreover we have*

$$c_2 \in (0, \tfrac{1}{4}). \tag{2.39}$$

PROOF. Let us first observe that $\lim_{c \to 0_+} h(c) = 1$, thus h is a continuous function on $[0, +\infty)$ if we define $h(0) := 1$. Next we observe that

$$h(1/4) \overset{(2.33)}{=} \left(\frac{3-\sqrt{5}}{3+\sqrt{5}} \right)^{\frac{1}{\sqrt{5}}} < 1. \tag{2.40}$$

We will show that

$$\exists\, \widetilde{c} \in (0, +\infty) \; : \; h'(c) > 0 \text{ if } c \in (0, \widetilde{c}), \text{ but } h'(c) < 0 \text{ if } c \in (\widetilde{c}, +\infty). \tag{2.41}$$

Once we have this, the statement of Lemma 29 will follow from the facts that $h(0) = 1$ and $h(1/4) < 1$.

It remains to prove (2.41). Let us define

$$k(c) := \ln(h(c/16)), \qquad r(c) := (1+c)^{3/2} k'(c). \tag{2.42}$$

Let us observe that in order to prove (2.41), it is enough to prove

$$\exists\, \widehat{c} \in (0, +\infty) \; : \; r(c) > 0 \text{ if } c \in (0, \widehat{c}), \text{ but } r(c) < 0 \text{ if } c \in (\widehat{c}, +\infty), \tag{2.43}$$

where actually $\widehat{c} = 16\widetilde{c}$. It remains to prove (2.43). First note that we have

$$k'(c) = \frac{-c + (1+2c)^{-1/2} - 1}{c^2 + c} - \frac{1}{2}(1+c)^{-3/2} \ln\left(\frac{\sqrt{1+2c} - \sqrt{1+c}}{\sqrt{1+2c} + \sqrt{1+c}} \right), \tag{2.44}$$

thus $\lim_{c \to 0_+} k'(c) = +\infty$. Also note that $\exists\, c : k'(c) < 0$, since $k(0) = 0$ and $k(4) < 0$ by (2.40) and (2.42). These observations imply that the function $c \mapsto r(c)$ takes both positive and negative values. Thus in order to prove (2.43), it is enough to prove that $r : (0, +\infty) \to \mathbb{R}$ is a decreasing function.

$$r'(c) = \frac{\sqrt{1+c}}{2c^2(2c+1)^{3/2}} q(c), \quad \text{where} \qquad q(c) := \sqrt{2c+1}(2 - 2c^2 + 3c) - 2 - 6c. \tag{2.45}$$

It remains to check that $q(c) < 0$ for all $c > 0$. This readily follows after we observe that $q(0) = 0$, $q'(0) = -1$ and $q''(c) = \frac{-15c}{\sqrt{2c+1}}$ for any $c > 0$. □

REMARK 30. Although it is just elementary calculus, the proof of the uniqueness part of Lemma 29 is one of the trickiest of the paper. Since ultimately, the uniqueness of the nontrivial scale invariant fixed point of Theorem 12 hinges on this, one would like to find a more elegant and insightful proof. It is tempting to try and prove that the function h, or the function $c \mapsto ch(c)$, are either convex or concave on the entire positive axis, but this is *not true*. The function $c \mapsto f_c(1)^2 - \frac{1}{2} f_c(1)$ that occurs in (2.2) (iii) appears to be concave, but we have been unable to prove so.

2.5. *Non-trivial solution of the bivariate RDE .*

PROOF OF LEMMA 16. We apply Lemma 19 to the function f_{c_2}. Condition (i) is satisfied since

$$f_{c_2}(1) \stackrel{(2.35)}{=} \tfrac{1}{4}\left(1 + \sqrt{1 + 32c_2}\right) \stackrel{(2.39)}{<} \tfrac{1}{4}\left(1 + \sqrt{9}\right) = 1. \tag{2.46}$$

Conditions (ii), (iii) and (v) of Lemma 19 are satisfied by Lemma 27, so it remains to check condition (iv), which requires $2f'_{c_2}(1) = f_{c_2}(1)$. Using (2.21) (ii), we can rewrite this as

$$\frac{2c_2}{f_{c_2}(1) - \frac{1}{2}} = f_{c_2}(1), \tag{2.47}$$

which is satisfied by Lemmas 28 and 29. □

REMARK 31. Formula (2.47) shows that condition (2.2) (iii) is equivalent to the statement that the measure $\rho^{(2)}$ associated with f puts no mass on the diagonal $\big\{(r, r) : r \in [0, 1]\big\}$.

3. Frozen percolation.

3.1. *Outline.* In the previous section, we have proved Theorem 12, which implies that the RTP $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map χ from (1.30) and law ρ from (1.31) is nonendogenous. In the present section, we provide the proofs of our remaining results, which are Theorems 2, 3, 6, and 7, as well as Lemma 8, Proposition 9, and Lemma 11.

In Subsection 3.2, we show that there is a one-to-one correspondence between solutions to the RDEs (1.12) and (1.32), under which the measure ν from (1.16) corresponds to the measure ρ from (1.31). We also prove a correspondence between solutions to the associated bivariate RDEs and use this to derive Theorem 7 from Theorem 12.

In Subsection 3.3, we classify all solutions to the RDE (1.32). Using results from the preceding subsection, this also leads to a description of general solutions to the RDE (1.12).

Theorem 6 is proved in Subsections 3.4 and 3.5. In Subsection 3.4, we use the classification of solutions to (1.32) to prove a version of Theorem 6 for frozen percolation on the MBBT. In Subsection 3.5 this is then translated into a result for the oriented binary tree using a coupling between two RTPs, one for frozen percolation on the MBBT, and the other for the oriented binary tree.

In Subsection 3.6, we prove Lemma 8 as well as Proposition 9 and Lemma 11 about scale invariance of the MBBT. Lemma 11 allows us to identify the nontrivial solution $\rho_2^{(2)}$ of the bivariate RDE from Theorem 12 as $\underline{\rho}^{(2)}$. Using results from Subsection 3.3, we use this to obtain an explicit formula for $\underline{\nu}^{(2)}$ based on our formula for $\underline{\rho}^{(2)}$.

In Subsection 3.7 we mainly rely on arguments from [Ald00] to translate results about frozen percolation on the oriented binary tree into results about frozen percolation on the 3-regular tree. In particular, we derive Theorem 2 from Theorem 6 and Theorem 3 from Theorem 7.

3.2. *Equivalence of RDEs.* In this subsection, we show that there is a one-to-one correspondence between solutions to the RDEs (1.12) and (1.32), under which the measure ν from (1.16) corresponds to the measure ρ from (1.31). We also prove a correspondence between solutions to the associated bivariate RDEs and use this to derive Theorem 7 from Theorem 12. We start with a simple observation.

LEMMA 32 (No burning before the critical point). *Every solution μ to the RDE (1.12) is concentrated on $I' := [\frac{1}{2}, 1] \cup \{\infty\}$.*

PROOF. If μ solves the RDE (1.12), then we can construct an RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map γ from (1.5) and μ. Then by (1.15),

$$\mu\big([0,\tfrac{1}{2}]\big) = \mathbb{P}[X_\varnothing \leq \tfrac{1}{2}] \leq \mathbb{P}[X^\uparrow_\varnothing \leq \tfrac{1}{2}] = \mathbb{P}\big[\varnothing \overset{\mathbb{T}^{1/2}\setminus\mathbb{F}}{\longrightarrow} \infty\big] \leq \mathbb{P}\big[\varnothing \overset{\mathbb{T}^{1/2}}{\longrightarrow} \infty\big] = 0, \tag{3.1}$$

where the last equality follows from the fact that a branching process with a binomial offspring distribution with parameters $2, \frac{1}{2}$ is critical and hence dies out a.s. $\square$

The next lemma, which is the first main result of the present subsection, says that there is a one-to-one correspondence between solutions to the RDEs (1.12) and (1.32). The idea behind the proof (and in particular the occurrence of the geometric distribution in (3.6)) will become more clear in Section 3.5 below.

LEMMA 33 (Equivalence of RDEs). *Let $I' := [\frac{1}{2}, 1] \cup \{\infty\}$ and let $H : I \to I'$ be the bijection defined by $H(t) := 1/(2-t)$ $(t \in [0,1])$ and $H(\infty) := \infty$. If μ solves the RDE (1.32), then its image under the map H solves the RDE (1.12). Conversely, if μ' solves the RDE (1.12), then its image under the map H^{-1} solves the RDE (1.32).*

PROOF. Let $T_{\rm y}$ be defined as in (1.7) but for the map χ in (1.30), i.e.,

$$T_{\rm y}(\mu) := \text{ the law of } \chi[\tau_\varnothing, \kappa_\varnothing](Y_1, Y_2) \tag{3.2}$$

where Y_1, Y_2 are i.i.d. with law μ and independent of $(\tau_\varnothing, \kappa_\varnothing)$. Then we can write

$$T_{\rm y} = \tfrac{1}{2} T_\Phi + \tfrac{1}{2} T_{\tt min}, \tag{3.3}$$

where

$$T_\Phi(\mu) := \text{ the law of } \Phi[\tau_\varnothing](Y_1) \quad \text{and} \quad T_{\tt min}(\mu) := \text{ the law of } Y_1 \wedge Y_2, \tag{3.4}$$

and $\Phi : [0,1] \times I \to I$ denotes the function

$$\Phi[t](x) := \begin{cases} x & \text{if } x > t, \\ \infty & \text{if } x \leq t. \end{cases} \tag{3.5}$$

Note that the map T_Φ is linear, but $T_{\tt min}$ is not. Let us define

$$T_{\rm z} := \sum_{n=1}^{\infty} 2^{-n} T_\Phi^{n-1} T_{\tt min}. \tag{3.6}$$

We claim that μ is a fixed point of $T_{\rm y}$ if and only if it is a fixed point of $T_{\rm z}$. Indeed, $T_{\rm y}(\mu) = \mu$ implies $T_{\rm min}(\mu) = 2\mu - T_\Phi(\mu)$ and hence, using the linearity of T_Φ,

$$T_{\rm z}(\mu) = \sum_{n=1}^{\infty} 2^{-n} T_\Phi^{n-1} \big(2\mu - T_\Phi(\mu)\big) = \mu. \tag{3.7}$$

Conversely, since

$$T_{\rm z} = \tfrac{1}{2} T_{\tt min} + \tfrac{1}{2} T_\Phi \circ T_{\rm z} \tag{3.8}$$

$T_{\rm z}(\mu)=\mu$ implies $\mu=\frac{1}{2}T_{\tt min}(\mu)+\frac{1}{2}T_\Phi(\mu)=T_{\rm y}(\mu)$.

