ON THE KREĬN-VON NEUMANN AND FRIEDRICHS EXTENSION OF POSITIVE OPERATORS

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Dedicated to Seppo Hassi on the occasion of his 60th birthday

1 Introduction

In his profound paper (von Neumann, 1931), J. von Neumann introduced the concept of the adjoint of a densely defined possibly unbounded operator $J : \mathcal{K} \to \mathcal{H}$ between two Hilbert spaces as the operator $J^* : \mathcal{H} \to \mathcal{K}$, having the domain

$$\operatorname{dom} J^* = \{g \in \mathcal{H} : \exists k' \in \mathcal{K} \text{ such that } (Jk \,|\, g) = (k \,|\, k') \ \forall k \in \operatorname{dom} J\},$$

by setting

$$J^*g := k', \quad g \in \operatorname{dom} J^*.$$

Although the adjoint operator behaves nicer than the original one (because it is always closed), it is not necessarily densely defined. An essential question arises therefore: when is the domain $\operatorname{dom} J^*$ a dense subspace of $\mathcal{H}$? Von Neumann himself gave an elegant answer to that question. Namely, he proved that J^* is densely defined if and only if J is a closable operator. Moreover, in that case the second adjoint J^{**} of J exists and it is equal to the closure $\overline{J}$ of J:

$$\overline{J} = J^{**}.$$

At the same time, $J^{**}J^*$ and J^*J^{**} are positive self-adjoint operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Note also that we have

$$\operatorname{dom}(J^{**}J^*)^{1/2} = \operatorname{dom} J^* \quad \text{and} \quad \operatorname{dom}(J^*J^{**})^{1/2} = \operatorname{dom} J^{**}$$

on the domains, and

$$\operatorname{ran}(J^{**}J^*)^{1/2} = \operatorname{ran} J^{**} \quad \text{and} \quad \operatorname{ran}(J^*J^{**})^{1/2} = \operatorname{ran} J^*$$

on the ranges. Here, for a given positive self-adjoint operator A, $A^{1/2}$ denotes the unique positive self-adjoint square root of A; see, e.g., (Sebestyén & Tarcsay, 2017).

However, if J is not closed, then J^*J and JJ^* are not self-adjoint operators in general. In fact, it is not even clear whether those operators are densely defined, and therefore it is also a non-trivial question whether they have any positive self-adjoint extensions at all. From classical works by Friedrichs, Kreĭn, and von Neumann, we know that a densely defined positive and symmetric operator may be extended to a positive self-adjoint operator, see, e.g., (Friedrichs, 1934; Kreĭn, 1947; von Neumann, 1931). In that case, there exist two distinguished self-adjoint extensions A_N and A_F of any positive symmetric operator A, such that

$$A_N \leq A_F,$$

and every positive self-adjoint extension $\widetilde{A}$ of A is between A_N and A_F: $A_N \leq \widetilde{A} \leq A_F$. The smallest extension A_N of A is called the *Kreĭn-von Neumann extension*, while the largest extension A_F of A is called the *Friedrichs extension*.

The problem of the existence of positive self-adjoint extensions has its relevance even in the non-densely defined case. Although the Friedrichs extension exists only for a densely defined operator, the smallest extension always exists if there exists any extension, see, e.g., (Sebestyén & Stochel, 1991) and also (Sebestyén & Stochel, 2007; Hassi, 2004).

In the present paper we revise the main result Theorem 1 of Sebestyén & Stochel (1991) and give some new characterizations for a not necessarily densely defined positive symmetric operator to admit positive self-adjoint extensions. More specifically, in Section 2 we collect some new properties for an operator to be closable. Based on this new characterization of closability, we establish in Section 3 the correct version of the "duality theorem" stated in Jorgensen, Pearse & Tian (2018: Theorem 5). In Section 4 we give a short proof of the fact that the "modulus square" operator T^*T of any densely defined operator T always has a positive self-adjoint extension, cf. (Sebestyén & Tarcsay, 2012: Theorem 2.1). At the same time, we shall see that this is not the case with TT^*; that operator might be even non-closable. However, we are going to establish necessary and sufficient conditions for the extendibility of TT^*. In particular, our construction of the Kreĭn-von Neumann extension in Section 4 will be used to exhibit a counterexample to (Jorgensen, Pearse & Tian, 2018: Theorem 5). Finally, in Section 5 we treat the problem of the existence of the Friedrichs extension of a densely defined positive symmetric operator. In particular, we discuss there the case when the Friedrichs extension of the operator T^*T is identical with T^*T^{**}.

