

LIST COLORING OF TWO MATROIDS THROUGH REDUCTION TO PARTITION MATROIDS*

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Abstract. In the list coloring problem for two matroids, we are given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on the same ground set S , and the goal is to determine the smallest number k such that given arbitrary lists L_s of k colors for $s \in S$, it is possible to choose a color from each list so that every monochromatic set is independent in both M_1 and M_2 . When both M_1 and M_2 are partition matroids, Galvin's celebrated list coloring theorem for bipartite graphs gives the answer. However, not much is known about the general case. One of the main open questions is to decide if there exists a constant c such that if the coloring number is k (i.e., the ground set can be partitioned into k common independent sets), then the list coloring number is at most $c \cdot k$. In the present paper, we consider matroid classes that appear naturally in combinatorial and graph optimization problems, specifically graphic matroids, paving matroids and gammoids. We show that if both matroids are from these fundamental classes, then the list coloring number is at most twice the coloring number.

The proof is based on a new approach that reduces a matroid to a partition matroid without increasing its coloring number too much, and might be of independent combinatorial interest. In particular, we show that if $M = (S, \mathcal{I})$ is a matroid in which S can be partitioned into k independent sets, then there exists a partition matroid $N = (S, \mathcal{J})$ with $\mathcal{J} \subseteq \mathcal{I}$ in which S can be partitioned into (A) k independent sets if M is a transversal matroid, (B) $2k - 1$ independent sets if M is a graphic matroid, (C) $\lceil kr/(r - 1) \rceil$ independent sets if M is a paving matroid of rank r , and (D) $2k - 2$ independent sets if M is a gammoid. It should be emphasized that in cases (A), (B) and (D) the rank of N is the same as that of M .

We further extend our results to a much broader family by showing that taking direct sum, homomorphic image or truncation of matroids from these classes results in a matroid admitting a reduction to a partition matroid with coloring number at most twice the original one.

Key words. List coloring, Matroids, Packing common bases, Weak maps

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1. Introduction. Given a graph $G = (V, E)$, a *proper edge coloring* of G is an assignment of colors to the edges so that no two adjacent edges have the same color. The *edge coloring number* is the smallest integer k for which G has a proper edge coloring by k colors. A classical result of König [19] states that the edge coloring number of bipartite graphs is equal to its maximum degree, while for arbitrary graphs Vizing's [29] celebrated theorem gives an upper bound.

Assume now that a list L_e of colors is given for each edge $e \in E$. A *proper list edge coloring* of G is a proper edge coloring such that every edge e receives a color from its list L_e . The *list edge coloring number* is the smallest integer k for which G has a proper list edge coloring whenever $|L_e| \geq k$ for every $e \in E$. The List Coloring Conjecture [15, 30] states that for any graph, the list edge coloring number equals the edge coloring number. The conjecture is wide open, and only partial results are known. Probably the most famous one is the celebrated result of Galvin [10], which shows that the conjecture holds for bipartite multigraphs.

THEOREM 1.1 (Galvin). *The list edge coloring number of a bipartite graph is*

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equal to its edge coloring number, that is, to its maximum degree.

Matchings in a bipartite graph $G = (S, T; E)$ form the common independent sets of two matroids; to see this, define a subset of edges to be independent in the first matroid if each vertex in S has degree at most one in it, and define a subset of edges to be independent in the second matroid if each vertex in T has degree at most one in it. Hence one might consider matroidal generalizations of list coloring. For a loopless matroid¹ $M = (S, r)$, let $\chi(M)$ denote the *coloring number* of M , that is, the minimum number of independent sets of M needed to cover the ground set. We call a matroid k -colorable if $\chi(M) \leq k$. If a list L_s of colors is given for each element $s \in S$, then a *proper list edge coloring* of M is a coloring such that every element s receives a color from its list L_s . The *list coloring number* is the smallest integer k for which M has a proper list coloring whenever $|L_s| \geq k$ for every $s \in S$.

Analogously, the *coloring number* $\chi(M_1, M_2)$ of the intersection of two matroids M_1 and M_2 on the same ground set S is the minimum number of common independent sets needed to cover S . If a list L_s of colors is given for each element $s \in S$, then a *proper list edge coloring* of the intersection of M_1 and M_2 is a coloring such that every element s receives a color from its list L_s and elements of the same color form an independent set in both matroids. The *list coloring number* $\chi_\ell(M_1, M_2)$ is the smallest integer k for which the intersection of M_1 and M_2 has a proper list coloring whenever $|L_s| \geq k$ for every $s \in S$. Hence Theorem 1.1 states that if both M_1 and M_2 are partition matroids then $\chi_\ell(M_1, M_2) = \max\{\chi(M_1), \chi(M_2)\}$.

Previous work.. Seymour observed [28] that the list coloring theorem holds for a single matroid.

THEOREM 1.2 (Seymour). *The list coloring number of a matroid is equal to its coloring number.*

Lasoń [20] gave a generalization of the theorem when the sizes of the lists are not necessarily equal. As a common generalization of Theorems 1.1 and 1.2, it is tempting to conjecture that $\chi(M_1, M_2) = \chi_\ell(M_1, M_2)$ for every pair of matroids [17]. No pair M_1, M_2 is known for which the conjecture fails. However, there are only a few matroid classes for which the problem has been settled. Király and Pap [18] verified the conjecture for transversal matroids, for matroids of rank two, and if the common bases are the arborescences of a digraph which is the disjoint union of two spanning arborescences rooted at the same vertex. It is worth mentioning that a similar statement does not hold for the case of three matroids as shown by the following example due to Király [16]. Let $S = \{a, b, c, d, e, f\}$ be a ground set of size six, and let M_1, M_2 and M_3 be partition matroids with circuit sets $\mathcal{C}_1 = \{\{a, d\}, \{b, e\}, \{c, f\}\}$, $\mathcal{C}_2 = \{\{a, e\}, \{b, f\}, \{c, d\}\}$ and $\mathcal{C}_3 = \{\{a, f\}, \{b, d\}, \{c, e\}\}$, respectively. Then $\{a, b, c\}$ and $\{d, e, f\}$ is a partition into two common bases. However, if $L_a = L_d = \{1, 2\}$, $L_b = L_e = \{1, 3\}$ and $L_c = L_f = \{2, 3\}$, then there is no proper list coloring.

In [16], Király proposed a weakening of the problem.

PROBLEM 1.3 (Király). *Does there exist a constant c such that if the coloring number of the intersection of two matroids is k , then the list coloring number is at most $c \cdot k$?*

¹The ground set of a matroid containing a loop cannot be decomposed into independent sets. Therefore every matroid considered in the paper is assumed to be loopless without explicitly mentioning this. Nevertheless, parallel elements might exist.

For spanning arborescences, it was observed by Kobayashi [16] that the constructive characterization of k -arborescences implies that lists of size $\frac{3}{2}k + 1$ are sufficient.

Another motivation comes from the problem of approximating the minimum number of common independent sets of two matroids needed to cover the ground set. Aharoni and Berger [1] proved the following interesting result.

THEOREM 1.4 (Aharoni and Berger). *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids. If S can be decomposed into k_1 independent sets in M_1 and into k_2 independent sets in M_2 , then it can be decomposed into $2 \max\{k_1, k_2\}$ common independent sets.*

Note that in terms of approximating the coloring number, Theorem 1.4 states that $\chi(M_1, M_2) \leq 2 \max\{\chi(M_1), \chi(M_2)\}$. The proof of the theorem is highly non-trivial and uses topological arguments which do not directly give an algorithm for finding the decomposition in question. We propose a conjecture on reduction of matroids to partition matroids from which the theorem would easily follow.

The idea of reducing a matroid to a simpler one goes back to the late 60's. In [5], Crapo and Rota introduced the notion of weak maps. Weak maps were further investigated by Lucas [22, 23] who characterized rank-preserving weak maps for linear matroids. However, these results did not consider the possible increase in the coloring number of the matroid that plays a crucial role in our investigations.

Given matroids $M = (S, \mathcal{I})$ and $N = (S, \mathcal{J})$, we say that N is a *reduction* of M if $\mathcal{J} \subseteq \mathcal{I}$, that is, every independent set of N is independent in M . In notation, we will denote N being a reduction of M by $N \preceq M$. The reduction is *rank-preserving* if $r_M(S) = r_N(S)$. In that case, we write $N \preceq_r M$. In terms of weak maps, a reduction corresponds to the identity weak map on the common ground set of the two matroids.²

Bérczi and Schwarcz [4] studied rainbow circuit-free colorings of binary matroids, where a circuit is rainbow colored if no two of its elements have the same color. Such a coloring can be identified with a reduction to a partition matroid, hence the results presented there are closely related to our setting. They proved that every rainbow circuit-free coloring of a binary matroid contains a monochromatic cut, and gave lower and upper bounds on the maximum size of a color class and the number of colors, respectively, in any rainbow circuit-free coloring.