We observe that

$$T_{\rm z}(\mu):=\text{ the law of } \Phi[\tau_1]\circ\cdots\circ\Phi[\tau_N](Y_1\wedge Y_2)=\gamma[\tau_1\vee\cdots\vee\tau_N](Y_1,Y_2), \tag{3.9}$$

where $(\tau_k)_{k\geq 1}$ are uniformly distributed on $[0,1]$, the r.v.'s Y_1,Y_2 have law μ, the r.v. N is geometrically distributed with $\mathbb{P}[N=n]=2^{-n-1}$ $(n\geq 0)$, and all r.v.'s are independent. Since

$$\mathbb{P}\big[\tau_1\vee\cdots\vee\tau_N\leq t\big]=\sum_{n=0}^\infty 2^{-n-1}t^n=\frac{\frac{1}{2}}{1-\frac{1}{2}t}=H(t)\qquad\big(t\in[0,1]\big), \tag{3.10}$$

we have that $\tau:=H(\tau_1\vee\cdots\vee\tau_N)$ satisfies $\mathbb{P}[\tau=\frac{1}{2}]=\mathbb{P}[N=0]=\frac{1}{2}$ and

$$\mathbb{P}\big[\tau<t\big]=\mathbb{P}\big[\tau_1\vee\cdots\vee\tau_N<H^{-1}(t)\big]=H\big(H^{-1}(t)\big)=t\qquad\big(t\in[\tfrac{1}{2},1]\big). \tag{3.11}$$

Then, using the fact that

$$\gamma[H(t)]\big(H(x),H(y)\big)=H\big(\gamma[t](x,y)\big)\qquad\big(x,y\in I,\ t\in[0,1]\big) \tag{3.12}$$

and using also Lemma 32, we see that the law μ of an I-valued random variable Y solves the RDE $T_{\rm z}(\mu)=\mu$ or equivalently

$$Y\stackrel{\rm d}{=}\gamma[\tau_1\vee\cdots\vee\tau_N](Y_1,Y_2), \tag{3.13}$$

if and only if $X:=H(Y)$, $X_1:=H(Y_1)$, and $X_2:=H(Y_2)$ solve the RDE (1.12). □

LEMMA 34 (Equivalence of special solutions). *The measure ν in (1.16) is the image of the measure ρ in (1.31) under the map H.*

PROOF. Since $H^{-1}(t)=2-1/t$ $(t\in[\frac{1}{2},1])$ is the inverse of $H(t):=1/(2-t)$ $(t\in[0,1])$, we see that

$$\rho\big([0,H^{-1}(t)]\big)=\tfrac{1}{2}H^{-1}(t)=1-\frac{1}{2t}=\nu\big([0,t]\big)\qquad\big(t\in[\tfrac{1}{2},1]\big), \tag{3.14}$$

which shows that ν is the image of ρ under H. □

We next turn our attention to the bivariate RDEs.

LEMMA 35 (Equivalence of bivariate RDEs). *Let $H:I\to I'$ be the map defined in Lemma 33. Let $T^{(2)}_{\rm x}$ and $T^{(2)}_{\rm y}$ be the bivariate maps defined as in (1.8) for the maps γ in (1.5) and χ in (1.30), respectively. Then a measure $\mu^{(2)}\in\mathcal{P}(I^2)$ solves the bivariate RDE $T^{(2)}_{\rm y}(\mu^{(2)})=\mu^{(2)}$ if and only if its image $\nu^{(2)}$ under the map $(y_1,y_2)\mapsto\big(H(y_1),H(y_2)\big)$ solves the bivariate RDE $T^{(2)}_{\rm x}(\nu^{(2)})=\nu^{(2)}$.*

PROOF. Let $T_H : \mathcal{P}(I) \to \mathcal{P}(I')$ be the function that maps a measure on I to its image under the map H. The proof of Lemma 33 consisted of showing that for any $\mu \in \mathcal{P}(I)$, one has $T_{\rm y}(\mu) = \mu$ if and only if $T_{\rm z}(\mu) = \mu$, and moreover $T_{\rm x} T_H = T_H T_{\rm z}$. With exactly the same proof, these statements remain true if we replace the maps $T_{\rm x}, T_{\rm y}, T_{\rm z}$, and T_H with their bivariate versions $T^{(2)}_{\rm x}, T^{(2)}_{\rm y}, T^{(2)}_{\rm z}$, and $T^{(2)}_H$. It follows that $\mu^{(2)} \in \mathcal{P}(I^2)$ solves $T^{(2)}_{\rm y}(\mu^{(2)}) = \mu^{(2)}$ if and only if $\nu^{(2)} := T^{(2)}_H(\mu^{(2)})$ solves $T^{(2)}_{\rm x}(\nu^{(2)}) = \nu^{(2)}$, which is the claim of the lemma. □

Our results so far allow us to prove Theorem 7.

PROOF OF THEOREM 7. By Theorem 12, the bivariate map $T^{(2)}_{\rm y}$ has a fixed point $\rho^{(2)}_2 \in \mathcal{P}(I^2)_\rho$ that is not concentrated on the diagonal $\{(y,y) : y \in I\}$. Let $\nu^{(2)}_2$ denote the image of $\rho^{(2)}_2$ under the map $(y_1, y_2) \mapsto \big(H(y_1), H(y_2)\big)$. Then $\nu^{(2)}_2 \in \mathcal{P}(I^2)_\nu$ by Lemma 34. By Lemma 35, $\nu^{(2)}_2$ is a fixed point of $T^{(2)}_{\rm x}$. Since $\nu^{(2)}_2$ is not concentrated on the diagonal, Theorem 4 (i) and (iii) imply that the RTP associated with ν is nonendogenous. □

Each solution μ to an RDE defines an RTP, which through (1.10) defines a special solution $\underline{\mu}^{(2)}$ to the corresponding bivariate RDE. In particular, we define $\underline{\nu}^{(2)}$ and $\underline{\rho}^{(2)}$ in this way starting from the measures ρ and ν defined in (1.31) and (1.16). The final result of this subsection relates these measures to each other.

LEMMA 36 (Nontrivial solutions to bivariate RDE). *Let $(Y_\varnothing, Y'_\varnothing)$ be a random variable with law $\underline{\rho}^{(2)}$ and let H be the function from (3.56). Then $\big(H(Y_\varnothing), H(Y'_\varnothing)\big)$ has law $\underline{\nu}^{(2)}$.*

PROOF. We will use a characterization of $\underline{\rho}^{(2)}$ and $\underline{\nu}^{(2)}$ from [MSS18]. We first need some abstract definitions. Let I be a Polish space. If ξ is a random probability law on I, and $\eta \in \mathcal{P}(\mathcal{P}(I))$ is the law of ξ, then

$$\eta^{(n)} := \mathbb{E}\big[\underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}}\big] \tag{3.15}$$

is called the *n-th moment measure* of η. In [MSS18, Lemma 2] it was shown that for each map T of the form (1.7), there exists a *higher level map* $\check{T} : \mathcal{P}(\mathcal{P}(I)) \to \mathcal{P}(\mathcal{P}(I))$ that is uniquely characterised by

$$\check{T}(\eta)^{(n)} = T^{(n)}(\eta^{(n)}) \qquad \big(n \geq 1,\ \eta \in \mathcal{P}(\mathcal{P}(I))\big), \tag{3.16}$$

where $T^{(n)}$ is the associated *n-variate map*. Let ν be a solution to the RDE (1.6) and let $\mathcal{P}(\mathcal{P}(I))_\nu$ denote the space of all $\eta \in \mathcal{P}(\mathcal{P}(I))$ with $\eta^{(1)} = \nu$. In [MSS18, Prop 3], it was shown that the set $\{\eta \in \mathcal{P}(\mathcal{P}(I))_\nu : \check{T}(\eta) = \eta\}$, equipped with the *convex order*, has a unique minimal element $\underline{\nu}$ and maximal element $\overline{\nu}$. Moreover, by [MSS18, Lemma 2 and Props 3 and 4], the measures $\underline{\nu}^{(2)}$ and $\overline{\nu}^{(2)}$ from (1.9) and (1.10) are the second moment measures of $\underline{\nu}$ and $\overline{\nu}$.

We now return to our special setting with $I = [0,1] \cup \{\infty\}$. Let $T_{\rm x}, T_{\rm y}$, and T_H be as in the proof of Lemma 35. In Lemma 33, we have proved that $\mu \in \mathcal{P}(I)$ satisfies $T_{\rm y}(\mu) = \mu$ if and only if $\nu := T_H(\mu)$ satisfies $T_{\rm x}(\nu) = \nu$. In Lemma 35, we have shown that the same is true for the bivariate maps $T^{(2)}_{\rm x}, T^{(2)}_{\rm y}$, and $T^{(2)}_H$. The argument carries over without a change for general n-variate maps and therefore, by (3.16), the statement is also true for the associated higher-level maps $\check{T}_{\rm x}, \check{T}_{\rm y}$, and $\check{T}_H$. In particular, using also Lemma 34, we obtain that the image of the set

$$A := \big\{\eta \in \mathcal{P}(\mathcal{P}(I))_\rho : \check{T}_{\rm y}(\eta) = \eta\big\} \tag{3.17}$$

under the higher-level map $\check{T}_H$ is the set

$$B := \big\{\eta \in \mathcal{P}(\mathcal{P}(I')))_\nu : \check{T}_{\rm x}(\eta) = \eta\big\}. \tag{3.18}$$

Since by [MSS18, Prop 3], higher-level maps are monotone w.r.t. the convex order, $\check{T}_H$ maps the minimal element of A, which is $\underline{\rho}$, into the minimal element of B, which is $\underline{\nu}$. By (3.16), this implies that the bivariate map $T^{(2)}_H$ maps $\underline{\rho}^{(2)}$ to $\underline{\nu}^{(2)}$, which is the claim we wanted to prove. □

3.3. *General solution of the RDE.* In this subsection, we classify all solutions to the RDE (1.32). Through Lemma 33, this then also implies the form of a general solution of the RDE (1.12), significantly extending [Ald00, Lemma 3], who only considered solutions without atoms in $[0,1]$.

Let $O \subset (0,1]$ be open. Then O is a countable union of disjoint open intervals $(O_k)_{0 \le k < n+1}$ for some $0 \le n \le \infty$ (with $\infty + 1 := \infty$). Without loss of generality we can assume that $\emptyset \neq O_k \subset (0,1)$ for all $1 \le k < n+1$ while either $1 \in O_0$ or $O_0 = \emptyset$. We let $x_k \in (0,1)$ and $c_k > 0$ denote the center and radius of O_k, respectively, i.e., $O_k = (x_k - c_k, x_k + c_k)$, and we choose $x_0 \in (0,1] \cup \{2\}$ and $c_0 > 0$ such that $O_0 = (x_0 - c_0, x_0 + c_0) \cap (0,1]$. We define a measure μ on $[0,1]$ by

$$\mu(\mathrm{d}t) := \tfrac{1}{2} 1_{[0,1]\setminus O}(t)\mathrm{d}t + 1_{\{x_0 \neq 2\}} c_0 \delta_{x_0}(\mathrm{d}t) + \sum_{k=1}^{n} c_k \delta_{x_k}(\mathrm{d}t). \tag{3.19}$$

It is easy to see that $\mu([0,1]) \leq 1$, so we can unambiguously extend μ to a probability measure on $I = [0,1] \cup \{\infty\}$. We will prove the following result.

Proposition 37 (General solution to RDE). *The probability measure μ defined in (3.19) solves the RDE (1.32), and conversely, every solution of (1.32) is of this form.*

We need one preparatory lemma.