2 Closable operators

Let J be a densely defined operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Note that J is *closable* if for each sequence $(g_n)_{n\in\mathbb{N}} \subset \operatorname{dom} J$, such that $g_n \to 0$ and $Jg_n \to h$, it follows that $h = 0$. On the other hand, a profound theorem by von Neumann tells us that J is closable if and only if J^* is densely defined, that is,

$$(\operatorname{dom} J^*)^\perp = \{0\}.$$

In the following theorem, we give an extension of von Neumann's result and collect some new characteristic properties for an operator J to be closable. For further characterizations of closability and closedness, see, e.g., (Popovici & Sebestyén, 2014; Sebestyén & Tarcsay, 2016; 2019; 2020).

Theorem 2.1. *Let J be a densely defined operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Then the following properties are equivalent:*

(i) *J is closable;*

(ii) $(\operatorname{dom} J^*)^\perp = \{0\}$*;*

(iii) $(\operatorname{dom} J^*)^\perp \cap (\operatorname{ran} J)^{\perp\perp} = \{0\}$*;*

(iv) $(\operatorname{dom} J^*)^\perp \subseteq \operatorname{ran}(I + JJ^*)$.

Proof. (i) ⇒ (ii) Consider a vector $h \in (\text{dom } J^*)^\perp$, then

$$(0, h) \in \{\{-J^*k, k\} : k \in \text{dom } J^*\} = \overline{G(J)}.$$

Since J is closable, this implies $h = 0$.

(ii) ⇒ (iii) This implication is trivial.

(iii) ⇒ (i) Consider a sequence $(g_n)_{n\in\mathbb{N}} \subset \text{dom } J$ such that $g_n \to 0$, and $Jg_n \to h$. Then $h \in \overline{\text{ran } J} = (\text{ran } J)^{\perp\perp}$. On the other hand, for every $f \in \text{dom } J^*$

$$(f \mid h) = \lim_{n\to\infty} (f \mid Jg_n) = \lim_{n\to\infty} (J^*f \mid g_n) = 0,$$

which means that $h \in (\text{dom } J^*)^\perp$. Consequently, $h = 0$ by (ii) and therefore J is closable.

(ii) ⇒ (iv) This implication is clear.

(iv) ⇒ (ii) We are going to show that $\text{dom } J^*$ is dense in $\mathcal{H}$. To this aim, take $g \in (\text{dom } J^*)^\perp$. By (iv), there exists $h \in \text{dom } JJ^*$ such that $g = h + JJ^*h$. In particular, $h \in \text{dom } J^*$ and one has

$$0 = (g \mid h) = (h + JJ^*h \mid h) = (h \mid h) + (JJ^*h \mid h) = \|h\|^2 + \|J^*h\|^2,$$

so that $h = 0$. This implies that $g = 0$ and therefore (iv) implies (ii). □

3 Duality theorems

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with a common vector subspace $\mathcal{D}$. In Jorgensen, Pearse & Tian (2018: Theorem 5) a necessary and sufficient condition is stated for the existence of a positive and self-adjoint operator Δ on $\mathcal{H}_1$ with the duality property

$$(\Delta\varphi \mid \psi)_1 = (\varphi \mid \psi)_2, \qquad \varphi, \psi \in \mathcal{D},$$

cf. also (Jorgensen & Pearse, 2016: Theorem 4.1). Unfortunately, there is a simple but serious error in their proof and the statement itself is not true in that form either (a counterexample will be exhibited in Example 4.2 below). In Theorem 3.3 we are going to establish the correct form of that statement. Its proof depends on the following lemma.

Lemma 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $J : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator between them. Then the following three statement are equivalent:*

(i) $\text{ran } J \subseteq \text{dom } J^*$*;*

(ii) *J is closable and* $\text{dom } J \subseteq \text{dom } J^*J^{**}$*;*

(iii) *there exists a positive self-adjoint operator A in $\mathcal{K}$ such that* $\text{dom } J \subseteq \text{dom } A$ *and*

$$(Ag \mid k) = (Jg \mid Jk), \qquad g, k \in \text{dom } J. \tag{3.1}$$

Proof. (i) ⇒ (ii) Since $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$, it follows that

$$(\mathrm{ran}\ J)^{\perp\perp} \subseteq (\mathrm{dom}\ J^*)^{\perp\perp} = \mathcal{H} \ominus (\mathrm{dom}\ J^*)^{\perp},$$

and, consequently,

$$(\mathrm{dom}\ J^*)^{\perp} \cap (\mathrm{ran}\ J)^{\perp\perp} = \{0\}.$$

Applying Theorem 2.1 we see that J is closable. On the other hand, $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$ implies that $\mathrm{dom}\ J = \mathrm{dom}\ J^*J \subseteq \mathrm{dom}\ J^*J^{**}$.