It is worth mentioning that every matroid $M = (S, \mathcal{I})$ has a reduction to a partition matroid $N = (S, \mathcal{J})$ of the same rank (see e.g. [4]). The sketch of the proof is as follows: Fix an arbitrary basis $B = \{s_1, \dots, s_r\}$ of M , and add s_i to the i th partition class. Then for an arbitrary element $s \in S - B$, consider the fundamental circuit $C(s, B)$ of s with respect to B , and add s to the partition class containing the element of $C(s, B) \cap B$ with the smallest index. If we pick exactly one element from every class of the partition thus obtained, we get a basis of the matroid. This can be verified using the circuit axioms, not discussed in this paper. Nevertheless, this algorithm has no control over the sizes of the partition classes. It can happen that some of the classes have a large size compared to the coloring number of the original matroid, and such a reduction is not suitable for our purposes.

Our results.. In the present paper, we consider matroid classes that appear naturally in combinatorial and graph optimization problems, and show that if both matroids are from these fundamental classes then the constant c appearing in Prob-

²Just before the submission of the present paper, Im et al. [13] published a work that considers the matroid intersection cover problem. Their approach is based on partition decompositions of matroids, a generalization of reductions to partition matroids. Our results improve on their bounds and the present paper also provides tight examples.

lem 1.3 can be chosen to be roughly two. In fact, our proof extends to a much broader family as we show that taking direct sum, homomorphic image or truncation of matroids from these classes results in a matroid admitting a reduction to a partition matroid with coloring number at most twice the original one.

A *partition matroid*³ is a matroid $N = (S, \mathcal{I})$ such that $\mathcal{I} = \{X \subseteq S : |X \cap S_i| \leq 1 \text{ for } i = 1, \dots, q\}$ for some partition $S = S_1 \cup \dots \cup S_q$. Clearly, the coloring number of N is $\chi(N) = \max\{|S_i| : i = 1, \dots, q\}$. Note that the following are equivalent for a matroid $M = (S, \mathcal{I})$: (i) $N \preceq M$, (ii) every circuit of M intersects at least one of the S_i 's in more than one element, and (iii) $\{x_1, \dots, x_q\} \in \mathcal{I}$ whenever $x_i \in S_i$ for $i = 1, \dots, q$.

To illustrate the applicability of reduction, assume that M_1 and M_2 are matroids on the same ground set that are reducible to k -colorable partition matroids N_1 and N_2 , respectively. Then, by Theorem 1.1, $\chi_\ell(N_1, N_2) \leq k$. As $N_1 \preceq M_1$ and $N_2 \preceq M_2$, this in turn implies that $\chi_\ell(M_1, M_2) \leq k$.

As a first step towards understanding reducibility to partition matroids, we concentrate on matroid classes that appear naturally in combinatorial and graph optimization problems, specifically graphic matroids, paving matroids, and gammoids; see the corresponding sections for the precise definitions. These classes also include uniform matroids, laminar matroids and transversal matroids, hence the presented results also apply to those. We show that matroids from these fundamental classes admit a reduction to a partition matroid with coloring number at most twice the coloring number of the original matroid.

The first two results are for transversal and graphic matroids and are based on easy observations. Although transversal matroids are special cases of gammoids, we discuss them separately as transversal matroids admit an optimal reduction in terms of coloring number.

THEOREM 1.5. *Let $M = (S, \mathcal{I})$ be a k -colorable transversal matroid. Then there exists a k -colorable partition matroid N with $N \preceq_r M$.*

THEOREM 1.6. *Let $M = (S, \mathcal{I})$ be a k -colorable graphic matroid. Then there exists a $(2k - 1)$ -colorable partition matroid N with $N \preceq_r M$, and the bound for the coloring number of N is tight.*

The next result is for paving matroids.

THEOREM 1.7. *Let $M = (S, \mathcal{I})$ be a k -colorable paving matroid of rank $r \geq 2$. Then there exists a $\lceil \frac{rk}{r-1} \rceil$ -colorable partition matroid N with $N \preceq M$.*

For $r = 2$, the bound on the coloring number of N can be improved and the reduction can be chosen to be rank-preserving.

THEOREM 1.8. *Let $M = (S, \mathcal{I})$ be a k -colorable paving matroid of rank 2. Then there exists a $\lfloor \frac{4k}{3} \rfloor$ -colorable partition matroid N with $N \preceq_r M$, and the bound for the coloring number of N is tight.*

It is not difficult to see that every loopless matroid of rank 2 is paving, hence Theorem 1.8 gives a tight bound on the coloring number of the reduction N of such matroids.

³In the literature, partition matroids are defined more generally in the sense that the upper bounds on the intersection might be different for the different partition classes. As all the partition matroids used in the paper have all-ones upper bounds, we make this restriction without explicitly mentioning it.

For $r = 3$, we can provide rank-preserving reductions at the price of increasing the coloring number of N .

THEOREM 1.9. *Let $M = (S, \mathcal{I})$ be a k -colorable paving matroid of rank 3. Then there exists a $(2k - 1)$ -colorable partition matroid N with $N \preceq_r M$.*

Gammoids form a subfamily of strongly base orderable matroids, a class of matroids with distinguished structural properties related to basis exchanges. Determining the coloring number of the intersection of two strongly base orderable matroids was completely solved by Davies and McDiarmid [6]. However, their result does not imply any bound on the constant c in Problem 1.3 for the case of gammoids. The main contribution of the paper is a proof that any k -colorable gammoid can be reduced to a $(2k - 2)$ -colorable partition matroid for $k \geq 2$. This in turn implies that the constant c in question is at most 2. The assumption $k \geq 2$ is not restrictive, as if $k = 1$ then M is the free matroid on S which is already a partition matroid.

THEOREM 1.10. *Let $M = (S, \mathcal{I})$ be a k -colorable gammoid ($k \geq 2$). Then there exists a $(2k - 2)$ -colorable partition matroid N with $N \preceq_r M$, and the bound for the coloring number of N is tight.*

We are not aware of any k -coverable matroid that does not admit a (not necessarily rank-preserving) $2k$ -coverable reduction. Based on the above results, we believe that the following conjecture holds in general.

CONJECTURE 1.11. *Every k -colorable matroid can be reduced to a $2k$ -colorable partition matroid.*

It is worth emphasizing that the conjecture asks for a reduction which is not necessarily rank-preserving. In [4], a subset of the authors showed that in certain cases a constant factor loss in the rank is unavoidable if the covering number of the reduction is ought to be bounded, even in the case of binary matroids.

Recall that the problem proposed by Király asks for the existence of a constant c such that if the coloring number $\chi(M_1, M_2)$ is k , then the list coloring number $\chi_\ell(M_1, M_2)$ is at most $c \cdot k$. Conjecture 1.11 and Theorem 1.1 would imply $\chi_\ell(M_1, M_2) \leq \chi_\ell(N_1, N_2) = \max\{\chi(N_1), \chi(N_2)\} \leq 2 \max\{\chi(M_1), \chi(M_2)\}$, where N_1 and N_2 are the reductions of M_1 and M_2 , respectively, provided by the conjecture. Therefore we would get a value of 2 for the constant c . By the same idea, Conjecture 1.11 and König's classical theorem together would immediately provide a new, hopefully algorithmic proof of Theorem 1.4.

We extend our results to broader classes of matroids by investigating the behaviour of reducibility under standard matroid operations, specifically for direct sum, homomorphic image, and truncation.

THEOREM 1.12. *Let $M = (S, \mathcal{I})$ be a matroid such that for any $S' \subseteq S$ the restriction $M' = M|_{S'}$ can be reduced to a $2\chi(M')$ -colorable partition matroid. Then any homomorphic image has the same property.*

THEOREM 1.13. *The family of matroids M that can be reduced to a $2\chi(M)$ -colorable partition matroid is closed under truncation.*

The rest of the paper is organized as follows. Basic definitions and notation are introduced in Section 2. Results for the transversal, graphic and paving cases are presented in Section 3. The main result of the paper, Theorem 1.10 is proved in Section 4. Finally, the connection between matroid operations and reducibility is discussed in Section 5.