Lemma 38 (RDE for MBBT). *A probability measure μ on I solves the RDE (1.32) if and only if*

$$\int_{[0,t]} \mu(\mathrm{d}s)\, s = \mu\big([0,t]\big)^2 \qquad \big(t \in [0,1]\big). \tag{3.20}$$

Proof. Let Φ be the function defined in (3.5). Then

$$\chi[\tau,1](x,y) = \Phi[\tau](x) \quad \text{and} \quad \chi[\tau,2](x,y) = x \wedge y. \tag{3.21}$$

Using this and the fact that the function $F(t) := \mu\big([0,t]\big)$ $(t \in [0,1])$ uniquely characterizes μ, we see that (1.32) is equivalent to

$$\begin{aligned} F(t) = \mathbb{P}[Y_\varnothing \leq t] &= \tfrac{1}{2}\int_0^1 \mathrm{d}s\, \mathbb{P}\big[\Phi[s](Y_1) \leq t\big] + \tfrac{1}{2}\mathbb{P}[Y_1 \wedge Y_2 \leq t] \\ &= \tfrac{1}{2}\int_0^1 \mathrm{d}s\, \mathbb{P}[s < Y_1 \leq t] + \tfrac{1}{2}\big(1 - \mathbb{P}[Y_1 > t]^2\big) \\ &= \tfrac{1}{2}\int_0^t \mathrm{d}s\, \big\{F(t) - F(s)\big\} + \tfrac{1}{2}\big(1 - (1 - F(t))^2\big) \\ &= \tfrac{1}{2}tF(t) - \tfrac{1}{2}\int_0^t \mathrm{d}s\, F(s) + F(t) - \tfrac{1}{2}F(t)^2, \end{aligned} \tag{3.22}$$

which can be rewritten as

$$\big(t - F(t)\big)F(t) = \int_0^t \mathrm{d}s\, F(s) \qquad \big(t \in [0,1]\big). \tag{3.23}$$

Using the fact that

$$tF(t) = \int_{[0,t]} \mathrm{d}(sF(s)) = \int_{[0,t]} s\, \mathrm{d}F(s) + \int_{[0,t]} F(s)\, \mathrm{d}s, \tag{3.24}$$

we can rewrite (3.23) as (3.20). □

PROOF OF PROPOSITION 37. We first prove that the measure in (3.19) solves (3.20). With x_k and c_k as in (3.19), we will prove that the measure μ' on $[0,\infty)$ defined by

$$\mu'(\mathrm{d}t) := \tfrac{1}{2}1_{[0,\infty)\backslash O}(t)\mathrm{d}t + \sum_{k=0}^{n} c_k \delta_{x_k}(\mathrm{d}t) \tag{3.25}$$

solves (3.20) for all $t \geq 0$. Restricting μ' to $[0,1]$ we then see that μ satisfies (3.20) for all $t \in [0,1]$.

If $\mu'(\mathrm{d}s) := \frac{1}{2}\mathrm{d}s$ then the left-hand side of (3.20) is $\frac{1}{2}\cdot\frac{1}{2}t^2$ while the right-hand side is $(\frac{1}{2}t)^2$, so (3.20) holds. Next, if we modify μ' by concentrating all the mass in an interval of the form $(x-c, x+c)$ in the middle of that interval, then (3.20) remains true for all $t \leq x-c$ and $t \geq x+c$. Applying this observation inductively and taking the limit, we see that μ' solves (3.20) for all $t \in [0,\infty)\backslash O$. But the left- and right-hand sides of (3.20) are constant on the intervals $[x_k - c_k, x_k)$ and $[x_k, x_k + c_k]$ $(k \geq 0)$ so (3.20) holds for all $t \geq 0$.

The proof that all solutions of (3.20) are of the form (3.19) goes in a number of steps. Taking increasing limits, we observe that (3.20) implies

$$\int_{[0,t)} \mu(\mathrm{d}s)\, s = \mu\big([0,t)\big)^2 \qquad \big(t \in (0,1]\big). \tag{3.26}$$

We next claim that:

(3.27)

If μ solves (3.20) and $\mu\big([0,t)\big) = \frac{1}{2}u$ with $0 \leq t \leq u$, then $\mu\big([t,u]\big) = 0$.

Indeed, we obtain from (3.20) that

$$\int_{[0,t)} \mu(\mathrm{d}s)\, s + \int_{[t,u]} \mu(\mathrm{d}s)\, s = \big[\mu\big([0,t)\big) + \mu\big([t,u]\big)\big]^2, \tag{3.28}$$

which using (3.26) and our assumption that $\mu\big([0,t)\big) = \frac{1}{2}u$ yields

$$\mu\big([t,u]\big)^2 = \int_{[t,u]} \mu(\mathrm{d}s)\, s - u\mu\big([t,u]\big) \leq 0. \tag{3.29}$$

Our next claim is that:

$$\begin{array}{l}\text{If } \mu \text{ solves (3.20) and } c := \mu(\{t\}) > 0 \text{ for some } t \in [0,1], \\ \text{then } c = 2\big[\tfrac{1}{2}t - \mu\big([0,t)\big)\big].\end{array} \tag{3.30}$$

Indeed, (3.20) implies

$$\int_{[0,t)} \mu(\mathrm{d}s)\, s + ct = \big[\mu\big([0,t)\big) + c\big]^2, \tag{3.31}$$

which using (3.26) implies

$$ct = 2c\mu\big([0,t)\big) + c^2. \tag{3.32}$$

Using our assumption that $c > 0$, we arrive at (3.30). Let F denote the function $F(t) := \mu\big([0,t]\big)$ $(t \in [0,1])$. We need one more claim, which says that:

$$\begin{array}{l}\text{If } \mu \text{ solves (3.20) and has no atoms in } [s,u),\\ \text{then } \mu\big([0,s)\big) < \tfrac{1}{2}s \text{ implies } \mu\big([s,u)\big) = 0.\end{array} \tag{3.33}$$

Indeed, if μ has no atoms in $[s,u)$, then the function $F(t) := \mu\big([0,t]\big)$ $(t \in [0,1])$ solves

$$t\mu(\mathrm{d}t) = t\mathrm{d}F(t) \overset{(3.20)}{=} \mathrm{d}\big(F(t)^2\big) = 2F(t)\mathrm{d}F(t) = 2F(t)\mu(\mathrm{d}t) \tag{3.34}$$

on $[s,u)$, which shows that the restriction of μ to $[s,u)$ is concentrated on $\{t \in [s,u) : F(t) = \frac{1}{2}t\}$. Now if (3.33) would not hold, then $\tau := \inf\{t \in [s,u) : F(t) = F(s) + \varepsilon\}$ would satisfy $s < \tau < u$ for some $\varepsilon > 0$. But then $\mu\big([s,\tau]\big) = 0$ and hence $F(\tau) = F(s)$, which is a contradiction.

Claim (3.27) says that if $F(t) > \frac{1}{2}t$, then F must stay constant until the next time when $F(t) = \frac{1}{2}t$. Claim (3.33) says that if $F(t) < \frac{1}{2}t$, then F must stay constant until the next time when it makes a jump. Claim (3.30) says that if F makes a jump at time t, then it jumps from $\frac{1}{2}t - \frac{1}{2}c$ to $\frac{1}{2}t + \frac{1}{2}c$ for some $c > 0$. Using these facts, it is easy to see that μ must be of the form (3.19). □

3.4. *Frozen percolation on the MBBT.* In this subsection, we prove a version of Theorem 6 for frozen percolation on the MBBT, from which in the next subsection we will derive Theorem 6. We first need some definitions concerning general RTPs corresponding to the RDE (1.32), similar to those introduced in Subsection 1.4 for general RTPs corresponding to the RDE (1.12).

Let $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an RTP corresponding to the map χ from (1.30) and a general solution μ to the RDE (1.32). Generalising the definition in (1.25), we set

$$\mathbb{S}_{\mathbf{i}} := \big\{\mathbf{i}j_1\cdots j_n \in \mathbb{T} : j_m \le \kappa_{\mathbf{i}j_1\cdots j_{m-1}}\ \forall 1 \le m \le n\big\}. \tag{3.35}$$

Modifying the definition of $\mathbb{T}^t$ in (1.13), in the present context, we set

$$\begin{array}{c}\mathbb{T}^t := \big\{\mathbf{i} \in \mathbb{T} : \kappa_{\mathbf{i}} = 2 \text{ or } \tau_{\mathbf{i}} \le t\big\}, \quad \mathbb{S}^t_{\mathbf{i}} := \mathbb{T}^t \cap \mathbb{S}_{\mathbf{i}},\\ \text{and} \quad \mathbb{F}_{\mathrm{y}} := \big\{\mathbf{i} \in \mathbb{T} : \kappa_{\mathbf{i}} = 1,\ \tau_{\mathbf{i}} \ge Y_{\mathbf{i}1}\big\}.\end{array} \tag{3.36}$$

Similar to (1.14), we define I-valued random variables $(Y^{\uparrow}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ by

$$Y^{\uparrow}_{\mathbf{i}} := \inf\big\{t\in[0,1] : \mathbf{i} \stackrel{\mathbb{S}^t_{\mathbf{i}}\backslash\mathbb{F}_{\rm y}}{\longrightarrow} \infty\big\}, \tag{3.37}$$

with $\inf\emptyset := \infty$. Note that if $\mathbf{i}\in\mathbb{S}$, then in (3.37) we can equivalently replace $\mathbb{S}^t_{\mathbf{i}}$ by $\mathbb{S}^t_{\varnothing} =: \mathbb{S}^t$. At time $t\in[0,1]$, we call points in $\mathbb{T}^t\backslash\mathbb{F}_{\rm y}$ *open*, points in $\mathbb{T}^t\cap\mathbb{F}_{\rm y}$ *frozen*, and all other points in $\mathbb{T}$ *closed*. We call $\tau_{\mathbf{i}}$ the *activation time* of $\mathbf{i}$ and refer to $Y_{\mathbf{i}}$ and $Y^{\uparrow}_{\mathbf{i}}$ as its *burning time* and *percolation time*, respectively. Note that our modified definition of $\mathbb{T}^t$ has the effect that *branching points*, i.e., points $\mathbf{i}$ for which $\kappa_{\mathbf{i}}=2$, are always open. The remaining *blocking points*, i.e., points $\mathbf{i}$ for which $\kappa_{\mathbf{i}}=1$ are initially closed. At its activation time, a blocking point $\mathbf{i}$ either freezes or opens, depending on whether at that moment $\mathbf{i}1$ is burnt or not.

It follows from the inductive relation (1.29) that if $\kappa_{\mathbf{i}}=1$, then $Y_{\mathbf{i}}>\tau_{\mathbf{i}}$, i.e., a blocking point can only burn after its activation time. We see from the definition of $\mathbb{F}_{\rm y}$ in (3.36) and the definition of the map χ in (1.30) that if a blocking point $\mathbf{i}$ burns at some time $Y_{\mathbf{i}}\in[0,1]$, then $\mathbf{i}$ must be open at that time. Formula (1.30) moreover implies that if a point $\mathbf{i}\in\mathbb{T}$ burns at some time $Y_{\mathbf{i}}\in[0,1]$, then starting at $\mathbf{i}$ there must be a ray in $\mathbb{S}_{\mathbf{i}}$ consisting of points that burn at the same time as $\mathbf{i}$. By our earlier remark and since branching points are always open, such a ray must be open, which proves that (compare (1.15))

$$Y^{\uparrow}_{\mathbf{i}} \leq Y_{\mathbf{i}} \quad \text{a.s.} \quad (\mathbf{i}\in\mathbb{T}). \tag{3.38}$$

The next proposition says that the opposite inequality holds only if μ is the special solution ρ to the RDE defined in (1.31).

PROPOSITION 39 (Percolation probability). *Let $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}},Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an RTP corresponding to the map χ from (1.30) and a solution μ to the RDE (1.32). Then*

$$\mathbb{P}\big[Y^{\uparrow}_{\mathbf{i}}\leq t\big] = F(t)\vee\big(t-F(t)\big) \qquad \big(t\in[0,1]\big), \tag{3.39}$$

where $F(t):=\mu\big([0,t]\big)$ $(t\in[0,1])$. Moreover, one has $Y^{\uparrow}_{\varnothing}=Y_{\varnothing}$ a.s. if and only if μ is the measure ρ in (1.31).

The proof of Proposition 39 needs some preparations. We will be interested in the law of the open connected component of the root conditional on the root not being burnt. In the next lemma we condition on the origin not being burnt and calculate the probability that (i) the root is a branching point, (ii) the root is a blocking point and its descendant is not burnt,

(iii) the root is a blocking point and its descendant is burnt. We show that conditional on the event (ii), the activation time of the root is uniformly distributed.