(ii) ⇒ (iii) If J is closable, then $A := J^*J^{**}$ is a positive self-adjoint operator in $\mathcal{H}$, and by (ii) one has $\mathrm{dom}\ J \subseteq \mathrm{dom}\ A$. On the other hand,

$$(Ag \,|\, k) = (J^*J^{**}g \,|\, k) = (J^{**}g \,|\, J^{**}k) = (Jg \,|\, Jk),$$

for every $g, k \in \mathrm{dom}\ J$.

(iii) ⇒ (i) Suppose that A is a positive operator with $\mathrm{dom}\ J \subseteq \mathrm{dom}\ A$ which satisfies (3.1). Let $k \in \mathrm{dom}\ J$ be arbitrary, then for every $g \in \mathrm{dom}\ J$

$$(Jg \,|\, Jk) = (Ag \,|\, k) = (g \,|\, Ak),$$

which implies $Jk \in \mathrm{dom}\ J^*$. Therefore, $\mathrm{ran}\ J \subseteq \mathrm{dom}\ J^*$. □

Remark 3.2. Let J be a closed operator. Then the inclusion

$$\mathrm{dom}\ J \subseteq \mathrm{dom}\ J^*J^{**} \tag{3.2}$$

is only possible if J is continuous and everywhere defined on $\mathcal{H}_1$, see, e.g., (Tarcsay, 2012: Lemma 2.1). This suggests that Lemma 3.1 is only relevant if J is a closable but not a closed operator.

The erroneous observation in the proof of (Jorgensen, Pearse & Tian, 2018: Theorem 5) is that (3.2) holds true provided that both J and J^* are densely defined. This makes it necessary to provide the following correct version of (Jorgensen, Pearse & Tian, 2018: Theorem 5), which can also be considered as a noncommutative version of the Lebesgue-Radon-Nikodym decomposition theorem.

Theorem 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real or complex Hilbert spaces which contain a common linear manifold $\mathcal{D}$ as a vector space. Suppose that $\mathcal{D}$ is dense in $\mathcal{H}_1$ and set*

$$\mathcal{D}^* := \big\{h \in \mathcal{H}_2 : \exists C_h \geq 0 \ \text{ such that } \ |(\varphi \,|\, h)_2| \leq C_h\|\varphi\|_1 \ \ \forall \varphi \in \mathcal{D}\big\}.$$

Then the following two conditions are equivalent:

(i) *$\mathcal{D} \subseteq \mathcal{D}^*$ in $\mathcal{H}_2$;*

(ii) *there exists a positive self-adjoint operator Δ in $\mathcal{H}_2$ such that $\mathcal{D} \subseteq \mathrm{dom}\ \Delta$ in $\mathcal{H}_1$ and*

$$(\Delta\varphi \,|\, \psi)_1 = (\varphi \,|\, \psi)_2, \qquad \varphi, \psi \in \mathcal{D}. \tag{3.3}$$

Proof. Let J be the operator from $\mathcal{D} \subseteq \mathcal{H}_1$ to $\mathcal{H}_2$ defined by the identification $J\varphi := \varphi$, $\varphi \in \mathcal{D}$. Then J is a densely defined operator such that its adjoint J^* has domain $\mathcal{D}^*$: $\mathrm{dom}\ J^* = \mathcal{D}^*$. The desired equivalence follows now from Lemma 3.1. □