2. Preliminaries. We follow the notation and terminology of [8]. Given a ground set S together with subsets $X, Y \subseteq S$, the *difference* of X and Y is denoted by $X - Y$. We denote the *symmetric difference* of X and Y by $X \triangle Y := (X - Y) \cup (Y - X)$. If Y consists of a single element y , then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively.

For a graph $G = (V, E)$ and a subset $X \subseteq V$ of vertices, the *set of edges induced by X* is denoted by $E[X]$, while the *graph induced by X* is denoted by $G[X]$. Given a connected component K of G , a *co-circuit* of a *cut* of K is a subset of edges in $E[K]$ whose deletion disconnects K . The component is *k-edge-connected* if the minimum size of a cut in K is at least k . The graphs obtained by deleting a subset $X \subseteq V$ of vertices or a subset $F \subseteq E$ of edges are denoted by $G - X$ and $G - F$, respectively. The *degree* of a vertex v with respect to $F \subseteq E$ is denoted by $d_F(v)$.

Let $G = (A, B; E)$ be a bipartite graph with vertex classes A and B and edge set E , and let $F \subseteq E$ be a subset of edges. For a set $X \subseteq A$, the *set of neighbors of X with respect to F* is denoted by $\Gamma_F(X)$, that is, $\Gamma_F(X) = \{b \in B : \text{there exists an edge } ab \in F \text{ with } a \in X\}$. We will drop the subscript F when F is the whole edge set. We denote the *set of vertices in X incident to edges in F* by $X(F)$. A forest (or tree) $F \subseteq E$ is a *B_2 -forest* (or *B_2 -tree*, respectively) if $d_F(b) = 2$ for every $b \in B$. The existence of a B_2 -forest was characterized by Lovász [21].

THEOREM 2.1 (Lovász). *Let $G = (A, B; E)$ be a bipartite graph. Then there exists a B_2 -forest in G if and only if the strong Hall condition holds for every non-empty subset of B , that is,*

$$|\Gamma(X)| \geq |X| + 1 \quad \text{for all } \emptyset \neq X \subseteq B.$$

Matroids were introduced by Whitney [32] and independently by Nakasawa [26] as abstract generalizations of linear independence in vector spaces. A matroid M is a pair $(S, \mathcal{I})$ where S is the *ground set* of the matroid and $\mathcal{I} \subseteq 2^S$ is the family of *independent sets* that satisfy the following *independence axioms*: (I1) $\emptyset \in \mathcal{I}$, (I2) $X \subseteq Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$, and (I3) $X, Y \in \mathcal{I}$, $|X| < |Y| \Rightarrow \exists e \in Y - X$ s.t. $X + e \in \mathcal{I}$. The *rank* of a set $X \subseteq S$ is the maximum size of an independent subset of X and is denoted by $r_M(X)$. The maximal independent sets of M are called *bases*. A *cut* is an inclusionwise minimal set $X \subseteq S$ that intersects every basis. A *loop* is an element that is non-independent on its own. Two non-loop elements $e, f \in S$ are *parallel* if $\{e, f\}$ is non-independent.

Given a matroid $M = (S, \mathcal{I})$, its *restriction* to a subset $S' \subseteq S$ is the matroid $M|_{S'} = (S', \mathcal{I}')$ where $\mathcal{I}' = \{I \in \mathcal{I} : I \subseteq S'\}$. Adding a *parallel copy* of an element $s \in S$ results in a matroid $M' = (S', \mathcal{I}')$ on ground set $S' = S + s'$ where $\mathcal{I}' = \{X \subseteq S' : \text{either } X \in \mathcal{I}, \text{ or } s \notin X, s' \in X \text{ and } X - s' + s \in \mathcal{I}\}$. The *dual* of M is the matroid $M^* = (S, \mathcal{I}^*)$ where $\mathcal{I}^* = \{X \subseteq S : S - X \text{ contains a basis of } M\}$. The *k-truncation* of a matroid $M = (S, \mathcal{I})$ is the matroid $(S, \mathcal{I}_k)$ with $\mathcal{I}_k = \{X \in \mathcal{I} : |X| \leq k\}$. We denote the *k-truncation* of M by $(M)_k$. For a set T and map $\varphi: S \rightarrow T$, the *homomorphic image* of M is the matroid $M_\varphi = (T, \mathcal{I}_\varphi)$ where $\mathcal{I}_\varphi = \{X \subseteq T : \text{there is a set } Y \in \mathcal{I} \text{ for which } \varphi(Y) = X\}$, that is, $\mathcal{I}_\varphi$ consists of those subsets of T that are obtained as the image of an independent set of M .

The *direct sum* $M_1 \oplus M_2$ of matroids $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ on disjoint ground sets is the matroid $M = (S_1 \cup S_2, \mathcal{I})$ whose independent sets are the disjoint unions of an independent set of M_1 and an independent set of M_2 . The *sum* $M_1 + M_2$ of $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on the same ground set is the matroid $M = (S, \mathcal{I})$ whose independent sets are the disjoint unions of an independent set of M_1 and an

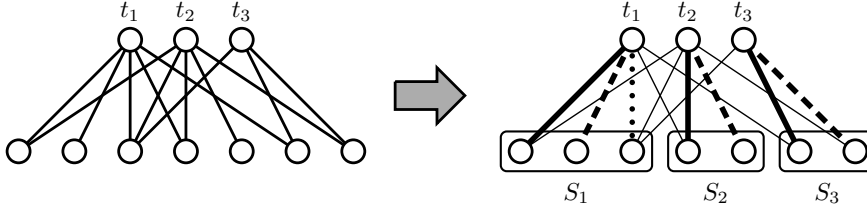


Fig. 1: An illustration of the proof of Theorem 1.5. Thick, dashed and dotted edges are corresponding to three matchings covering S .

independent set of M_2 . The rank of the sum of k matroids was characterized by Edmonds and Fulkerson [7].

THEOREM 2.2 (Edmonds and Fulkerson). *Let $M_1 = (S, \mathcal{I}_1), \dots, M_k = (S, \mathcal{I}_k)$ be matroids on the same ground set S with rank functions $r_1, \dots, r_k$, respectively. Then the maximum size of an independent set of $M_1 + \dots + M_k$ is $\min\{\sum_{i=1}^k r_i(X) + |S - X| : X \subseteq S\}$.*

As a corollary, we get a characterization for the partitionability of the ground set into k independent sets in a matroid.

COROLLARY 2.3. *Let $M = (S, \mathcal{I})$ be a matroid with rank function r . Then S can be partitioned into k independent sets if and only if $|X| \leq k \cdot r(X)$ for every $X \subseteq S$.*

Another corollary is a characterization for the existence of k disjoint bases of a matroid.

COROLLARY 2.4. *Let $M = (S, \mathcal{I})$ be a matroid with rank function r . Then M has k pairwise disjoint bases if and only if $|S - X| \geq k \cdot (r(S) - r(X))$ for every $X \subseteq S$.*

3. Transversal, graphic and paving matroids. First we consider three basic cases: transversal, graphic, and paving matroids. Although the proofs are simple, they might help the reader to get familiar with the notion of reduction. Also, we show the connection to some earlier results such as Gallai colorings of complete graphs.

3.1. Transversal matroids. Given a bipartite graph $G = (S, T; E)$, a set $X \subseteq S$ is *matchable* if there is a matching of G covering X . The matchable sets satisfy the independence axioms; the matroid obtained this way is a *transversal matroid*. It is an easy exercise to show that the size of T can be chosen to be r where r denotes the rank of the matroid (see e.g. [8]). In Section 4, we will use that the rank of a subset $X \subseteq S$ in the transversal matroid is $r(X) = \min\{|X| - |Y| + |\Gamma(Y)| : Y \subseteq X\}$ by the Frobenius-König-Hall theorem [19, 11, 9].