LEMMA 40 (Law conditioned on not being burnt). *Let* $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ *be an RTP corresponding to the map* χ *from (1.30) and a solution* μ *to the RDE (1.32). Then*

$$\begin{aligned}
&\text{(i)} && \mathbb{P}\big[\kappa_\varnothing = 2 \,\big|\, Y_\varnothing > t\big] = \tfrac{1}{2}\big(1 - F(t)\big),\\
&\text{(ii)} && \mathbb{P}\big[\kappa_\varnothing = 1,\ Y_1 > t \,\big|\, Y_\varnothing > t\big] = \tfrac{1}{2},\\
&\text{(iii)} && \mathbb{P}\big[\kappa_\varnothing = 1,\ Y_1 \leq t \,\big|\, Y_\varnothing > t\big] = \tfrac{1}{2}F(t),
\end{aligned} \tag{3.40}$$

where $F(t) := \mu\big([0,t]\big)$ $(t \in [0,1])$. *Moreover,*

$$\mathbb{P}\big[\tau_\varnothing \leq s \,\big|\, \kappa_\varnothing = 1,\ Y_1 > t,\ Y_\varnothing > t\big] = s \qquad \big(s, t \in [0,1]\big). \tag{3.41}$$

PROOF. One has

$$\begin{aligned}
\mathbb{P}\big[\kappa_\varnothing = 2,\ Y_\varnothing > t\big] &= \mathbb{P}\big[\kappa_\varnothing = 2,\ Y_1 > t,\ Y_2 > t\big] = \tfrac{1}{2}\big(1 - F(t)\big)^2,\\
\mathbb{P}\big[\kappa_\varnothing = 1,\ Y_1 > t,\ Y_\varnothing > t\big] &= \tfrac{1}{2}\mathbb{P}[Y_1 > t] = \tfrac{1}{2}\big(1 - F(t)\big).
\end{aligned} \tag{3.42}$$

Dividing by $\mathbb{P}[Y_\varnothing > t] = 1 - F(t)$ yields (3.40) (i) and (ii), and the remaining formula follows since the total probability is one. Since $\kappa_\varnothing = 1$ and $Y_1 > t$ a.s. imply $Y_\varnothing > t$, and since $\tau_\varnothing$ is independent of $Y_1, \kappa_\varnothing$ and uniformly distributed, we also obtain (3.41). □

For $t \in [0,1]$, we inductively define $(\mathbb{O}^t_n)_{n\geq 0}$ by $\mathbb{O}^t_0 := \{\varnothing\}$ and

$$\mathbb{O}^t_n := \big\{\mathbf{i}j : \mathbf{i} \in (\mathbb{O}^t_{n-1} \cap \mathbb{T}^t)\backslash\mathbb{F}_{\mathrm{y}},\ 1 \leq j \leq \kappa_{\mathbf{i}}\big\}. \tag{3.43}$$

We call $\mathbb{O}^t := \bigcup_{n=0}^\infty \mathbb{O}^t_n$ the *open component of the root*. Note that $\mathbb{O}^t_n$ consists of all descendants of open elements of $\mathbb{O}^t_{n-1}$, while elements of $\mathbb{O}^t_{n-1}$ that are closed or frozen produce no offspring. As a result, the root percolates at time $t \in [0,1]$ if and only if $\mathbb{O}^t$ is infinite. The next lemma says that conditional on the event that the root is not burnt, $(\mathbb{O}^t_n)_{n\geq 0}$ is a branching process that can be subcritical, critical, or supercritical, depending on t and our choice of the solution μ to the RDE (1.32).

LEMMA 41 (The open unburnt component of the root). *Fix* $t \in [0,1]$ *and write* $\mathbb{O}^t_n = \{\mathbf{i}j : \mathbf{i} \in \mathbb{O}^t_{n-1},\ 1 \leq j \leq \lambda^t_{\mathbf{i}}\}$ *with* $\lambda^t_{\mathbf{i}} \in \{0,1,2\}$. *If* $(\mathbb{U}_k)_{0\leq k<n}$ *is a possible realization of* $(\mathbb{O}^t_k)_{0\leq k<n}$*, then conditional on the event* $\mathcal{A}^t := \{Y_\varnothing >$

$t,\ (\mathbb{O}^t_k)_{0\leq k<n}=(\mathbb{U}_k)_{0\leq k<n}\big\}$, the random variables $(\lambda^t_{\mathbf{i}})_{\mathbf{i}\in\mathbb{U}_{n-1}}$ are i.i.d. with law

$$\begin{gathered}\mathbb{P}[\lambda^t_{\mathbf{i}}=0\,|\,\mathcal{A}^t]=\tfrac{1}{2}\big(1-t+F(t)\big),\quad \mathbb{P}[\lambda^t_{\mathbf{i}}=1\,|\,\mathcal{A}^t]=\tfrac{1}{2}t,\\ \mathbb{P}[\lambda^t_{\mathbf{i}}=2\,|\,\mathcal{A}^t]=\tfrac{1}{2}\big(1-F(t)\big),\end{gathered}\tag{3.44}$$

where $F(t):=\mu\big([0,t]\big)$ $(t\in[0,1])$.

Proof. Fix $t\in[0,1]$. We claim that $Y_\varnothing>t$ implies $Y_{\mathbf{i}}>t$ for all $\mathbf{i}\in\mathbb{O}^t$. Indeed, if $\mathbf{i}\in\mathbb{O}^t_{n-1}$ is open and not burnt, then all its descendants must be unburnt, while elements that are not open have no descendants in $\mathbb{O}^t_n$, so the claim follows by induction.

Fix $(\mathbb{U}_k)_{0\leq k<n}$ and define $\mathcal{A}^t$ as in the lemma, which by what we have just proved is the same as the event

$$\mathcal{A}^t=\big\{Y_{\mathbf{i}}>t\ \forall\mathbf{i}\in\mathbb{U},\ (\mathbb{O}^t_k)_{0\leq k<n}=(\mathbb{U}_k)_{0\leq k<n}\big\},\tag{3.45}$$

where $\mathbb{U}:=\bigcup_{0\leq k<n}\mathbb{U}_k$. By Lemma 40, independently for each $\mathbf{i}\in\mathbb{U}_{n-1}$,

$$\begin{array}{ll}\text{(i)} & \mathbb{P}\big[\kappa_{\mathbf{i}}=2\,\big|\,\mathcal{A}^t\big]=\tfrac{1}{2}\big(1-F(t)\big),\\ \text{(ii)} & \mathbb{P}\big[\kappa_{\mathbf{i}}=1,\ \tau_{\mathbf{i}}\leq t,\ Y_{\mathbf{i}1}>t\,\big|\,\mathcal{A}^t\big]=\tfrac{1}{2}t,\\ \text{(iii)} & \mathbb{P}\big[\kappa_{\mathbf{i}}=1,\ \tau_{\mathbf{i}}>t,\ Y_{\mathbf{i}1}>t\,\big|\,\mathcal{A}^t\big]=\tfrac{1}{2}(1-t),\\ \text{(iv)} & \mathbb{P}\big[\kappa_{\mathbf{i}}=1,\ Y_{\mathbf{i}1}\leq t\,\big|\,\mathcal{A}^t\big]=\tfrac{1}{2}F(t),\end{array}\tag{3.46}$$

which are the conditional probabilities that (i) $\mathbf{i}$ is a branching point, (ii) $\mathbf{i}$ is an open blocking point, (iii) $\mathbf{i}$ is a closed blocking point and its descendant is not burnt, (iv) $\mathbf{i}$ is a blocking point and its descendant is burnt, which is only possible if $\mathbf{i}$ is closed or frozen. Since $\lambda^t_{\mathbf{i}}=2$ in case (i), $\lambda^t_{\mathbf{i}}=1$ in case (ii), and $\lambda^t_{\mathbf{i}}=0$ in the remaining cases, the lemma follows. □

Proof of Proposition 39. By (3.38),

$$\begin{aligned}\mathbb{P}\big[Y^\uparrow_\varnothing\leq t\big]&=\mathbb{P}\big[Y_\varnothing\leq t\big]+\mathbb{P}\big[Y_\varnothing>t\big]\mathbb{P}\big[Y^\uparrow_\varnothing\leq t\,\big|\,Y_\varnothing>t\big]\\ &=F(t)+\big(1-F(t)\big)\mathbb{P}\big[\mathbb{O}^t_n\neq\emptyset\ \forall n\geq 0\,\big|\,Y_\varnothing>t\big].\end{aligned}\tag{3.47}$$

By Lemma 41, the probability

$$p:=\mathbb{P}\big[\mathbb{O}^t_n\neq\emptyset\ \forall n\geq 0\,\big|\,Y_\varnothing>t\big]\tag{3.48}$$

is the survival probability of a branching process with offspring distribution as in (3.44). It is well-known [AN72, Thm III.4.1] that the survival probability is the largest solution in $[0,1]$ of the equation $\Psi(p)=p$, where (compare formula (A.1) in the appendix)

$$\Psi(p):=\tfrac{1}{2}\big(1-F(t)\big)p(1-p)-\tfrac{1}{2}\big(1-t+F(t)\big)p.\tag{3.49}$$

Assuming that $F(t) < 1$, it follows that

$$p = 0 \vee \Big\{1 - \frac{1 - t + F(t)}{1 - F(t)}\Big\} = 0 \vee \frac{t - 2F(t)}{1 - F(t)}. \tag{3.50}$$

Inserting this into (3.47) we arrive at (3.39). This argument does not work if $F(t) = 1$, which by Proposition 37 is only possible if $t = 1$ and $\mu = \delta_1$. In this case, no freezing takes place until at time $t = 1$ all $\mathbf{i} \in \mathbb{T}$ are open, so the left- and right-hand sides of (3.39) are both trivially equal to one.

Formula (3.38) says that $Y^{\uparrow}_{\varnothing} \leq Y_{\varnothing}$ a.s., so we have $Y^{\uparrow}_{\varnothing} = Y_{\varnothing}$ a.s. if and only if

$$\mathbb{P}\big[Y^{\uparrow}_{\varnothing} \leq t\big] = \mathbb{P}[Y_{\varnothing} \leq t] = F(t) \qquad \big(t \in [0, 1]\big), \tag{3.51}$$

which by (3.39) happens if and only if $F(t) \geq \frac{1}{2}t$ $(t \in [0, 1])$. By Proposition 37, the only solution to the RDE (1.32) with this property is the measure ρ in (1.31). □

3.5. *Frozen percolation on the binary tree.* In this subsection we derive Theorem 6 from Proposition 39. Our main tool is a coupling between, one the one hand, an RTP $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map χ from (1.30), and on the other hand, an RTP $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map γ from (1.5). We first describe the main idea of the construction and then fill in the technical details.

It is easy to see that for an RTP corresponding to the map χ from (1.30), the number of blocking points between two consecutive branching points is geometrically distributed with parameter $1/2$. Imagine, for the moment, that instead there would always be exactly one blocking point between two consecutive branching points. Then, comparing (1.5) and (1.30), one can check that the inductive relation satisfied by the burning times $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S},\ \kappa_{\mathbf{i}}=1}$ of blocking points would be exactly the same as the inductive relation satisfied by the burning times $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ of arbitrary points in an RTP corresponding to the map γ from (1.5). Inspired by this, starting from an RTP corresponding to the map χ from (1.30), we will construct an associated RTP corresponding to the map γ from (1.5) along the following steps:

(i) If there are two or more blocking points between two consecutive branching points, then we replace them by one point, whose new activation time is the maximum of the activation times of the blocking points it replaces.
(ii) If there are no blocking points between two consecutive branching points, then we add one such point, and assign it an activation time that is uniformly distributed on $[-1, 0]$.