4 Von Neumann's problem on positive self-adjoint extendibility

Given a positive symmetric operator A in a real or complex Hilbert space $\mathcal{K}$, the question arises whether there exists a positive self-adjoint extension of A. If the operator in question is densely defined, then we know from classical papers by Friedrichs, Kreĭn, and von Neumann that the operator has a positive self-adjoint extension, see (Friedrichs, 1934; Kreĭn, 1947; von Neumann, 1931); cf. also (Ando & Nishio, 1970; Arlinskiĭ et al., 2001; Prokaj & Sebestyén, 1996a;b; Schmüdgen, 2012). However, uniqueness of the extension occurs only in the very special case when the operator in question is essentially self-adjoint. In all other cases, the set of positive self-adjoint extensions is an operator interval $[A_N, A_F]$, where A_N is the smallest (the so-called Kreĭn-von Neumann) extension, while A_F is the largest (the so-called Friedrichs) extension of A. Recall that the partial ordering among the set of positive self-adjoint operators is given by

$$A \leq B \quad \iff \quad (I+B)^{-1} \leq (I+A)^{-1}.$$

Equivalently, by means of the square roots, one has $A \leq B$ if and only if

$$\operatorname{dom} B^{1/2} \subseteq \operatorname{dom} A^{1/2} \quad \text{and} \quad \|A^{1/2}k\|^2 \leq \|B^{1/2}k\|^2, \qquad \forall k \in \operatorname{dom} B^{1/2}.$$

The problem of the existence of positive self-adjoint extensions has its relevance even in the non-densely defined case, and was treated in detail by Sebestyén & Stochel (1991), see also (Sebestyén & Stochel, 2007; Hassi, 2004).

In the next result we revise (Sebestyén & Stochel, 1991: Theorem 1) on the existence of the Kreĭn-von Neumann extension of a positive and symmetric operator A. In this case it is convenient to introduce the linear space $\mathcal{D}_*(A)$ by

$$\mathcal{D}_*(A) := \{k \in \mathcal{K} :\ \sup\{|(Ag\,|\,k)| :\ g \in \operatorname{dom} A,\ (Ag\,|\,g) \leq 1\} < +\infty\}. \tag{4.1}$$

Theorem 4.1. *Let A be a positive and symmetric operator on a real or complex Hilbert space $\mathcal{K}$. Then the following statements are equivalent:*

(i) $\mathcal{D}_*(A)$ *as in* (4.1) *is dense in* $\mathcal{K}$*;*

(ii) *for every sequence* $(g_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A$ *and* $k \in \mathcal{K}$ *such that*

$$(Ag_n\,|\,g_n)_{\mathcal{K}} \to 0 \quad \textit{and} \quad Ag_n \to k,$$

it follows that $k = 0$*;*

(iii) *there exist a Hilbert space* $\mathcal{E}$ *and a densely defined linear operator* $V : \mathcal{K} \to \mathcal{E}$ *such that* $\operatorname{dom} A \subseteq \operatorname{dom} V$, $(V(\operatorname{dom} A))^\perp = \{0\}$, *and*

$$\langle Vg, Vh\rangle_{\mathcal{E}} = (Ag\,|\,h)_{\mathcal{K}}, \qquad g \in \operatorname{dom} A,\ h \in \operatorname{dom} V; \tag{4.2}$$

(iv) *there exists a positive self-adjoint extension of A.*

If any, and hence all, assertions of (i)-(iv) *are satisfied, then there exists the smallest positive extension A_N of A.*

Proof. (i) $\Rightarrow$ (ii) Assume that $(Ag \,|\, g) = 0$ for some $g \in \mathrm{dom}\ A$. Then $\sup |(Ag, k)| < \infty$ for all $k \in \mathcal{D}_*(A)$, which implies $(Ag, k) = 0$. Since $\mathcal{D}_*(A) \subseteq \mathcal{K}$ is dense by (i), it follows that $Ag = 0$.

This means that

$$\langle Ag, Ah\rangle_{\mathcal{E}} := (Ag \,|\, h)_{\mathcal{K}}, \qquad g, h \in \mathrm{dom}\ A, \tag{4.3}$$

defines an inner product on $\mathrm{ran}\ A$. Denote by $\mathcal{E}$ the completion of that space and consider the natural inclusion operator $J_A : \mathcal{E} \supseteq \mathrm{ran}\ A \to \mathcal{K}$,

$$J_A(Ag) := Ag \in \mathcal{K}, \qquad g \in \mathrm{dom}\ A. \tag{4.4}$$

Clearly, $\mathrm{ran}\ A$ forms a dense linear manifold in $\mathcal{E}$ by definition, so that J_A is densely defined. On the other hand, one has

$$\mathrm{dom}\ J_A^* = \mathcal{D}_*(A), \tag{4.5}$$

thanks to the identities

$$(J_A(Ag) \,|\, h)_{\mathcal{K}} = (Ag \,|\, h)_{\mathcal{K}}, \qquad g \in \mathrm{dom}\ A,\ h \in \mathcal{K},$$

and

$$\langle Ag, Ag\rangle_{\mathcal{E}} = (Ag \,|\, g)_{\mathcal{K}}, \qquad g \in \mathrm{dom}\ A.$$

From (4.5) and (i) we see that J_A^* is densely defined and therefore J_A is closable. From this it follows that A fulfills (ii).