THEOREM 1.5. *Let $M = (S, \mathcal{I})$ be a k -colorable transversal matroid. Then there exists a k -colorable partition matroid N with $N \preceq_r M$.*

Proof. Let $G = (S, T; E)$ a bipartite graph where $T = \{t_1, \dots, t_r\}$, r being the rank of the transversal matroid on S . By assumption, the transversal matroid is k -colorable, so there exist k matchings $F_1, \dots, F_k$ covering every vertex in S exactly once. We may assume that none of these matchings is empty. Let $S_i = \bigcup_{j=1}^k \Gamma_{F_j}(t_i)$ for $i = 1, \dots, r$ (see Figure 1). Then $S_1 \cup \dots \cup S_r$ is a partition of S with classes of size at most k . Pick an arbitrary element $s_j \in S_j$ for $j = 1, \dots, r$. The edge set $\{t_j s_j : j = 1, \dots, r\}$ shows that the picked elements form a matchable set, hence the

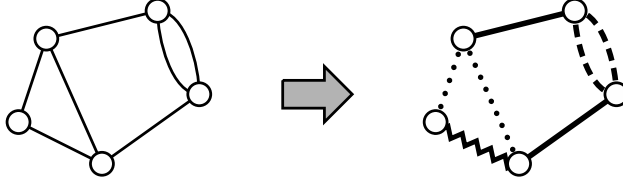


Fig. 2: An illustration of the proof of Theorem 1.6. The graph $G = (V, E)$ can be decomposed into three forests. Let S_1, S_2, S_3 and S_4 denote the sets of thick, dashed, dotted and zigzag edges, respectively. Then S_{i+1} is a minimum cut in one of the components of $G - \bigcup_{j=1}^i S_j$ for $i = 0, \dots, 3$. Observe that there is no rainbow colored cycle in G (in which any two edges receive different colors).

partition matroid defined by the partition is a k -colorable rank-preserving reduction of the transversal matroid. $\square$

3.2. Graphic matroids. For a graph $G = (V, E)$, the *graphic matroid* $M = (E, \mathcal{I})$ of G is defined on the edge set by considering a subset $F \subseteq E$ to be independent if it is a forest, that is, $\mathcal{I} = \{F \subseteq E : F \text{ does not contain a cycle}\}$. Nash-Williams [25] gave a characterization for G being decomposable into k forests, or in other words, for the graphic matroid of G being k -colorable.

THEOREM 3.1 (Nash-Williams). *Given a graph $G = (V, E)$, the edge set can be decomposed into k forests if and only if $|E[X]| \leq k(|X| - 1)$ for every non-empty subset X of V .*

THEOREM 1.6. *Let $M = (S, \mathcal{I})$ be a k -colorable graphic matroid. Then there exists a $(2k - 1)$ -colorable partition matroid N with $N \preceq_r M$, and the bound for the coloring number of N is tight.*

Proof. Let $G = (V, E)$ be a graph whose graphic matroid $M = (E, \mathcal{I})$ is k -colorable and let $K \subseteq V$ be a connected component of G of size at least 2. We claim that there exists a cut in K of size at most $2k - 1$. Indeed, if every cut of K contains at least $2k$ edges then K is a $2k$ -edge-connected component and so $|E[K]| \geq k|K|$ by counting the edges around each vertex in K . By Theorem 3.1, this contradicts the k -colorability of M .

Set $S_0 := \emptyset$ and $i := 0$. As long as there exists a connected component K in $G - \bigcup_{j=0}^i S_j$ of size at least 2, let $S_{i+1} \subseteq E$ be a minimum cut of K (see Figure 2), and update $i := i + 1$. By the above, $|S_{i+1}| \leq 2k - 1$. Let $E = S_1 \cup \dots \cup S_q$ denote the partition thus obtained. We claim that the partition matroid corresponding to this partition is a reduction of M . In order to see this, we have to show that every cycle of G intersects at least one of the partition classes in at least two elements. Given a cycle C , let i be the smallest index with $|S_i \cap C| > 0$. Then $C \subseteq \bigcup_{j \geq i}^q S_j$ and S_i is a cut in $\bigcup_{j \geq i}^q S_j$, hence $|S_i \cap C| \geq 2$. As the deletion of S_i increases the number of components of $G - \bigcup_{j=0}^{i-1} S_j$ by exactly one for $i = 1, \dots, q$, the rank of the partition matroid thus obtained is the same as the rank of the graphic matroid of G , concluding the first half of the theorem.

To show that the given bound is tight, let $G = (V, E)$ be a complete graph on $2k$ vertices. By Nash-Williams' theorem, the coloring number of the graphic matroid of G is k . Observe that reducing the graphic matroid of G to a partition matroid is

equivalent to coloring the edges of the graph in such a way that there is no cycle whose edges are colored with completely different colors. An edge coloring of a complete graph is called a *Gallai coloring* if no triangle is colored with three distinct colors, which is a weaker restriction than the above. Bialostocki, Dierker and Voxman [2] showed that every Gallai coloring contains a monochromatic spanning tree. This means that for any reduction of the graphic matroid of G to a partition matroid, there is a partition class of size at least $2k - 1$. $\square$

Remark 3.2. Theorem 1.6 can be proved in a similar way by observing that any graph that can be decomposed into k forests contains a vertex of degree at most $2k - 1$. The advantage of the proof based on cuts is twofold: it provides a rank-preserving reduction, and it can be straightforwardly extended to arbitrary matroids in the following sense.

THEOREM 3.3. *If $M = (S, \mathcal{I})$ is a matroid so that $M|S'$ has a cut of size at most k for any $S' \subseteq S$, then M can be reduced to a k -colorable partition matroid.*

The proof of Theorem 3.3 is based on the fact that the intersection of a circuit and a cut in a matroid cannot have size 1.

3.3. Paving matroids. A matroid $M = (S, \mathcal{I})$ of rank r is called *paving* if every set of size at most $r - 1$ is independent, or in other words, every circuit of the matroid has size at least r .

Although paving matroids have a very restricted structure and so are quite well-understood, they play a fundamental role among matroids. After Blackburn, Crapo and Higgs [3] enumerated all matroids up to eight elements, it was observed that most of these matroids are paving matroids. Crapo and Rota suggested that perhaps paving matroids dominate the enumeration of matroids [5]. This statement was made precise by Mayhew, Newman, Welsh and Whittle in [24]. They conjectured that the asymptotic fraction of matroids on n elements that are paving tends to 1 as n tends to infinity. Although this remains open, a similar statement on the asymptotic ratio of the logarithms of the numbers of matroids and sparse paving matroids has been proven in [27].

First we consider paving matroids of arbitrary rank.

THEOREM 1.7. *Let $M = (S, \mathcal{I})$ be a k -colorable paving matroid of rank $r \geq 2$. Then there exists a $\lceil \frac{rk}{r-1} \rceil$ -colorable partition matroid N with $N \preceq M$.*

Proof. Consider any partition $S = S_1 \cup \dots \cup S_{r-1}$ into $r - 1$ parts of almost equal sizes, that is, $|S_i| = \lfloor |S|/(r - 1) \rfloor$ or $|S_i| = \lceil |S|/(r - 1) \rceil$ for $i = 1, \dots, r - 1$. As M is k -colorable, we have $|S| \leq kr$ and so $|S_i| \leq \lceil kr/(r - 1) \rceil$. As M is paving, any set of size at most $r - 1$ is independent, hence the partition matroid N defined by the partition $S_1 \cup \dots \cup S_{r-1}$ is a $\lceil kr/(r - 1) \rceil$ -colorable reduction of M , as required. $\square$

The bound on the coloring number can be improved when $r = 2$, and the reduction can be chosen to be rank-preserving.

THEOREM 1.8. *Let $M = (S, \mathcal{I})$ be a k -colorable paving matroid of rank 2. Then there exists a $\lfloor \frac{4k}{3} \rfloor$ -colorable partition matroid N with $N \preceq_r M$, and the bound for the coloring number of N is tight.*

Proof. Let $S = T_1 \cup \dots \cup T_q$ denote the partition of the ground set into classes of parallel elements, that is, for every $x \in T_i$ and $y \in T_j$ the set $\{x, y\}$ is independent if and only if $i \neq j$. We may assume that $|T_1| \geq \dots \geq |T_q|$. Note that $|T_1| \leq k$ as the matroid is k -colorable. Let i denote the smallest index such that $|T_1 \cup \dots \cup T_i| \geq$

$|S|/3$ holds, and consider the partition $S = S_1 \cup S_2$ where $S_1 = T_1 \cup \dots \cup T_i$ and $S_2 = T_{i+1} \cup \dots \cup T_q$. If $i = 1$, then $|S_1| = |T_1| \leq k$, otherwise

$$|S_1| = (|T_1| + \dots + |T_{i-1}|) + |T_i| < \frac{|S|}{3} + |T_i| \leq \frac{|S|}{3} + |T_1| < \frac{2|S|}{3} \leq \frac{4k}{3},$$

where we used that $|S| \leq 2k$ holds as M is k -colorable and $r = 2$. By the definition of i , we have $|S_2| \leq 2|S|/3 \leq 4k/3$ as well. Thus $\max\{|S_1|, |S_2|\} \leq 4k/3$ always holds, hence the partition matroid N defined by the partition $S_1 \cup S_2$ is a $\lfloor 4k/3 \rfloor$ -colorable reduction of M .