(iii) We transform the activation times that we obtain by this procedure using a monotone mapping from $[-1,1]$ to $[0,1]$, which has the result that the transformed times are uniformly distributed on $[0,1]$.

We now formulate this a little more precisely. Let

$$1_n := \underbrace{1\cdots 1}_{n \text{ times}} \tag{3.52}$$

denote the word of length $n \geq 0$ that contains only 1's. For each $\mathbf{i} \in \mathbb{S}$, we set

$$b(\mathbf{i}) := \mathbf{i}1_{N_\mathbf{i}} \quad \text{with} \quad N_\mathbf{i} := \inf\{n \geq 0 : \kappa_{\mathbf{i}1_n} = 2\}. \tag{3.53}$$

In words, $b(\mathbf{i})$ is the next branching point above $\mathbf{i}$ (which may be $\mathbf{i}$ itself). We inductively define a map $\psi : \mathbb{T} \to \mathbb{S}$ by $\psi(\varnothing) = \varnothing$ and

$$\psi(\mathbf{i}j) := b\big(\psi(\mathbf{i})\big)j \quad (\mathbf{i} \in \mathbb{T},\ j = 1,2). \tag{3.54}$$

Note that points of the form $\psi(\mathbf{i})$ with $\mathbf{i} \in \mathbb{T}\backslash\{\varnothing\}$ are direct descendants of branching points, and $N_{\psi(\mathbf{i})}$ is the number of steps we have to walk up from $\psi(\mathbf{i})$ to reach the next branching point.

We let $(\tilde{\tau}_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$ be an i.i.d. collection of uniformly distributed $[-1,0]$-valued random variables, independent of everything else. For each $\mathbf{i} \in \mathbb{T}$, we define

$$\sigma_\mathbf{i} := \begin{cases} \max\big\{\tau_{\psi(\mathbf{i})1_n} : 0 \leq n \leq N_{\psi(\mathbf{i})} - 1\big\} & \text{if } N_{\psi(\mathbf{i})} \geq 1, \\ \tilde{\tau}_\mathbf{i} & \text{otherwise,} \end{cases} \tag{3.55}$$

i.e., $\sigma_\mathbf{i}$ is the maximum of the activation times of blocking points that lie directly below the branching point $b(\psi(\mathbf{i}))$, if there are any, and $\sigma_\mathbf{i} = \tilde{\tau}_\mathbf{i}$ otherwise. For each $\mathbf{i} \in \mathbb{T}$, the number $N_{\psi(\mathbf{i})}$ of blocking points that lie below the branching point $b(\psi(\mathbf{i}))$ is geometrically distributed with parameter $1/2$, and the values of their activation times are i.i.d. uniformly distributed on $[0,1]$ and independent of $N_{\psi(\mathbf{i})}$. These quantities are moreover independent for different $\mathbf{i} \in \mathbb{T}$. As a result, the $(\sigma_\mathbf{i})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. with distribution function

$$\mathbb{P}[\sigma_\mathbf{i} < s] = H(s) := \begin{cases} \frac{1}{2}(1+s) & \text{if } s \in [-1,0], \\ \dfrac{1}{2-s} & \text{if } s \in [0,1], \end{cases} \tag{3.56}$$

where we have used the calculation in (3.10) and we extend the function $H : [0,1] \to [\frac{1}{2},1]$ from Lemma 33 into a function $H : [-1,1] \to [0,1]$.

PROPOSITION 42 (Coupling of RTPs). *Let $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}},Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an RTP corresponding to the map χ from (1.30) and any solution to the RDE (1.32). Let $(\tilde\tau_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an independent i.i.d. collection of uniformly distributed $[-1,0]$-valued random variables, and let $\psi:\mathbb{T}\to\mathbb{T}$, $(\sigma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and $H:[-1,1]\to[0,1]$ be defined as in (3.54), (3.55), and (3.56). Then setting*

$$\overline{\tau}_{\mathbf{i}}:=H(\sigma_{\mathbf{i}})\quad\text{and}\quad X_{\mathbf{i}}:=H(Y_{\psi(\mathbf{i})})\qquad(\mathbf{i}\in\mathbb{T})\tag{3.57}$$

defines an RTP $(\overline{\tau}_{\mathbf{i}},X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map γ from (1.5). Moreover, any RTP corresponding to γ is equal in distribution to an RTP constructed in this way. Finally, one has

$$X^{\uparrow}_{\mathbf{i}}:=H(Y^{\uparrow}_{\psi(\mathbf{i})})\qquad(\mathbf{i}\in\mathbb{T}),\tag{3.58}$$

where $X^{\uparrow}_{\mathbf{i}}$ is defined in (1.14) and $Y^{\uparrow}_{\psi(\mathbf{i})}$ is defined in (3.37).

PROOF. We claim that $(Y_{\psi(\mathbf{i})})_{\mathbf{i}\in\mathbb{T}}$ satisfy the inductive relation

$$Y_{\psi(\mathbf{i})}=\gamma[\sigma_{\mathbf{i}}]\big(Y_{\psi(\mathbf{i}1)},Y_{\psi(\mathbf{i}2)}\big)\qquad(\mathbf{i}\in\mathbb{T}),\tag{3.59}$$

where we define $\gamma[t](x,y)$ as in (1.5) also for negative t. Indeed, if $N_{\psi(\mathbf{i})}=0$, then $\sigma_{\mathbf{i}}\le 0$ while $Y_{\psi(\mathbf{i}1)},Y_{\psi(\mathbf{i}2)}>0$ a.s., and

$$Y_{\psi(\mathbf{i})}=\chi[2]\big(Y_{\psi(\mathbf{i})1},Y_{\psi(\mathbf{i})2}\big)=Y_{\psi(\mathbf{i}1)}\wedge Y_{\psi(\mathbf{i}2)}.\tag{3.60}$$

On the other hand, if $N_{\psi(\mathbf{i})}\ge 1$, then

$$\begin{aligned}Y_{\psi(\mathbf{i})}&=\chi[\tau_{\psi(\mathbf{i})},1]\circ\cdots\circ\chi[\tau_{\psi(\mathbf{i})1_{N_{\psi(\mathbf{i})}}},1]\circ\chi[2]\big(Y_{\psi(\mathbf{i})1_{N_{\psi(\mathbf{i})}}1},Y_{\psi(\mathbf{i})1_{N_{\psi(\mathbf{i})}}2}\big)\\&=\gamma[\tau_{\psi(\mathbf{i})}\vee\cdots\vee\tau_{\psi(\mathbf{i})1_{N_{\psi(\mathbf{i})}}}]\big(Y_{\psi(\mathbf{i}1)},Y_{\psi(\mathbf{i}2)}\big).\end{aligned}\tag{3.61}$$

Using (3.59) and (3.12), we conclude that $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ satisfy the inductive relation (1.4). By (3.56), the random variables $(\overline{\tau}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ are i.i.d. and uniformly distributed on $[0,1]$. Moreover, for any finite rooted subtree $\mathbb{U}\subset\mathbb{T}$, the r.v.'s $(X_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ are independent of $(\overline{\tau}_{\mathbf{i}})_{\mathbf{i}\in\partial\mathbb{U}}$ and i.i.d.

This completes the proof that $(\overline{\tau}_{\mathbf{i}},X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is an RTP corresponding to the map γ from (1.5). Using Lemma 33, we see that every RTP $(\tau_{\mathbf{i}},X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ corresponding to the map γ from (1.5) and some solution μ to the RDE (1.12) is equal in distribution to an RTP constructed in this way.

To prove also (3.58), we observe that the frozen set F_{x} from (1.13) for the RTP $(\overline{\tau}_{\mathbf{i}},X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ is given by

$$\begin{aligned}\mathbb{F}&=\big\{\mathbf{i}\in\mathbb{T}:\overline{\tau}_{\mathbf{i}}\ge X_{\mathbf{i}1}\wedge X_{\mathbf{i}2}\big\}=\big\{\mathbf{i}\in\mathbb{T}:\sigma_{\mathbf{i}}\ge Y_{\psi(\mathbf{i}1)}\wedge Y_{\psi(\mathbf{i}2)}\big\}\\&=\big\{\mathbf{i}\in\mathbb{T}:N_{\psi(\mathbf{i})}\ge 1,\ \tau_{\psi(\mathbf{i})1_n}\ge Y_{\psi(\mathbf{i})1_{n+1}}\text{ for some }0\le n<N_{\psi(\mathbf{i})}\big\}\\&=\big\{\mathbf{i}\in\mathbb{T}:N_{\psi(\mathbf{i})}\ge 1,\ \psi(\mathbf{i})1_n\in\mathbb{F}_{\mathrm{y}}\text{ for some }0\le n<N_{\psi(\mathbf{i})}\big\},\end{aligned}\tag{3.62}$$

and hence at time t there exists a ray in $\mathbb{S}^t_{\psi(\mathbf{i})}\backslash\mathbb{F}_{\mathrm{y}}$ starting at $\psi(\mathbf{i})$ if and only if at time $s := H(t)$ there exists a ray in $\mathbb{T}^s\backslash\mathbb{F}$ starting at $\mathbf{i}$. $\square$

Proof of Theorem 6. By Lemma 32, μ is concentrated on $I' = [\frac{1}{2}, 1] \cup \{\infty\}$. Let μ' be the image of μ under the inverse of the map $H : I \to I'$ defined in Lemma 33. Then μ' solves the RDE (1.32). Let $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to the map χ from (1.30) and the measure μ'. We couple this RTP to $(\tau_{\mathbf{i}}, X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ as in Proposition 42. Since the function H is strictly increasing, we see that $X^{\uparrow}_{\varnothing} = X_{\varnothing}$ a.s. if and only if $Y^{\uparrow}_{\varnothing} = Y_{\varnothing}$ a.s. By Proposition 39 this is equivalent to μ' being the measure ρ in (1.31), which by Lemma 34 is equivalent to μ being the measure ν in (1.16). $\square$

3.6. *Scale invariance of the MBBT.* The aim of the present subsection is to prove Proposition 9 and Lemma 11 about scale invariance of (frozen percolation on) the MBBT. Lemma 11, in particular, allows us to identify the nontrivial fixed point $\rho^{(2)}_2$ from Theorem 12 as $\underline{\rho}^{(2)}$. Combining this with Lemma 36, we also obtain an explicit expression for $\underline{\nu}^{(2)}$. As a preparation for this, we first prove Lemma 8.

Proof of Lemma 8. It is well-known [AN72, Thm III.4.1] that the survival probability is the largest solution in $[0, 1]$ of the equation $\Psi(p) = p$, where (compare formula (A.1) in the appendix)

$$\Psi(p) = \{(1-p) - (1-p)^2\} + (1-t)\{(1-p) - (1-p)^0\} = p(1-p) - (1-t)p. \tag{3.63}$$

Since $\Psi(p) = 0$ has two roots, $p = 0$ and $p = t$, we conclude that the survival probability is t. $\square$

We next turn our attention to the proof of Proposition 9. Let $(\mathcal{T}, \Pi)$ be the MBBT. If we cut $\mathcal{T}$ at points in Π_t, then the connected component of the root is the family tree of a continuous-time branching process where particles split into two with rate one and die with rate $1 - t$. The tree $\mathcal{T}'$ defined in (1.22) is the *skeleton* of this process. It is well-known that $\mathcal{T}'$ is the family tree of a branching process, which is known as the *skeletal process*. There exist standard ways to find the skeletal process associated with a given branching process. Using these, it is easy to check that $\mathcal{T}'$ is the family tree of a binary branching process with branching rate t. In Appendix A, we outline a proof of this fact along these lines, with references to the relevant literature.