(ii) $\Rightarrow$ (iii) Note that the condition in (ii) implies that (4.3) defines an inner product. With the notation as in the proof of the implication (i) $\Rightarrow$ (ii), (ii) expresses that the canonical inclusion operator $J_A : \mathcal{E} \to \mathcal{K}$ is closable. Its adjoint $J_A^* : \mathcal{K} \to \mathcal{E}$ is therefore a densely defined operator such that

$$\langle J_A^* g, Ah\rangle_{\mathcal{E}} = (g \,|\, J_A(Ah))_{\mathcal{K}} = (g \,|\, Ah)_{\mathcal{K}} = \langle Ag, Ah\rangle_{\mathcal{E}}, \qquad g, h \in \mathrm{dom}\ A,$$

whence we conclude that

$$J_A^* g = Ag \in \mathcal{E}, \qquad g \in \mathrm{dom}\ A. \tag{4.6}$$

As a conseqence, J_A^* provides a factorization for A in the sense of (iii):

$$\langle J_A^* g, J_A^* h\rangle_{\mathcal{E}} = \langle Ag, J_A^* h\rangle_{\mathcal{E}} = (Ag \,|\, h)_{\mathcal{K}}, \qquad g \in \mathrm{dom}\ A,\ h \in \mathcal{D}_*(A). \tag{4.7}$$

Moreover, by (4.6) we see that

$$J_A^*(\mathrm{dom}\ A) = \{Ag :\ g \in \mathrm{dom}\ A\},$$

where the right-hand side is dense in $\mathcal{H}$ by definition. Hence, $V := J_A^*$ satisfies all requirements of (iii).

(iii) $\Rightarrow$ (iv) Let $V : \mathcal{K} \to \mathcal{E}$ be a densely defined closable operator satisfying the properties in (iii). By (4.2) we conclude that $Vg \in \mathrm{dom}\ V^*$ for every $g \in \mathrm{dom}\ A$ and that

$$V^* V g = Ag, \qquad g \in \mathrm{dom}\ A. \tag{4.8}$$

This means that $\mathrm{dom}\ V^*$ includes the dense set $V(\mathrm{dom}\ A)$, and therefore V is closable. Moreover, by (4.8) we see that $A \subset V^* V \subset V^* V^{**}$, i.e., the positive self-adjoint operator $V^* V^{**}$ extends A.

(iv) $\Rightarrow$ (i): Let B be a positive self-adjoint extension of A. Then for every $k \in \mathrm{dom}\ B^{1/2}$ and $g \in \mathrm{dom}\ A$ with $(Ag \,|\, g) \leq 1$, we obtain that

$$\begin{aligned} |(Ag \,|\, k)| &= |(Bg \,|\, k)| = |(B^{1/2}g \,|\, B^{1/2}k)| \\ &\leq \|B^{1/2}g\|\|B^{1/2}k\| = (Ag \,|\, g)^{1/2}\|B^{1/2}k\| \leq \|B^{1/2}k\|, \end{aligned}$$

whence $k \in \mathcal{D}_*(A)$. This implies that

$$\mathrm{dom}\ B^{1/2} \subseteq \mathcal{D}_*(A), \tag{4.9}$$

where the former subspace is dense in $\mathcal{K}$. Hence, $\mathcal{D}_*(A)$ is dense in $\mathcal{K}$, i.e., (i) holds.

Finally, let any, and hence all, assertions of (i)-(iv) be satisfied. First note that the operator J_A defined in (4.4) is closable by (i). Hence, from (4.6) and (4.7) it follows that

$$A_N := J_A^{**} J_A^* \tag{4.10}$$

is a positive self-adjoint extension of A. Furthermore we have

$$\mathcal{D}_*(A) = \mathrm{dom}\ J_A^* = \mathrm{dom}\,(J_A^{**} J_A^*)^{1/2} \tag{4.11}$$

and the density of $\mathrm{ran}\ A$ in $\mathcal{H}$ implies for every $k \in \mathcal{D}_*(A)$ that