The bound $\lfloor 4k/3 \rfloor$ on the coloring number of N is tight. Let S be a set of size $2k$ and take a partition $S = S_1 \cup S_2 \cup S_3$ where $\lceil |S|/3 \rceil = |S_1| \geq |S_2| \geq |S_3| = \lfloor |S|/3 \rfloor$. Consider the laminar matroid $M = (S, \mathcal{I})$ defined by the laminar family $\{S, S_1, S_2, S_3\}$ where $X \subseteq S$ is independent if and only if $|X| \leq 2$ and $|X \cap S_i| \leq 1$ for $i = 1, 2, 3$. It is not difficult to see that the coloring number of M is k . Suppose that M is reducible to a partition matroid N . The rank of N is either 1 or 2, as M has rank 2. In the former case $\chi(N) = 2k$, while in the latter case N is defined by a partition $S = P_1 \cup P_2$. Then every S_i is a subset of either P_1 or P_2 , as otherwise there exists two elements $x, y \in S_i$ such that $x \in P_1$ and $y \in P_2$, implying that $\{x, y\}$ is independent in N but dependent in M , a contradiction. Thus P_1 or P_2 contains at least two of the S_i 's, and so has size at least $|S_2| + |S_3| = |S| - |S_1| = 2k - \lceil 2k/3 \rceil = \lfloor 4k/3 \rfloor$, proving $\chi(N) \geq \lfloor 4k/3 \rfloor$. $\square$

For the proof of Theorem 1.9, we will need two technical statements. The first lemma describes the structure of paving matroids [12, 31, 8].

LEMMA 3.4. *Let $r \geq 2$ be an integer and S a set of size at least r . Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) family of proper subsets of S in which every set H_i has at least r elements and the intersection of any two of them has at most $r - 2$ elements. Then the set system $\mathcal{B}_{\mathcal{H}} = \{X \subseteq S : |X| = r, X \not\subseteq H_i \text{ for } i = 1, \dots, q\}$ forms the set of bases of a paving matroid. Moreover, every paving matroid can be obtained in this form.*

The next lemma characterizes the coloring number of paving matroids.

LEMMA 3.5. *Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) family satisfying the conditions of Lemma 3.4, and let $M = (S, \mathcal{I})$ be the paving matroid defined by $\mathcal{H}$. Then*

$$\chi(M) = \max \left\{ \left\lceil \frac{|S|}{r} \right\rceil, \left\lceil \frac{|H_1|}{r-1} \right\rceil, \dots, \left\lceil \frac{|H_q|}{r-1} \right\rceil \right\}.$$

Proof. Corollary 2.3 implies that $\chi(M) = \max\{\lceil |X|/r(X) \rceil : \emptyset \neq X \subseteq S\}$. Since $r(S) = r$ and $r(H_i) = r - 1$ (as every set of size at most $r - 1$ is independent), we get that $\chi(M) \geq \max\{\lceil |S|/r \rceil, \lceil |H_1|/(r-1) \rceil, \dots, \lceil |H_q|/(r-1) \rceil\}$.

To see the reverse inequality, take an arbitrary subset $X \subseteq S$. If $|X| \leq r - 1$, then $r(X) = |X|$ holds as the matroid is paving, therefore $|X|/r(X) = 1$. If $|X| \geq r$ and $X \subseteq H_i$ for some i , then $r(X) = r - 1$ and so $|X|/r(X) \leq |H_i|/(r - 1)$. Finally, if $|X| \geq r$ and none of the H_i 's contains X , then $r(X) = r$ and so $|X|/r(X) \leq |S|/r$, proving our claim. $\square$

Now we are ready to prove Theorem 1.9.

THEOREM 1.9. *Let $M = (S, \mathcal{I})$ be a k -colorable paving matroid of rank 3. Then there exists a $(2k - 1)$ -colorable partition matroid N with $N \preceq_r M$.*

Proof. Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) family satisfying the conditions of Lemma 3.4 that defines M . Without loss of generality, we may assume that

$|H_1| \geq \dots \geq |H_q|$. We distinguish two cases.

Case 1. $|S|/r \leq |H_1|/(r-1)$.

Consider any partition $H_1 = S_1 \cup \dots \cup S_{r-1}$ into $r-1$ parts of almost equal sizes, that is, $|S_i| = \lfloor |H_1|/(r-1) \rfloor$ or $|S_i| = \lceil |H_1|/(r-1) \rceil$ for $i = 1, \dots, r-1$, and let $S_r = S - H_1$. Note that none of $S_1, \dots, S_r$ is empty since H_1 is a proper subset of S of size at least $r-1$. Taking any elements $s_1 \in S_1, \dots, s_r \in S_r$ the set $X = \{s_1, \dots, s_r\}$ is independent in M as $X \not\subseteq H_i$ ($i = 1, \dots, q$) by $|X \cap H_1| = r-1$ and $|H_1 \cap H_i| \leq r-2$ ($i = 2, \dots, q$). Thus the partition matroid $N = (S, \mathcal{J})$ defined by the partition $S_1 \cup \dots \cup S_r$ is a rank-preserving reduction of M . N is clearly $\chi(M)$ -colorable as $|S_i| \leq \lceil |H_1|/(r-1) \rceil = \chi(M)$ for $i = 1, \dots, r-1$ and $|S_r| = |S| - |H_1| \leq r|H_1|/(r-1) - |H_1| = |H_1|/(r-1) \leq \chi(M)$.

Case 2. $|S|/r > |H_1|/(r-1)$.

Pick an arbitrary $s \in S$, let $H_{i_1}, \dots, H_{i_l}$ denote the sets of the family $\mathcal{H}$ containing s and let $H'_j = H_{i_j} - s$ for $j = 1, \dots, l$. The sets $H'_1, \dots, H'_l$ are disjoint as $|H_i \cap H_j| \leq r-2 = 1$ for $i \neq j$. We may assume that $|H'_1| \geq \dots \geq |H'_l|$. Note that for any set $T \subseteq S - s$ which does not intersect any H'_j properly, the partition $S = \{s\} \cup T \cup (S - T - s)$ defines a partition matroid $N = (S, \mathcal{J})$ which is a reduction of M .

If $|H'_1| + \dots + |H'_l| < |S|/3$, let $T \subseteq S - s$ be a set of size $\lfloor |S|/2 \rfloor$ containing $H'_1 \cup \dots \cup H'_l$. Then $\chi(N) = \max\{|T|, |S| - |T| - 1\} \leq |S|/2 < 2|S|/3 \leq 2\chi(M)$. If $|H'_1| + \dots + |H'_l| \geq |S|/3$, then let j denote the smallest index such that $|H'_1| + \dots + |H'_j| \geq |S|/3$ and let $T = H'_1 \cup \dots \cup H'_j$. If $j = 1$, then $|T| = |H'_1| < 2|S|/3$ by our assumption $|S|/r > |H_1|/(r-1)$ and $r = 3$. Otherwise $|H_1| < |S|/3$ and so $|T| \leq |S|/3 + |H'_j| \leq |S|/3 + |H'_1| < 2|S|/3$. Thus $\chi(N) = \max\{|T|, |S| - |T| - 1\} < 2|S|/3 \leq 2\chi(M)$. $\square$

Remark 3.6. Note that Case 1 of the proof does not rely on the fact that $r = 3$. That is, any paving matroid satisfying the assumption of Case 1 has a rank-preserving reduction $N \preceq_r M$ with $\chi(N) = \chi(M)$.

4. Gammoids. The aim of this section is to prove the main result of the paper, Theorem 1.10. A generalization of transversal matroids can be obtained with the help of directed graphs. Given a directed graph $D = (V, A)$ and two sets $X, Y \subseteq V$, we say that X is *linked* to Y if $|X| = |Y|$ and there exists $|X|$ vertex-disjoint directed paths from X to Y . Let $S \subseteq V$ be a set of starting vertices and $T \subseteq V$ be a set of destination vertices. Then the family $\mathcal{I} = \{Y \subseteq T : \exists X \subseteq S \text{ s.t. } X \text{ is linked to } Y\}$ forms the independent sets of a matroid that is called a *gammoid*. The gammoid is a *strict gammoid* if $T = V$. That is, a gammoid is obtained by restricting a strict gammoid to a subset of its elements.

Transversal matroids and gammoids are closely related. Ingleton and Piff [14] showed that strict gammoids are exactly the duals of transversal matroids, hence every gammoid is a restriction of the dual of a transversal matroid. Moreover, gammoids are the smallest class of matroids that includes transversal matroids and is closed under duality and taking minors.