To prove Proposition 9, we need a bit more, however, since we need to determine the joint law of $\mathcal{T}'$ and Π'. To prove also Lemma 11, we will moreover need a scaling property of RTPs corresponding to the map χ in (1.30) and law ρ from (1.31). In view of this, we find it more convenient to give self-contained proofs of Proposition 9 and Lemma 11, not referring to the abstract theory of skeletal processes.

Let $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to the map χ from (1.30) and law ρ from (1.31), and let $(\ell_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an independent i.i.d. collection of exponentially distributed random variables with mean $1/2$. As in Subsection 1.6, we use the random variables $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, \ell_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ to define an MBBT $(\mathcal{T}, \Pi)$. In particular, $\mathcal{T}$ is the family tree of a branching process $(\nabla\mathbb{S}_h)_{h\geq 0}$ where $\mathbb{S}$, defined in (1.25), is the collection of all individuals that will ever live.

We fix $0 < t \leq 1$ and define

$$Y^*_{\mathbf{i}} := \begin{cases} t^{-1} Y_{\mathbf{i}} & \text{if } Y_{\mathbf{i}} \leq t, \\ \infty & \text{otherwise.} \end{cases} \tag{3.64}$$

We also define $(\mathcal{T}', \Pi')$ as in (1.22) and define $(\mathcal{T}^*, \Pi^*)$ by

$$\mathcal{T}^* := \big\{(x, th) : (x, h) \in \mathcal{T}'\big\}, \quad \Pi^* := \big\{(x, th, t^{-1}\tau_{(x,h)}) : (x, h, \tau_{(x,h)}) \in \Pi'\big\}. \tag{3.65}$$

As in Proposition 9, we view $(\mathcal{T}', \Pi')$ and $(\mathcal{T}^*, \Pi^*)$ as marked metric spaces, i.e., we do not care about the precise labeling of elements of $\mathcal{T}'$ or $\mathcal{T}^*$. Proposition 9 can be rephrased by saying that the conditional law of $(\mathcal{T}^*, \Pi^*)$ given $\varnothing \stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow} \infty$ is equal to the original law of $(\mathcal{T}, \Pi)$. The following lemma says that in a sense, $(\mathcal{T}^*, \Pi^*)$ contains all relevant information about $Y^*_\varnothing$.

Lemma 43 (Relevant information). *One has*

$$\mathbb{P}\big[Y^*_\varnothing \in \cdot \,\big|\, (\tau_{\mathbf{i}}, \kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big] = \mathbb{P}\big[Y^*_\varnothing \in \cdot \,\big|\, (\mathcal{T}^*, \Pi^*)\big] \quad \text{a.s.} \tag{3.66}$$

The following proposition extends Proposition 9 to a scaling property of the joint law of $(Y^*_\varnothing, \mathcal{T}^*, \Pi^*)$. In particular, this implies Proposition 9.

Proposition 44 (Scaling of the joint law). *One has*

$$\mathbb{P}\big[(Y^*_\varnothing, \mathcal{T}^*, \Pi^*) \in \cdot \,\big|\, \mathcal{T}^* \neq \emptyset\big] = \mathbb{P}\big[(Y_\varnothing, \mathcal{T}, \Pi) \in \cdot\,\big]. \tag{3.67}$$

Before we prove Lemma 43 and Proposition 44, we first show how they imply Lemma 11.

PROOF OF LEMMA 11. Conditional on $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, let $(Y'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an independent copy of $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Then, according to the definitions in (1.9) and (1.10)

$$\overline{\rho}^{(2)}=\mathbb{P}\big[(Y_\varnothing,Y_\varnothing)\in\,\cdot\,\big]\quad\text{and}\quad\underline{\rho}^{(2)}=\mathbb{P}\big[(Y_\varnothing,Y'_\varnothing)\in\,\cdot\,\big].\tag{3.68}$$

Clearly, these measures are symmetric and their one-dimensional marginals are given by ρ. It remains to show that they have the scaling property (1.34). The claim for $\overline{\rho}^{(2)}$ follows easily from the fact that $Y_\varnothing$ has the law ρ in (1.31). It remains to prove the statement for $\underline{\rho}^{(2)}$.

Fix $r,s,t\in[0,1]$. Since $Y_\varnothing=\infty$ a.s. on the complement of the event $\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty$, we have

$$\begin{aligned}&\mathbb{P}\big[(Y_\varnothing,Y'_\varnothing)\in[0,tr]\times[0,ts]\big]\\&\quad=\mathbb{P}\big[(Y_\varnothing,Y'_\varnothing)\in[0,tr]\times[0,ts]\,\big|\,\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty\big]\,\mathbb{P}\big[\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty\big].\end{aligned}\tag{3.69}$$

Here $\mathbb{P}\big[\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty\big]=t$ by Lemma 8, so to show that $\underline{\rho}^{(2)}$ has the scaling property (1.34), it suffices to show that

$$\mathbb{P}\big[(Y_\varnothing,Y'_\varnothing)\in[0,tr]\times[0,ts]\,\big|\,\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty\big]=\mathbb{P}\big[(Y_\varnothing,Y'_\varnothing)\in[0,r]\times[0,s]\big].\tag{3.70}$$

Since $Y_\varnothing$ and $Y'_\varnothing$ are conditionally independent given the σ-field generated by $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and since the event that $\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty$ is measurable w.r.t. this σ-field, we can rewrite the left-hand side of (3.70) as

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{P}\big[Y_\varnothing\in[0,tr]\,\big|\,\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty,\ (\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big]\\&\qquad\cdot\mathbb{P}\big[Y_\varnothing\in[0,ts]\,\big|\,\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty,\ (\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big]\Big]\\&\quad\stackrel{1}{=}\mathbb{E}\Big[\mathbb{P}\big[Y^*_\varnothing\in[0,r]\,\big|\,\mathcal{T}^*\neq\emptyset,\ (\mathcal{T}^*,\Pi^*)\big]\mathbb{P}\big[Y^*_\varnothing\in[0,s]\,\big|\,\mathcal{T}^*\neq\emptyset,\ (\mathcal{T}^*,\Pi^*)\big]\Big]\\&\quad\stackrel{2}{=}\mathbb{E}\Big[\mathbb{P}\big[Y_\varnothing\in[0,r]\,\big|\,(\mathcal{T},\Pi)\big]\mathbb{P}\big[Y_\varnothing\in[0,s]\,\big|\,(\mathcal{T},\Pi)\big]\Big]\\&\quad\stackrel{3}{=}\mathbb{E}\Big[\mathbb{P}\big[Y_\varnothing\in[0,r]\,\big|\,(\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big]\mathbb{P}\big[Y_\varnothing\in[0,s]\,\big|\,(\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big]\Big],\end{aligned}\tag{3.71}$$

which equals the right-hand side of (3.70). Here, in step 1, we have used the definition of $Y^*_\varnothing$ in (3.64), as well as the fact that the event $\{\varnothing\stackrel{\mathcal{T}\backslash\Pi_t}{\longrightarrow}\infty\}$ is the same as the event $\{\mathcal{T}^*\neq\emptyset\}$, which is measurable with respect to the σ-fields generated by $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ and $(\mathcal{T}^*,\Pi^*)$, and we have applied Lemma 43. Step 2 follows from Proposition 44. In step 3 we have again applied Lemma 43 but this time for $t=1$, in which case $(Y^*_\varnothing,\mathcal{T}^*,\Pi^*)=(Y_\varnothing,\mathcal{T},\Pi)$. □

Proof of Propositions 9 and 44. Let $(\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be the RTP corresponding to the map χ from (1.30) and law ρ from (1.31), and let $(\ell_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be an independent i.i.d. collection of exponentially distributed random variables with mean $1/2$. Fix $t \in (0,1]$. For any $A \subset \mathbb{T}$ and $\mathbf{i} \prec \mathbf{j} \in \mathbb{T}$, we write $\mathbf{i} \xrightarrow{A} \mathbf{j}$ if there exist $\mathbf{i}_0, \ldots, \mathbf{i}_n \in A$, $n \geq 0$, such that $\mathbf{i}_0 = \mathbf{i}$, $\mathbf{i}_n = \mathbf{j}$, and $\overleftarrow{\mathbf{i}}_k = \mathbf{i}_{k-1}$ $(k = 1, \ldots, n)$. Let us say that $\mathbf{i} \in \mathbb{T}$ is *active* if it is either open or frozen, i.e., if $\kappa_{\mathbf{i}} = 2$ or $\tau_{\mathbf{i}} \leq t$, and let

$$\mathbb{A} := \big\{\mathbf{i} \in \mathbb{T} : \varnothing \xrightarrow{\mathbb{S}^t} \mathbf{i} \xrightarrow{\mathbb{S}^t} \infty\big\}, \tag{3.72}$$

with $\mathbb{S}^t$ as in (3.36) denote the collection of points that lie on an active ray in $\mathbb{S}$ starting at the root. Note that by Lemma 8, the probability that $\mathbb{A}$ is not empty is t. We give each $\mathbf{i} \in \mathbb{A}$ a *type* $\omega_{\mathbf{i}} \in [0,t) \cup \{1,2\}$, which is defined as follows:

$$\omega_{\mathbf{i}} := \begin{cases} \tau_{\mathbf{i}} & \text{if } \kappa_{\mathbf{i}} = 1, \\ 1 & \text{if } \kappa_{\mathbf{i}} = 2 \text{ and } \{\mathbf{i}1, \mathbf{i}2\} \cap \mathbb{A} \text{ has precisely one element}, \\ 2 & \text{if } \kappa_{\mathbf{i}} = 2 \text{ and } \mathbf{i}1, \mathbf{i}2 \text{ are both elements of } \mathbb{A}. \end{cases} \tag{3.73}$$

Let $\mathbb{A}_n := \{\mathbf{i} \in \mathbb{A} : |\mathbf{i}| = n\}$. We claim that conditional on the event that $\mathbb{A} \neq \emptyset$, the process $(\mathbb{A}_n)_{n\geq 0}$ with the types assigned to its elements is a multitype branching process with the following description. In each generation, we first assign types to the particles that are alive in an i.i.d. fashion according to the law

$$\mathbf{P}[\omega \leq s] := \tfrac{1}{2}s \quad \big(s \in [0,t]\big), \quad \mathbf{P}[\omega = 1] := 1 - t, \quad \text{and} \quad \mathbf{P}[\omega = 2] = \tfrac{1}{2}t, \tag{3.74}$$

and then let particles of type 2 produce two offspring while all other particles produce one offspring. To see this, observe that by Lemma 8, for each $\mathbf{i} \in \mathbb{T}$ and $s \in [0,t]$,

$$\begin{aligned} \mathbb{P}\big[\kappa_{\mathbf{i}} = 1,\ \tau_{\mathbf{i}} \leq s,\ \mathbf{i}1 \xrightarrow{\mathbb{S}^t_{\mathbf{i}}} \infty\big] &= \tfrac{1}{2}st, \\ \mathbb{P}\big[\kappa_{\mathbf{i}} = 2,\ \mathbf{i}1 \xrightarrow{\mathbb{S}^t_{\mathbf{i}}} \infty \text{ or } \mathbf{i}2 \xrightarrow{\mathbb{S}^t_{\mathbf{i}}} \infty \text{ but not both}\big] &= t(1-t), \\ \mathbb{P}\big[\kappa_{\mathbf{i}} = 2,\ \mathbf{i}1 \xrightarrow{\mathbb{S}^t_{\mathbf{i}}} \infty \text{ and } \mathbf{i}2 \xrightarrow{\mathbb{S}^t_{\mathbf{i}}} \infty\big] &= \tfrac{1}{2}t^2. \end{aligned} \tag{3.75}$$

If we condition on $(\mathbb{A}_k)_{0\leq k\leq n}$ and also on the types of particles in generations $0, \ldots, n-1$, then the types of particles in the n-th generation are i.i.d. and their law is the distribution in (3.75) normalised to make it a probability law, which is the distribution $\mathbf{P}$ in (3.74).