$$\begin{aligned} \|(J_A^{**} J_A^*)^{1/2} k\|_{\mathcal{K}}^2 &= \|J_A^* k\|_{\mathcal{E}}^2 \\ &= \sup \left\{ |\langle Ag, J_A^* k\rangle_{\mathcal{E}}|^2 : g \in \mathrm{dom}\ A, \langle Ag, Ag\rangle_{\mathcal{E}} \leq 1 \right\} \\ &= \sup \left\{ |(J_A(Ag) \,|\, k)_{\mathcal{K}}|^2 : g \in \mathrm{dom}\ A,\ (Ag \,|\, g)_{\mathcal{K}} \leq 1 \right\} \\ &= \sup \left\{ |(Ag \,|\, k)_{\mathcal{K}}|^2 : g \in \mathrm{dom}\ A,\ (Ag \,|\, g)_{\mathcal{K}} \leq 1 \right\}. \end{aligned}$$

Next we show that A_N as in (4.10) is the smallest self-adjoint extension of A. Let therefore B be any positive self-adjoint extension of A. Since the positive self-adjoint operator B has no proper self-adjoint extension, applying the above construction for B, we infer that $B = J_B^{**} J_B^*$. By the inclusion (4.9) we have $\mathrm{dom}\ B^{1/2} \subseteq \mathrm{dom}\ A_N^{1/2}$, see (4.10) and (4.11), and from the above calculation we obtain that, for every $k \in \mathrm{dom}\ B^{1/2}$,

$$\begin{aligned} \|A_N^{1/2} k\|^2 = \|(J_A^{**} J_A^*)^{1/2} k\|^2 &= \sup \left\{ |(Ag \,|\, k)|^2 : g \in \mathrm{dom}\ A,\ (Ag \,|\, g) \leq 1 \right\} \\ &\leq \sup \left\{ |(Bg \,|\, k)|^2 : g \in \mathrm{dom}\ B,\ (Bg \,|\, g) \leq 1 \right\} \\ &= \|(J_B^{**} J_B^*)^{1/2} k\|_{\mathcal{K}}^2 = \|B^{1/2} k\|_{\mathcal{K}}^2. \end{aligned}$$

Hence $A_N \leq B$, as it is stated. □

As was mentioned in the previous section, the statement of (Jorgensen, Pearse & Tian, 2018: Theorem 5) is not correct, as with the notation used in Theorem 3.2, they assert that the existence of the positive self-adjoint operator Δ satisfying (3.3) is equivalent to $\mathcal{D}^*$ being dense in $\mathcal{H}_2$. Based on the preceding theorem and its proof, it will be shown by a counterexample that their assertion is not true in general.

Example 4.2. Consider an unbounded positive self-adjoint operator A in a Hilbert space $\mathcal{K}$ and set

$$\mathcal{D} := \mathrm{ran}\ A.$$

Denote by $\mathcal{E}$ the "energy space" associated with A and by J the corresponding inclusion operator $J : \mathcal{E} \supseteq \operatorname{ran} A \to \mathcal{K}$ as in the proof of Theorem 4.1. Then $\mathcal{D}$ is a common vector subspace of $\mathcal{E}$ and $\mathcal{K}$ such that $\mathcal{D} \subseteq \mathcal{E}$ is dense. Furthermore,

$$\begin{aligned}\mathcal{D}^* :=& \{k \in \mathcal{K} : \exists C_k \geq 0 \text{ such that } |(\varphi \,|\, k)^2_{\mathcal{K}}| \leq C_h \|\varphi\|^2_{\mathcal{E}} \ \ \forall \varphi \in \mathcal{D}\} \\ =& \{k \in \mathcal{K} : \exists C_k \geq 0 \text{ such that } |(Ah \,|\, k)^2_{\mathcal{H}}| \leq C_k (Ah \,|\, h)_{\mathcal{K}} \ \ \forall h \in \operatorname{dom} A\},\end{aligned}$$

from which we conclude that

$$\mathcal{D}^* = \mathcal{D}_*(A) = \operatorname{dom} J^*,$$

so $\mathcal{D}^* \subseteq \mathcal{K}$ is dense. Suppose that the conclusion of (Jorgensen, Pearse & Tian, 2018: Theorem 5) is true, then by that theorem the density of $\mathcal{D}^*$ in $\mathcal{K}$ implies that there exists a positive self-adjoint operator $\Delta : \mathcal{E} \to \mathcal{E}$, $\mathcal{D} \subseteq \operatorname{dom} \Delta$, such that

$$\langle \Delta\varphi, \varphi \rangle_{\mathcal{E}} = (\varphi \,|\, \varphi)_{\mathcal{K}}, \qquad \varphi \in \mathcal{D}.$$