THEOREM 1.10. *Let $M = (S, \mathcal{I})$ be a k -colorable gammoid ($k \geq 2$). Then there exists a $(2k-2)$ -colorable partition matroid N with $N \preceq_r M$, and the bound for the coloring number of N is tight.*

Proof. The bound on the coloring number of N is tight. Consider the laminar matroid $M = (S, \mathcal{I})$ defined by the laminar family $\{S, S_1, \dots, S_k\}$ where $S_1 \cup \dots \cup S_k$

is a partition of S into subsets of size $k - 1$. That is, the size of the ground set S is $k^2 - k$. We define a set $X \subseteq S$ to be independent in M if $|X \cap S_i| \leq 1$ for $i = 1, \dots, k$, and $|X| \leq k - 1$. It is not difficult to see that M is a strict gammoid with coloring number k .

We claim that if $N \preceq M$ is a partition matroid, then $\chi(N) \geq 2k - 2$. Let $P_1 \cup \dots \cup P_q$ denote the partition defining N . Then every S_i is a subset of some P_j , as otherwise there exists two elements $x, y \in S_i$ such that $x \in P_a$ and $y \in P_b$ for $a \neq b$, implying that $\{x, y\}$ is independent in N but dependent in M , a contradiction. As the rank of M is $k - 1$, we have $q \leq k - 1$. By the above, there exists a class P_j that contains at least two of the S_i 's, and so has size at least $2k - 2$, proving $\chi(N) \geq 2k - 2$.

Now let $M = (S, \mathcal{I})$ be a k -colorable gammoid where $k \geq 2$. By the result of Ingleton and Piff, M can be obtained as a restriction of the dual of a transversal matroid. Let R be such a transversal matroid, and choose R in such a way that its rank is as small as possible. Let $G = (A, B; E)$ be a bipartite graph defining R with $S \subseteq A$ and $|B|$ being the rank of R .

The high-level idea of the proof is the following. First we show that there exists a B_2 -forest F in G . Then, by using an alternating structure on the components of F , we prove that F can be chosen in such a way that every component contains at most $2k - 2$ vertices from S . Let $\mathcal{C}$ denote the set of the connected components of F , and let $N = (S, \mathcal{J})$ be the partition matroid corresponding to partition classes $S(C)$ for $C \in \mathcal{C}$. Every component C is a B_2 -tree, hence it contains a perfect matching between $B(C)$ and $A(C) - a$ for any $a \in A(C)$. That is, if we leave out exactly one vertex from $A(C)$ for each $C \in \mathcal{C}$, the remaining vertices of A form a basis of R , and so the set of deleted vertices forms a basis in the strict gammoid that is the dual of R . This implies that $N \preceq M$ with $\chi(N) \leq 2k - 2$, thus proving the theorem.

We start with an easy observation.

CLAIM 4.1. *G contains k matchings of size $|B|$ such that every vertex in S is covered by at most $k - 1$ of them.*

Proof. Observe that a set $X \subseteq S$ is independent in M if and only if $A - X$ contains a basis of R , that is, $G - X$ has a matching covering B . The assumption that M is k -colorable is equivalent to the condition that S can be partitioned into k independent sets of M , and the claim follows. $\square$

The following claim proves an inequality that we will rely on.

CLAIM 4.2. *$k \cdot (|A| - |B|) - |S - X| \geq k \cdot \max\{|Y| - |\Gamma(Y)| : Y \subseteq X\}$ for every $X \subseteq A$.*

Proof. Let $\bar{R}$ be the matroid that is obtained from R by adding $k - 1$ parallel copies of every element in $A - S$, and adding $k - 2$ parallel copies of every element in S . The ground set A' of $\bar{R}$ has size $(k - 1)|S| + k|A - S|$. Then Claim 4.1 states that $\bar{R}$ has k pairwise disjoint bases.

Let $X \subseteq A$ be an arbitrary set and let X' be the set consisting of all the parallel copies of the elements of X . Then $|X'| = (k - 1) \cdot |X \cap S| + k \cdot |X - S|$ and $r_{\bar{R}}(X') = r_R(X) = \min\{|X| - |Y| + |\Gamma(Y)| : Y \subseteq X\}$. Recall that $|A'| = (k - 1) \cdot |S| + k \cdot |A - S|$ and $r_{\bar{R}}(A') = |B|$, hence

$$\begin{aligned} |A'| - |X'| &= (k - 1) \cdot |S| + k \cdot |A - S| - (k - 1) \cdot |X \cap S| - k \cdot |X - S| \\ &= (k - 1) \cdot |A| + |A - S| - (k - 1) \cdot |X| - |X - S| \\ &= (k - 1) \cdot |A - X| + |A - S - X|, \end{aligned}$$

519 and

$$\begin{aligned} 520 \quad r_{\bar{R}}(A') - r_{\bar{R}}(X') &= |B| - \min\{|X| - |Y| + |\Gamma(Y)| : Y \subseteq X\} \\ 521 \quad &= |B| - |X| + \max\{|Y| - |\Gamma(Y)| : Y \subseteq X\}. \end{aligned}$$

523 By Corollary 2.4 and Claim 4.1, $|A'| - |X'| \geq k \cdot (r_{\bar{R}}(A') - r_{\bar{R}}(X'))$, thus we get

$$524 \quad (k-1) \cdot |A - X| + |A - S - X| \geq k \cdot (|B| - |X| + \max\{|Y| - |\Gamma(Y)| : Y \subseteq X\}).$$

526 After rearranging, we obtain

$$527 \quad k \cdot (|A| - |B|) - |S - X| \geq k \cdot \max\{|Y| - |\Gamma(Y)| : Y \subseteq X\}$$

529 as stated. $\square$

530 Our next goal is to show that there exists a B_2 -forest in G .

531 CLAIM 4.3. $G = (A, B; E)$ contains a B_2 -forest.

532 *Proof.* As G has a matching of size $|B|$, the Hall condition holds for every subset of
533 B , thus $|\Gamma(U)| \geq |U|$ for every $U \subseteq B$. Let us call a set $U \subseteq B$ *tight* if $|\Gamma(U)| = |U|$.
534 Assume that G does not have a B_2 -forest. Then, by Theorem 2.1, there exists a
535 non-empty tight set in B . For arbitrary tight sets $U, W \subseteq B$, we get

$$\begin{aligned} 536 \quad |U| + |W| &= |\Gamma(U)| + |\Gamma(W)| = |\Gamma(U) \cap \Gamma(W)| + |\Gamma(U) \cup \Gamma(W)| \\ 537 \quad &\geq |\Gamma(U \cap W)| + |\Gamma(U \cup W)| \geq |U \cap W| + |U \cup W| \\ 538 \quad &= |U| + |W|, \end{aligned}$$

540 hence equality holds throughout, and so $U \cap W$ and $U \cup W$ are also tight. This implies
541 that there is a unique maximal tight set $\emptyset \neq Z \subseteq B$.

542 Let $X = A - \Gamma(Z)$. As Z is a tight set, $\max\{|Y| - |\Gamma(Y)| : Y \subseteq X\} \geq$
543 $|X| - |\Gamma(X)| \geq |A - \Gamma(Z)| - |B - Z| = |A| - |B|$, thus $S - X = \Gamma(Z) \cap S = \emptyset$
544 by Claim 4.2. Furthermore, every matching of size $|B|$ provides a perfect matching
545 between Z and $\Gamma(Z)$. That is, R is the direct sum of the transversal matroids R' and
546 R'' defined by $G[Z \cup \Gamma(Z)]$ and $G[(B - Z) \cup (A - \Gamma(Z))]$, respectively. Therefore M
547 is the restriction of the dual of R'' to S , contradicting the minimal choice of R . $\square$

548 Take an arbitrary B_2 -forest F in G . We will need the following technical claim.

549 CLAIM 4.4. Every leaf of F is in S .

550 *Proof.* Suppose to the contrary that F has a leaf vertex $a \in A - S$. Let $b \in B$ be
551 the unique neighbor of a in F , and let $G' = G - \{a, b\}$ denote the graph obtained by
552 deleting vertices a and b from G . Let $M' = (S, \mathcal{I}')$ denote the restriction of the dual
553 of the transversal matroid defined by G' to S . As the strong Hall condition holds
554 for G , the maximum size of a matching of G' is $|B| - 1$. We claim that $M = M'$,
555 contradicting the minimality of G .

556 Take an arbitrary set $X \in \mathcal{I}'$. By definition, $G' - X$ has a matching P' covering
557 $B - b$. Then $P' + ab$ is a matching of $G - X$ covering B , showing that $\mathcal{I}' \subseteq \mathcal{I}$.