Let $(\mathcal{T},\Pi)$ be the MBBT constructed as in Subsection 1.6 from the random variables $(\tau_{\mathbf{i}},\kappa_{\mathbf{i}},\ell_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, and let $(\mathcal{T}',\Pi')$ be as in (1.22). Then $(\mathcal{T}',\Pi')$ is uniquely determined by the branching process $\mathbb{A}$ and the types $\omega_{\mathbf{i}}$ and lifetimes $\ell_{\mathbf{i}}$ of elements $\mathbf{i}\in\mathbb{A}$. However, $\mathbb{A}$ contains, in a sense, too much information, since points $\mathbf{i}\in\mathbb{A}$ with type $\omega_{\mathbf{i}}=1$ are not visible in $(\mathcal{T}',\Pi')$. To remedy this, we need a procedure to remove these points, which we describe now.

For $\mathbf{i}\in\mathbb{A}$ with $\omega_{\mathbf{i}}\neq 2$, let $f(\mathbf{i}):=\mathbf{i}j$ where j is the unique element of $\{1,2\}$ such that $\mathbf{i}j\in\mathbb{A}$, and let

$$b(\mathbf{i}):=f^{n(\mathbf{i})}(\mathbf{i})\quad\text{with}\quad n(\mathbf{i}):=\inf\{k\geq 0:\omega_{f^k(\mathbf{i})}\neq 1\}\tag{3.76}$$

denote the next point above $\mathbf{i}$ that is not of type 1. Let $\mathbb{B}:=\{\mathbf{i}\in\mathbb{A}:\omega_{\mathbf{i}}\neq 1\}$. We inductively define a map $\psi:\mathbb{B}\to\mathbb{T}$ by $\psi\big(b(\varnothing)\big):=\varnothing$ and

$$\begin{aligned}\psi\big(b(\mathbf{i}j)\big)&:=\psi(\mathbf{i})j\quad(j=1,2)&&\text{if }\omega_{\mathbf{i}}=2,\\ \psi\big(b(\mathbf{i}1)\big)&:=\psi(\mathbf{i})1&&\text{if }\omega_{\mathbf{i}}\in[0,t).\end{aligned}\tag{3.77}$$

We let $\mathbb{S}'$ denote the image of $\mathbb{B}$ under the map ψ and assign types to the elements of $\mathbb{S}'$ by

$$\omega'_{\psi(\mathbf{i})}:=\omega_{\mathbf{i}}\qquad(\mathbf{i}\in\mathbb{B}).\tag{3.78}$$

We also define new lifetimes by

$$\ell'_{\psi(\mathbf{i})}:=\sum_{k=0}^{n(f(\mathbf{i}))}\ell_{f^k(\mathbf{i})}\tag{3.79}$$

where $n(\mathbf{i})$ is defined as in (3.76). Then the set $\mathbb{S}'$ and the random variables $(\omega'_{\mathbf{i}},\ell'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}}$ contain precisely the information needed to construct $(\mathcal{T}',\Pi')$, and nothing more.

Let $\mathbb{S}'_n:=\{\mathbf{i}\in\mathbb{S}':|\mathbf{i}|=n\}$. The process $(\mathbb{S}'_n)_{n\geq 0}$ inherits the branching property from the process $(\mathbb{A}_n)_{n\geq 0}$. To get the new generation, we first assign i.i.d. types to the particles in the present generation according to the law

$$\mathbf{P}'[\omega\leq s]:=\frac{s}{2t}\quad\big(s\in[0,t]\big),\quad\mathbf{P}'[\omega=2]=\tfrac{1}{2},\tag{3.80}$$

which is the law in (3.74) conditioned on $\omega\neq 1$, and then let particles with type in $[0,t)$ and $\{2\}$ produce one or two offspring, respectively. Each lifetime $\ell'_{\mathbf{i}}$ is the sum of a geometric number of exponentially distributed random variables. From this, it is easy to see that conditional on $\mathbb{S}'$ and the types, the lifetimes $(\ell'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}'}$ are i.i.d. and exponentially distributed with mean $\frac{1}{2}t^{-1}$. Since the random tree $\mathcal{T}'$ is the family tree of the branching process $(\mathbb{S}'_n)_{n\geq 0}$ with the lifetimes $(\ell'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}'}$, and the Poisson set Π' records points with type $\omega_{\mathbf{i}}\in[0,t)$ together with their activation times $\tau'_{\mathbf{i}}:=\omega_{\mathbf{i}}\in[0,t)$, this completes proof of Proposition 9.

We could have obtained Proposition 9 faster by referring to the the abstract theory of skeletal processes (see Appendix A). The advantage of our explicit construction, however, is that it also easily yields the stronger statement of Proposition 44. To see this, we define $(Y'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}'}$ by

$$Y'_{\psi(\mathbf{i})} := \begin{cases} Y_{\mathbf{i}} & \text{if } Y_{\mathbf{i}}\leq t \\ \infty & \text{otherwise.}\end{cases} \qquad (\mathbf{i}\in\mathbb{B}). \tag{3.81}$$

Since we started from an RTP corresponding to the law ρ from (1.31), and since $Y_{\mathbf{i}}>t$ a.s. on the complement of the event $\mathbf{i}\xrightarrow{\mathbb{S}_t}\infty$, we see that conditional on $(\mathbb{S}'_k)_{0\leq k\leq n}$ and the types of particles in generations $0,\dots,n-1$, the random variables $(Y'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}'_n}$ are i.i.d. with law $\mathbb{P}[Y'_{\mathbf{i}}\leq s]=\frac{1}{2}s/t$ ($s\in[0,t]$). We claim that they satisfy the inductive relation

$$Y'_{\mathbf{i}} = \chi[\omega'_{\mathbf{i}}](Y'_{\mathbf{i}1},Y'_{\mathbf{i}2}) \qquad (\mathbf{i}\in\mathbb{S}'), \tag{3.82}$$

where (compare (1.30))

$$\chi[\omega](x,y) := \begin{cases} x & \text{if } \omega\in[0,t),\ x>\omega, \\ \infty & \text{if } \omega\in[0,t),\ x\leq\omega, \\ x\wedge y & \text{if } \omega=2. \end{cases} \tag{3.83}$$

Note that $\mathbf{i}2\notin\mathbb{S}'$ if $\omega_{\mathbf{i}}\in[0,t)$, but since in this case, $\chi[\omega_{\mathbf{i}}](x,y)$ does not depend on y, (3.82) is unambiguous. Indeed, (3.82) follows from the fact that the original random variables $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ satisfy the inductive relation (1.29) and, in view of (3.73), $Y_{\mathbf{i}}=Y_{\mathbf{i}1}$ if $\mathbf{i}\in\mathbb{A}$ is of type $\omega_{\mathbf{i}}=1$.

These observations imply the statement of Proposition 44. Indeed, if we set

$$Y^*_{\mathbf{i}} := t^{-1}Y'_{\mathbf{i}}, \quad \omega^*_{\mathbf{i}} := \begin{cases} t^{-1}\omega'_{\mathbf{i}} & \text{if } \omega'_{\mathbf{i}}\in[0,t), \\ 2 & \text{if } \omega'_{\mathbf{i}}=2, \end{cases} \qquad \ell^*_{\mathbf{i}} := t^{-1}\ell'_{\mathbf{i}}, \tag{3.84}$$

then the random variables $\mathbb{S}'$ and $(\omega^*_{\mathbf{i}},\ell^*_{\mathbf{i}})_{\mathbf{i}\in\mathbb{S}'}$ define a marked tree $(\mathcal{T}^*,\Pi^*)$ such that the joint law of $(Y^*_\varnothing,\mathcal{T}^*,\Pi^*)$, conditioned on $\mathcal{T}^*\neq\emptyset$, is equal to the joint law of $(Y_\varnothing,\mathcal{T},\Pi)$. $\square$

PROOF OF LEMMA 43. We use notation as in the proof of Propositions 9 and 44. We adapt the proof of [MSS20, Lemma 46] to our present setting. We set $\mathbb{T}_{(n)} := \{\mathbf{i} \in \mathbb{T} : |\mathbf{i}| < n\}$ and let $\overline{\mathcal{F}}_{(n)}$ and $\overline{\mathcal{F}}$ be the σ-fields generated by the random variables $\tau_{\mathbf{i}}, \kappa_{\mathbf{i}}$ with $\mathbf{i} \in \mathbb{T}_{(n)}$ and $\mathbf{i} \in \mathbb{T}$, respectively. We also set $\mathbb{S}'_{(n)} := \mathbb{S}' \cap \mathbb{T}_{(n)}$, we let $\mathcal{F}_{(n)}$ be the σ-field generated by the random variables $\mathbb{S}'_{(n)}$ and $(\omega'_{\mathbf{i}})_{\mathbf{i} \in \mathbb{S}'_{(n)}}$, and we define $\mathcal{F}$ similarly, with $\mathbb{S}'_{(n)}$ replaced by $\mathbb{S}'$. We observe that $\mathcal{F}_{(n)} \subset \overline{\mathcal{F}}_{(n)}$ $(n \geq 1)$.

The inductive relation (3.82) shows that conditional on $\mathcal{F}_{(n)}$, the state at the root $Y'_\varnothing$ is a deterministic function of $(Y'_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}_n}$. Since $(Y'_{\mathbf{i}})_{\mathbf{i} \in \mathbb{T}_n}$ are independent of $\overline{\mathcal{F}}_{(n)}$, it follows that $Y'_\varnothing$ is conditionally independent of $\overline{\mathcal{F}}_{(n)}$ given $\mathcal{F}_{(n)}$, i.e.,

$$\mathbb{P}\big[Y'_\varnothing \in A \,\big|\, \overline{\mathcal{F}}_{(n)}\big] = \mathbb{P}\big[Y'_\varnothing \in A \,\big|\, \mathcal{F}_{(n)}\big] \quad \text{a.s.} \tag{3.85}$$

for any measurable $A \subset \mathbb{R}$. Letting $n \to \infty$, using martingale convergence and observing that $Y'_\varnothing$ contains the same information as $Y^*_\varnothing$ while $(\mathcal{T}^*, \Pi^*)$ contains the same information as $\mathcal{F}$, the claim follows. $\square$

REMARK 45. It follows from Lemma 11 that $\underline{\rho}^{(2)} = \rho_2^{(2)}$, the nontrivial scale-invariant fixed point from Theorem 12. Therefore, combining Lemma 36 with formula (1.37), we obtain a formula for $\underline{\nu}^{(2)}$. Indeed,

$$\underline{\nu}^{(2)}\big([0,r] \times [0,s]\big) = 2 - \frac{1}{2r} - \frac{1}{2s} - \Big(2 - \frac{1}{s \vee r}\Big) f_{c_2}\left(\frac{2 - \frac{1}{s \wedge r}}{2 - \frac{1}{s \vee r}}\right) \tag{3.86}$$

$(\frac{1}{2} < r, s \leq 1)$, where f_{c_2} is the function defined in Theorem 12.

3.7. *Frozen percolation on the 3-regular tree.* In this subsection, we use methods from [Ald00] to derive Theorems 2 and 3, which are concerned with the unoriented 3-regular tree, from Theorems 6 and 7, which are concerned with the oriented binary tree. We start with a preparatory lemma.