From this we conclude that

$$(J(Ah) \,|\, Ak)_{\mathcal{K}} = (Ah \,|\, Ak)_{\mathcal{K}} = \langle Ah, \Delta(Ak) \rangle_{\mathcal{E}}, \qquad h \in \operatorname{dom} A,$$

which in turn means that $Ak \in \operatorname{dom} J^*$ and $J^*(Ak) = \Delta(Ak)$. As a consequence we see that $\operatorname{ran} A \subseteq \operatorname{dom} J^*$, and since $\operatorname{dom} A \subseteq \operatorname{dom} J^*$ holds true as well, we obtain that

$$\mathcal{K} = \operatorname{dom} A + \operatorname{ran} A \subseteq \operatorname{dom} J^*.$$

So J^* is an everywhere defined bounded operator on $\mathcal{K}$, and therefore so is $A = J^{**}J^*$. This is in contradiction to the assumption that A is an unbounded operator.

Thanks to a classical result of J. von Neumann (von Neumann, 1931) we know that T^*T and TT^* are positive self-adjoint operators whenever T is densely defined and closed. In Sebestyén & Tarcsay (2014) we proved the converse of that statement: if both T^*T and TT^* are self-adjoint then T is necessarily closed, see also (Gesztesy & Schmüdgen, 2019) and (Sandovici, 2018) for the case of linear relations. This means that if T is not closed (or not even closable), then either T^*T or TT^* (or even both) fail to be self-adjoint. In fact, TT^* might even be non-closable; however, surprisingly, T^*T behaves well. Namely, it was proved in Sebestyén & Tarcsay (2012: Theorem 2.1) that T^*T always has a positive self-adjoint extension. We provide a short proof of that result.

Theorem 4.3. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Then T^*T has a positive self-adjoint extension.*

Proof. Consider the positive symmetric operator $A := T^*T$. We are going to show that

$$\operatorname{dom} T \subseteq \mathcal{D}_*(A).$$

Indeed, for $g \in \operatorname{dom} A$ and $k \in \operatorname{dom} T$, we have

$$\begin{aligned}|(Ag \,|\, k)|^2 = |(T^*Tg \,|\, k)|^2 = |(Tg \,|\, Tk)|^2 &\leq (Tg \,|\, Tg)(Tk \,|\, Tk) \\ &= (T^*Tg \,|\, g)(Tk \,|\, Tk) = (Ag \,|\, g)(Tk \,|\, Tk).\end{aligned}$$

Hence $\mathcal{D}_*(A)$ is dense in $\mathcal{K}$. Thus A has by Theorem 4.1 a positive self-adjoint extension. □

In the next result we deal with the positive extendibility of TT^*.

Theorem 4.4. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined operator between the real or complex Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. Then the following two statements are equivalent:*

(i) *TT^* has a positive self-adjoint extension;*

(ii) *$T|_{\mathrm{dom}\, T \cap \mathrm{ran}\, T^*}$ is a closable operator.*

Proof. The positive symmetric operator $A := TT^*$ has a positive self-adjoint extension if and only if it satisfies condition (ii) of Theorem 4.1. That is, according to that result, T^*T has a positive self-adjoint extension if and only if for every sequence $(h_n) \subset \mathrm{dom}\, TT^*$ and every vector $f \in \mathcal{H}$ the conditions

$$(TT^*h_n \,|\, h_n) = \|T^*h_n\|^2 \to 0 \qquad \text{and} \qquad TT^*h_n \to f,$$

imply that $f = 0$. Evidently, this is equivalent to the closability of the restriction of T to the set $\mathrm{ran}\, T^* \cap \mathrm{dom}\, T$. □

In the following example, we show that TT^* may have a bounded positive self-adjoint extension in some cases even if T is not even closable.