558 To see the opposite direction, consider any set $X \in \mathcal{I}$. By definition, $G - X$ has
559 a matching P covering B . Take an arbitrary matching P' of G' covering $B - b$. Now
560 $|P| = |B| = |B - b| + 1 = |P'| + 1$, hence the symmetric difference $P \Delta P'$ contains an
561 alternating path Q whose first and last edges are in P , and one of the end vertices of
562 Q is b . Then $P \Delta Q$ is a matching of $G' - X$ covering $B - b$, implying $X \subseteq \mathcal{I}'$. $\square$

We denote the difference $|A| - |B|$ by q . As M is the restriction of the dual of R to S , $r_M(S) \leq q$ is clearly satisfied. Moreover, equality holds since, by Claim 4.4, every leaf of F is in S , and taking an arbitrary leaf in every component of F results a basis of M .

Let $\mathcal{C}$ denote the set of connected components of F . Note that a forest might have components consisting of a single vertex of A . We have $|\mathcal{C}| = |A| - |B| = q$ as $|A(C)| = |B(C)| + 1$ for each $C \in \mathcal{C}$. We call a component $C \in \mathcal{C}$ *large* if $|S(C)| \geq 2k - 1$, *normal* if $k \leq |S(C)| \leq 2k - 2$, and *small* if $|S(C)| \leq k - 1$. We say that a component $C' \in \mathcal{C}$ is *reachable* from a component $C'' \in \mathcal{C}$ if there exists an alternating sequence $C_1, b_1a_2, C_2, b_2a_3, \dots, b_{p-2}a_{p-1}, C_{p-1}, b_{p-1}a_p, C_p$ of components and edges such that $C_1 = C''$, $C_p = C'$, and $b_i \in B(C_i)$, $a_{i+1} \in A(C_{i+1})$ hold for $i = 1, \dots, p - 1$. Such an alternating sequence is called a *path*, the *length* of the path being $p - 1$. The *distance* of C' from C'' is the minimum length of a path from C'' to C' .

We define a potential function on the set of B_2 -forests as follows. Let $\nu \gg \mu_1 \gg \lambda_1 \gg \mu_2 \gg \lambda_2 \gg \dots \gg \mu_{q-1} \gg \lambda_{q-1}$ be a decreasing sequence of $2q - 1$ positive numbers such that the ratio between any two consecutive ones is at least $|A| + 2$. Recall that $|\mathcal{C}| = q$. For a component $C \in \mathcal{C}$, the minimum distance of C from a large component is denoted by $\text{dist}(C)$. We define $\text{dist}(C)$ to be $+\infty$ if C is not reachable from any of the large components. The *potential* of the B_2 -forest F is defined as

$$\begin{aligned} \varphi(F) = & \nu \cdot \sum_{C \in \mathcal{C}} \max\{|S(C)| - (2k - 2), 0\} & (\text{total violation}) \\ & - \sum_{i=1}^{q-1} \mu_i \cdot |\{C \in \mathcal{C} : \text{dist}(C) = i\}| & (\text{number of components at distance } i) \\ & + \sum_{i=1}^{q-1} \lambda_i \cdot \sum_{\substack{C \in \mathcal{C} \\ \text{dist}(C) = i}} |S(C)|. & (\text{number of } S\text{-vertices in components at distance } i) \end{aligned}$$

Let F be a B_2 -forest for which $\varphi(F)$ is as small as possible. The following claim concludes the proof of the theorem.

CLAIM 4.5. *F has no large components.*

Proof. Suppose indirectly that there exists a large component. By applying Claim 4.2 with $X = \emptyset$, $|S| \leq k \cdot (|A| - |B|) = k \cdot |\mathcal{C}|$, hence, by the pigeonhole principle, there exists a small component as well.

First we show that there exists a small component that is reachable from a large component. Suppose indirectly that this is not true, and let $\mathcal{C}' \subseteq \mathcal{C}$ denote the set of components that are not reachable from a large component. Note that $\mathcal{C}'$ consists of normal and small components. Define $X = \bigcup \{A(C) : C \in \mathcal{C}'\}$. By the definition of reachability, $\Gamma(X) = \bigcup \{B(C) : C \in \mathcal{C}'\}$ and so $|X| - |\Gamma(X)| = |\mathcal{C}'|$. As every component in $\mathcal{C} - \mathcal{C}'$ is either normal or large and there is at least one large component, $|S - X| \geq k \cdot |\mathcal{C} - \mathcal{C}'| + 1$. Then

$$\begin{aligned} k \cdot \max\{|Y| - |\Gamma(Y)| : Y \subseteq X\} & \geq k \cdot (|X| - |\Gamma(X)|) \\ & = k \cdot |\mathcal{C}'| \\ & = k \cdot (|\mathcal{C}| - |\mathcal{C} - \mathcal{C}'|) \\ & = k \cdot (|A| - |B|) - k \cdot |\mathcal{C} - \mathcal{C}'| \\ & \geq k \cdot (|A| - |B|) - |S - X| + 1, \end{aligned}$$

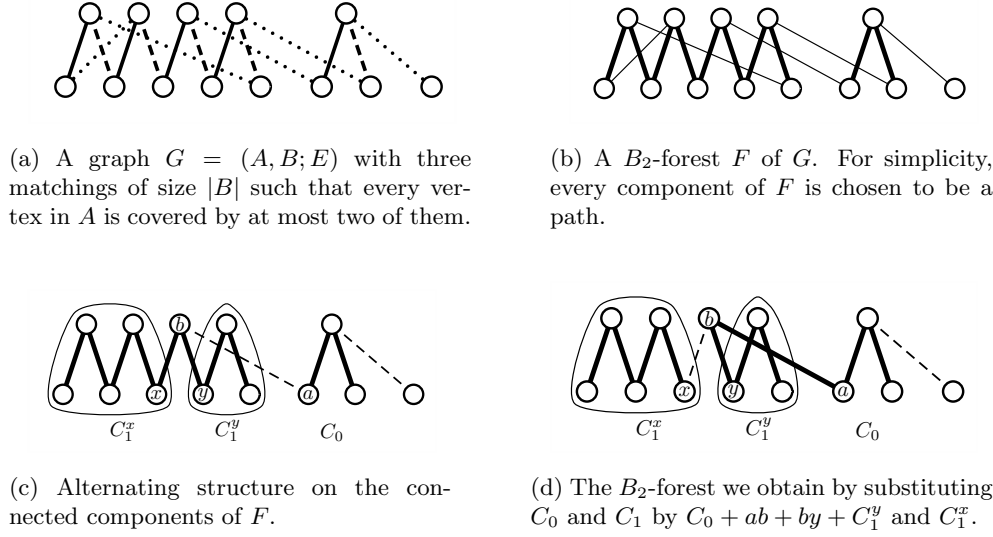


Fig. 3: An illustration of the proof of Theorem 1.10. In the example, $k = 3$ and $S = A$. The only large component of F is C_1 , all the other components are small.

contradicting Claim 4.2.

Let C_0 be a small component with $\text{dist}(C_0)$ being minimal. By the above, $\text{dist}(C_0) < +\infty$. Consider a shortest path from the set of large components to C_0 , and let C_1 be the last component on the path before C_0 . By the definition of a path, there exists an edge ab with $a \in A(C_0)$ and $b \in B(C_1)$. Let $x, y \in A(C_1)$ denote the neighbors of b in C_1 . The deletion of b from C_1 results in two connected components C_1^x and C_1^y such that $x \in C_1^x$ and $y \in C_1^y$ (see Figure 3).

Assume first that C_1 is a large component. As $|S(C_1)| \geq 2k - 1$ and $|S(C_0)| \leq k - 1$, either $|S(C_0 + ab + bx + C_1^x)| < |S(C_1)|$ or $|S(C_0 + ab + by + C_1^y)| < |S(C_1)|$ by Claim 4.4. Hence substituting C_0 and C_1 either by $C_0 + ab + bx + C_1^x$ and C_1^y or by $C_0 + ab + by + C_1^y$ and C_1^x decreases the total violation in $\varphi(F)$, a contradiction.

Therefore C_1 is a normal component, and there is another non-small component C_2 before C_1 on the shortest path from the set of large components to C_0 , together with an edge $b'a'$ with $a' \in A(C_1)$ and $b' \in A(C_2)$. We may assume that $a' \in C_1^x$. We distinguish two cases.