Let $(\mathcal{U}, \vec{F})$ satisfy properties (i)–(iii) of Theorem 2 and let F be defined in terms of $\vec{F}$ as in that theorem. Recall that we call edges in $E_t \backslash F$ open, edges in $E_t \cap F$ frozen, and all other edges closed. A similar convention applies in the oriented setting. For each $w \in T$ and $t \in [0,1]$, let $C_t(w)$ resp. $\vec{C}_t(w)$ denote the set of vertices that can at time t be reached by an open unoriented resp. oriented path starting at w.

LEMMA 46 (Finite unoriented clusters). *Almost surely, for all $t \in [0,1]$, if $C_t(w)$ is finite, then $C_t(w) = \vec{C}_t(w)$.*

Proof. Clearly $C_t(w) \subset \vec{C}_t(w)$ regardless of whether $C_t(w)$ is finite or not. To see that equality holds if $C_t(w)$ is finite, assume the converse. Then there must be $x \in C_t(w)$ and $y \notin C_t(w)$ such that the oriented edge (x,y) is open at time t. Among all such edges, we can choose the unique one for which $s := U_{\{x,y\}}$ is minimal. Since $y \notin C_t(w)$, the oriented edge must have frozen at time s, so by property (i) of Theorem 2, at time s there must be an open ray starting at x not using y. Such a ray must use an oriented edge to leave $C_t(w)$ that is open at time s and hence also at the later time t, contradicting the minimality of $U_{\{x,y\}}$. □

Proof of Theorem 2. We first prove uniqueness. Assume that $\vec{F}$ satisfies properties (i)–(iii). For each $(v,w) \in \vec{E}$, let

$$X_{(v,w)} := \inf\big\{t \in [0,1] : \exists \text{ ray } (v_n,w_n)_{n\geq 0} \text{ starting with } (v_0,w_0) = (v,w) \text{ such that } (v_n,w_n) \in \vec{F}\ \forall n \geq 0\big\}, \tag{3.87}$$

with $\inf\emptyset := \infty$. Let γ be the map in (1.5). Property (i) implies that

$$X_{(x,v)} := \gamma[U_{\{x,v\}}](X_{(v,y)}, X_{(v,z)}) \tag{3.88}$$

whenever $v \in T$ and x,y,z are the three neighbours of v. Let S be a finite subtree of (T,E). Then, for each $(v,w) \in \partial S$, the set $\vec{E}_{(v,w)}$ is naturally isomorphic to the oriented binary tree $\mathbb{T}$. Formula (3.88) and properties (ii) and (iii) imply that $(\mathcal{U}_{\{x,y\}}, X_{(x,y)})_{(x,y)\in\vec{E}_{(v,w)}}$ is an RTP corresponding to the map γ and some solution μ to the RDE (1.12). Property (i) and Theorem 6 imply that $\mu = \nu$, the measure defined in (1.16). By property (iii), the RTPs corresponding to different $(v,w) \in \partial S$ are independent. By (3.88), these RTPs uniquely determine $X_{(x,y)}$ for each $(x,y) \in \vec{E}$. This shows that the joint law of $\mathcal{U} = (U_{\{x,y\}})_{\{x,y\}\in E}$ and $(X_{(x,y)})_{(x,y)\in E}$ is uniquely determined. Since

$$(x,v) \in \vec{F} \text{ if and only if } U_{\{x,v\}} \geq X_{(v,y)} \wedge X_{(v,z)} \tag{3.89}$$

whenever $v \in T$ and x,y,z are the three neighbours of v, the joint law of $(\mathcal{U}, \vec{F})$ is also uniquely determined.

As Aldous already showed in [Ald00], existence follows basically from the same argument. We fix a finite subtree S of (T,E), construct independent RTPs corresponding to γ and ν for each $(v,w) \in \partial S$, inductively define $X_{(x,y)}$ for each $(x,y) \in \vec{E}$ by (3.88), and then define $\vec{F}$ by (3.89). It follows from the properties of RTPs that if we add a vertex to S or remove a vertex, then the law of the object we have just constructed does not change. As a result, our construction is independent of the choice of S, the law of $(\mathcal{U},\vec{F})$ is invariant under automorphisms of the tree, and property (iii) holds for general S. Property (i) now follows from Theorem 6, completing the proof that an object satisfying (i)–(iii) exists.

It is clear that $(\mathcal{U},F)$, defined in terms of $(\mathcal{U},\vec{F})$, is invariant under automorphisms of the tree. To see that it also satisfies property (i) of Theorem 1, we observe that by the way F has been defined in terms of $\vec{F}$ and property (i) of Theorem 2, $\{v,w\}\not\in F$ if and only if for each $t<U_{\{v,w\}}$, the oriented clusters $\vec{C}_t(v)$ and $\vec{C}_t(w)$ are both finite. By Lemma 46, this is equivalent to $C_t(v)$ and $C_t(w)$ being finite, proving property (i) of Theorem 1. $\square$

The following simple abstract lemma prepares for the proof of Theorem 3.

LEMMA 47 (Almost surely not equal). *Let $(\omega_{\mathbf{i}},X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be a nonendogenous RTP, where $\mathbb{T}$ denotes the space of all finite words made up from the alphabet $\{1,\ldots,d\}$, with $d\geq 2$. Let $(X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ be a copy of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$, conditionally independent given $(\omega_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$. Then $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\neq(X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}$ a.s.*

PROOF. Let ν denote the solution of the RDE used to construct the RTP. Let $\mathbb{T}_n:=\{\mathbf{i}\in\mathbb{T}:|\mathbf{i}|=n\}$. Then $(X_{\mathbf{i}},X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}_n}$ are i.i.d. with common law $\underline{\nu}^{(2)}$ as in (1.10). By Theorem 4, $\underline{\nu}^{(2)}\neq\overline{\nu}^{(2)}$, which implies that $p:=\mathbb{P}[X_{\mathbf{i}}\neq X'_{\mathbf{i}}]>0$ and hence

$$\mathbb{P}\big[(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\neq(X'_{\mathbf{i}})_{\mathbf{i}\in\mathbb{T}}\big]\leq\mathbb{P}\big[X_{\mathbf{i}}=X'_{\mathbf{i}}\text{ for all }\mathbf{i}\in\mathbb{T}_n\big]\leq(1-p)^{d^n}. \tag{3.90}$$

Since $d\geq 2$ and n is arbitrary, the claim follows. $\square$

PROOF OF THEOREM 3. We use the construction of $(\mathcal{U},\vec{F})$ in the proof of Theorem 2. We fix a finite subtree S of (T,E). Independently for each $(v,w)\in\partial S$, we construct an RTP $\big(\mathcal{U}_{\{x,y\}},X_{(x,y)}\big)_{(x,y)\in\vec{E}_{(v,w)}}$ corresponding to the map γ in (1.5) and measure ν in (1.16), and we let $\big(X'_{(x,y)}\big)_{(x,y)\in\vec{E}_{(v,w)}}$ be a copy of $\big(X_{(x,y)}\big)_{(x,y)\in\vec{E}_{(v,w)}}$, conditionally independent given the random variables $(\mathcal{U}_{\{x,y\}})_{\{x,y\}\in E_{(v,w)}}$. Using (3.88), we inductively define $X_{(x,y)}$ and $X'_{(x,y)}$ for all $(x,y)\in\vec{E}$ and in terms of these random variables we define $\vec{F}$ and $\vec{F}'$ as in (3.89), which are finally used to define F and F' as in Theorem 2. Then F and F' are conditionally independent given $\mathcal{U}$.

It follows from Theorem 7 and Lemma 47 that a.s. $X_{(x,y)}\neq X'_{(x,y)}$ for some $(x,y)\in\vec{F}$. By (3.87), this implies that $\vec{F}\neq\vec{F}'$ a.s. By Lemma 46 and property (i) of Theorem 2, the set $\vec{F}$ is a.s. determined by the pair $(\mathcal{U},F)$, and likewise $\vec{F}'$ is a.s. determined by $(\mathcal{U},F')$, so $\vec{F}\neq\vec{F}'$ a.s. implies $F\neq F'$ a.s. $\square$

APPENDIX A: SKELETAL BRANCHING PROCESSES

Informally speaking, the *skeletal process* of a branching process is the process consisting of those particles whose offspring will never die out. It is well-known that the skeletal process of a branching process is itself a branching process. For discrete time processes, a proof can be found in [AN72, Thm I.12.1]. There is also an extensive literature about skeletal processes of superprocesses, see [EKW15] and references therein. In this appendix, we show how the skeletal process of a continuous-time branching process can be calculated, and use this to sketch an alternative proof that $\mathcal{T}'$, defined in (1.22), is the family tree of a binary branching process with branching rate t.

Generalising our set-up, let $(Z_h)_{h\geq 0}$ be a continuous-time branching processes in which each particle is with rate $\mathbf{r}(k)$ replaced by k new particles. A sufficient condition for $(Z_h)_{h\geq 0}$ to be well-defined and nonexplosive is that $\sum_k \mathbf{r}(k)k < \infty$. A convenient tool is the *generating semigroup* $(U_h)_{h\geq 0}$ defined as $U_h\phi := u_h$ $(\phi \in [0,1])$, where $(u_h)_{h\geq 0}$ is the unique solution with initial state $u_0 = \phi$ to the differential equation

$$\begin{aligned}&\tfrac{\partial}{\partial h}u_h = \Psi(u_h) \quad (h \geq 0)\\ &\quad\text{with}\quad \Psi(u) := \sum_{k\geq 0}\mathbf{r}(k)\big\{(1-u) - (1-u)^k\big\} \quad \big(u \in [0,1]\big).\end{aligned} \tag{A.1}$$

The generating semigroup uniquely determines the transition probabilities of $(Z_h)_{h\geq 0}$ through the relation

$$\mathbb{E}\big[(1-\phi)^{Z_h}\big] = \mathbb{E}\big[(1-U_h\phi)^{Z_0}\big] \qquad (h \geq 0). \tag{A.2}$$

This can be deduced, for example, from [AN72, Sect. III.3], although the notation there is quite different.

Let p be the survival probability of $(Z_h)_{h\geq 0}$, which is the largest root in $[0,1]$ of the equation $\Psi(p) = 0$. Then we claim that setting

$$U'_h\phi := p^{-1}U_h(p\phi) \qquad \big(\phi \in [0,1]\big) \tag{A.3}$$

defines a generating semigroup, which corresponds to the skeletal process $(Z'_h)_{h\geq 0}$ of $(Z_h)_{h\geq 0}$. For discrete time processes, a proof can be found in [AN72, Thm I.12.1]. The statement for continuous-time processes can easily be derived from this by adding independent exponentially distributed lifetimes to the discrete time process. In particular, if $r(0) = 1-t$, $r(2) = 1$, and all other rates are zero, then the differential equation in (A.1) reads

$$\tfrac{\partial}{\partial h}u_h = \Psi(u_h) = u_h(1-u_h) - (1-t)u_h \qquad (h \geq 0), \tag{A.4}$$

and $(U'_h)_{h\geq 0}$ is given by the solutions to the differential equation
(A.5)
$$\frac{\partial}{\partial h}v_h = t^{-1}\Psi(tv_h) = t^{-1}\big(tv_h(1-tv_h)-(1-t)tv_h\big) = tv_h(1-v_h) \qquad (h\geq 0),$$

which we recognise as the generating semigroup of a branching process where particles split into two with rate t and never die.

The transformation in (A.3) can be traced back to [Har48] while the interpretation in terms of the skeletal process dates back to [AN72, Thm I.12.1]. See also [FS04, Thm 9] for a statement in the context of superprocesses. It is possible to go further and write $(Z_h)_{h\geq 0}$ as the union of skeletal and non-skeletal particles, which then form a two-type branching process. This sort of statements date back to [OCo93] and have been developed and exploited in a superprocess setting; see [EKW15] and references therein.

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