Case 1. $|S(C_1^x)| \geq |S(C_1)| - |S(C_0)|$

Modify F by substituting components C_0 and C_1 by $C_0 + ab + by + C_1^y$ and C_1^x , respectively. By the assumption, $|S(C_0 + ab + by + C_1^y)| = |S(C_0)| + |S(C_1)| - |S(C_1^x)| \leq 2 \cdot |S(C_0)| \leq 2k - 2$, thus no new large component appears. Furthermore, the set of components with distance less than $\text{dist}(C_1)$ does not change. The distance of C_1^x remains $\text{dist}(C_1)$ because of the edge $b'a'$. If the distance of $C_0 + ab + by + C_1^y$ is $\text{dist}(C_1)$, then the number of components at distance $\text{dist}(C_1)$ increases. Otherwise, the distance of $C_0 + ab + by + C_1^y$ is at least $\text{dist}(C_1) + 1$, hence the number of S -vertices in components at distance $\text{dist}(C_1)$ decreases by Claim 4.4. In both cases, $\varphi(F)$ decreases, a contradiction.

Case 2. $|S(C_1^x)| < |S(C_1)| - |S(C_0)|$

Modify F by substituting components C_0 and C_1 by $C_0 + ab + bx + C_1^x$ and C_1^y , respectively. By the assumption, $|S(C_0 + ab + bx + C_1^x)| \leq |S(C_0)| + |S(C_1)| - |S(C_0)| = |S(C_1)|$. As C_1 is normal, no new large component appears. Furthermore, the set of components with distance less than $\text{dist}(C_1)$ does not change. The distance of $C_0 + ab + bx + C_1^x$ remains $\text{dist}(C_1)$ because of the edge $b'a'$. The distance of C_1^y is either $\text{dist}(C_1)$ or $\text{dist}(C_0)$. In the former case, the number of components at distance $\text{dist}(C_1)$ increases, while in the latter case, the number of S -vertices in components at distance $\text{dist}(C_1)$ decreases as $|S(C_1^x)| + |S(C_0)| < |S(C_1)|$. In both cases, $\varphi(F)$ decreases, a contradiction. $\square$

By Claim 4.5, F has no large component. As we have seen before, the partition matroid $N = (S, \mathcal{J})$ corresponding to partition classes $S(C)$ for $C \in \mathcal{C}$ is a reduction of the original gammoid M with coloring number at most $2k - 2$. By Claim 4.4, $S(C)$ is non-empty for every $C \in \mathcal{C}$ and $r_M(S) = |\mathcal{C}| = q$, hence the reduction is rank-preserving. $\square$

For the first sight, the proof seems to provide a polynomial-time algorithm for determining the partition matroid, assuming that a digraph $D = (V, A)$ representing the gammoid is given. A bipartite graph $G = (A, B; E)$ representing R can be constructed from D (see e.g. [8]). The reductions appearing in the proofs of Claims 4.3 and 4.4 can be performed in polynomial time, hence we may assume that G contains a B_2 -forest F . Such a forest can be found by [21]. By using the alternating structure described in the proof of Claim 4.5, we can modify F to get a B_2 -forest in which every component contains at most $2k - 2$ vertices from S . However, it is not clear how to bound the number of augmentation steps as the coefficients in the potential function can be exponential. An interesting question is whether this procedure terminates after a polynomial number of steps.

5. Matroid operations and reducibility. In this section, we discuss the connection between reducibility and certain matroid operations such as direct sum, homomorphic image, and truncation. The first of these is straightforward: if matroids M_1 and M_2 are reducible to partition matroids N_1 and N_2 , respectively, then $M_1 \oplus M_2$ is reducible to the partition matroid $N_1 \oplus N_2$. Note that the coloring number of a direct sum is the maximum coloring number of its summands. In particular, if $\chi(N_1) \leq 2\chi(M_1)$ and $\chi(N_2) \leq 2\chi(M_2)$, then $\chi(N_1 \oplus N_2) \leq 2\chi(M_1 \oplus M_2)$.

THEOREM 1.12. *Let $M = (S, \mathcal{I})$ be a matroid such that for any $S' \subseteq S$ the restriction $M' = M|S'$ can be reduced to a $2\chi(M')$ -colorable partition matroid. Then any homomorphic image has the same property.*

Proof. Consider the homomorphic image $M_\varphi = (T, \mathcal{I}_\varphi)$ for a map $\varphi: S \rightarrow T$. We have to show that $M_\varphi|T'$ can be reduced to a $2\chi(M_\varphi|T')$ -colorable partition matroid for any $T' \subseteq T$. Let $T' = T_1 \cup \dots \cup T_k$ be a partition into $k = \chi(M_\varphi|T')$ independent sets of $M_\varphi|T'$. Clearly, the T_i s are also independent in M_φ . For $i = 1, \dots, k$, let S_i be an inclusionwise minimal independent set of M such that $\varphi(S_i) = T_i$. Then the restriction of φ is a bijection between $S' = S_1 \cup \dots \cup S_k$ and T' . By our assumptions, the restriction $M|S'$ can be reduced to a $2\chi(M|S')$ -colorable partition matroid N' . Then N'_φ is a reduction of $M_\varphi|T'$, and since φ is a bijection between S' and T' , it is a partition matroid with $\chi(N'_\varphi) = \chi(N') \leq 2\chi(M|S') \leq 2k$. $\square$

Note that the sum $M_1 + M_2$ of matroids M_1 and M_2 is the homomorphic image of

their direct sum $M_1 \oplus M_2$. Therefore, if M_1 and M_2 satisfy the reducibility assumption of Theorem 1.12, then so does $M_1 + M_2$.

In the previous theorem we could have replaced the bound $2\chi(M')$ by any function of $\chi(M')$. However, factor two plays a crucial role in our next result for truncation.

THEOREM 1.13. *The family of matroids M that can be reduced to a $2\chi(M)$ -colorable partition matroid is closed under truncation.*

Proof. Let $M = (S, \mathcal{I})$ denote a matroid of rank r that is reducible to a $2\chi(M)$ -colorable partition matroid N . As every k -truncation of M can be obtained by a series of $r-1, r-2, \dots, k$ -truncations, it suffices to prove that the $(r-1)$ -truncation M' of M is reducible to a $2\chi(M')$ -colorable partition matroid.

Let $S = S_1 \cup \dots \cup S_q$ denote the partition that defines N . We may assume that $|S_1| \geq \dots \geq |S_q|$. If $q \leq r-1$, then N is already a $2\chi(M)$ -colorable reduction of M' and the claim follows by $\chi(M') \geq \chi(M)$. Hence assume that $q = r$. Consider the partition matroid N' defined by the partition classes $S_1, \dots, S_{r-2}, S_{r-1} \cup S_r$. Then N' is a reduction of M' , hence it suffices to prove that N' is $2\chi(M')$ -colorable.

If $|S_{r-1}| + |S_r| \leq |S_1|$, then $\chi(N') = |S_1| = \chi(N) \leq 2\chi(M) \leq 2\chi(M')$. Otherwise $|S_{r-1}| + |S_r| > |S_1|$, and so $\chi(N') = |S_{r-1}| + |S_r|$. Using $|S| = |S_1| + \dots + |S_r|$ and $|S_i| \geq (|S_{r-1}| + |S_r|)/2$ for $i = 1, 2, \dots, r-2$, we get

$$\begin{aligned} |S| &\geq (r-2) \cdot \frac{|S_{r-1}| + |S_r|}{2} + |S_{r-1}| + |S_r| \\ &= \frac{r}{2} \cdot (|S_{r-1}| + |S_r|) \\ &= \frac{r}{2} \cdot \chi(N'). \end{aligned}$$

That is,

$$\chi(N') \leq \frac{2|S|}{r} < 2 \cdot \frac{|S|}{r-1} \leq 2\chi(M'),$$

concluding the proof of the theorem. $\square$

Remark 5.1. Note that an analogous statement holds if we replace $2\chi(M)$ by $2\chi(M) - 1$, as we proved $\chi(N') < 2\chi(M')$ in the second case. As laminar matroids can be obtained from free matroids by taking direct sums and truncations, Theorem 1.13 provides a simple proof that every k -colorable laminar matroid is reducible to a $(2k-1)$ -colorable partition matroid. As laminar matroids form a subclass of gammoids, Theorem 1.10 implies the slightly stronger bound of $2k-2$. However, it is not clear whether the analogue of Theorem 1.13 holds if we replace $2\chi(M)$ by $2\chi(M) - 2$.

As a consequence of our results, we get that Conjecture 1.11 holds for matroids that can be obtained from paving matroids, graphic matroids and gammoids by a series of direct sums, homomorphic images and truncations.

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