Example 4.5. Let $\mathcal{K}$ be a separable Hilbert space and consider two orthonormal bases in it

$$\{e_{n,m} : n, m \in \mathbb{N}\} \quad \text{and} \quad \{f_n : n \in \mathbb{N}\}.$$

Let us define the operator T on the vectors $e_{n,m}$ by setting

$$Te_{n,m} := mf_n, \qquad n, m \in \mathbb{N},$$

and then extend it by linearity to $\mathrm{dom}\, T := \mathrm{span}\, \{e_{n,m} : n, m \in \mathbb{N}\}$. It follows from this definition that $\mathrm{dom}\, T^* = \{0\}$. In order to see this, observe that for $z \in \mathrm{dom}\, T^*$ and $n \in \mathbb{N}$ we have

$$(z \,|\, f_n) = \frac{1}{m}(z \,|\, Te_{n,m}) = \frac{1}{m}(T^*z \,|\, e_{n,m}),$$

for any $m \in \mathbb{N}$. Letting $m \to \infty$ gives that $(z \,|\, f_n) = 0$ and, hence, $z = 0$. Consequently, T is non-closable (in fact, T is a maximal singular operator), but $A = 0$ is a (bounded) positive self-adjoint extension of TT^*.

The previous example demonstrated that TT^* can behave nicely even though T is singular. However, as the following example illustrates, there exists an operator T such that TT^* is non-closable.

Example 4.6. Consider a maximal singular operator T in a Hilbert space $\mathcal{K}$, that is, an operator such that $\mathrm{dom}\, T^* = \{0\}$ (take e.g. the operator T from Example 4.5). Consider the following operator

$$J : \mathcal{K} \supseteq \mathrm{dom}\, T \to \mathcal{K} \times \mathcal{K}, \qquad Jg := \{g, Tg\}.$$

Then it is easy to verify that $\mathrm{dom}\, J^* = \mathcal{K} \times \mathrm{dom}\, T^* = \mathcal{K} \times \{0\}$, and $J^*\{k, 0\} = k$. In particular, $\mathrm{dom}\, JJ^* = \mathrm{dom}\, T \times \{0\}$, and

$$JJ^*\{g, 0\} = \{g, Tg\}, \qquad g \in \mathrm{dom}\, T.$$

Furthermore, we claim that J is not closable. Indeed, take any nonzero $k \in \mathcal{K}$, then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}\ T$ such that $g_n \to 0$ and $Tg_n \to k$. Then

$$JJ^*\{g_n, 0\} = \{g_n, Tg_n\} \to \{0, k\},$$

which means that JJ^* may not be closable.

Theorem 4.7. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined closable linear operator, such that*

$$\mathrm{dom}\ T \subseteq \mathrm{ran}\ T^*. \tag{4.12}$$

Then $T^{**}T^*$ agrees with the Kreĭn-von Neumann extension of TT^*.

Proof. Denote by $\mathcal{E}$ the completion of $\mathrm{ran}\ TT^*$ under the inner product

$$\langle TT^*h, TT^*f\rangle := (TT^*h \mid f) = (T^*h \mid T^*f), \qquad h, f \in \mathrm{dom}\ TT^*.$$

By the construction of the proof of Theorem 4.1, the Kreĭn-von Neumann extension of TT^* is of the form $J^{**}J^*$, where J is the natural inclusion operator from $\mathcal{E} \supseteq \mathrm{ran}\ TT^*$ into $\mathcal{H}$:

$$J(TT^*h) := TT^*h, \qquad h \in \mathrm{dom}\ TT^*.$$

Note that by (4.12) we have the identity $\mathrm{dom}\ T = \{T^*g : g \in \mathrm{dom}\ TT^*\}$. Consequently,

$$\begin{aligned}
\mathrm{dom}\,(J^{**}J^*)^{1/2} &= \mathrm{dom}\ J^* = \mathcal{D}_*(TT^*) \\
&= \Big\{h \in \mathcal{H} : \sup\Big\{|(TT^*f \mid h)| : f \in \mathrm{dom}\ TT^*,\ \|T^*f\|^2 \le 1\Big\} < +\infty\Big\} \\
&= \mathrm{dom}\ T^* = \mathrm{dom}\,(T^{**}T^*)^{1/2}.
\end{aligned}$$

At the same time we have that

$$\begin{aligned}
\|(J^{**}J^*)^{1/2}h\|^2 &= \|J^*h\|^2_{\mathcal{E}} \\
&= \sup\left\{|\langle TT^*f, J^*h\rangle^2| : f \in \mathrm{dom}\ TT^*,\ \|T^*f\|^2 \le 1\right\} \\
&= \sup\left\{|(T^*f \mid T^*h)|^2 : f \in \mathrm{dom}\ TT^*,\ \|T^*f\|^2 \le 1\right\} \\
&= \|T^*h\|^2,
\end{aligned}$$

for every $h \in \mathrm{dom}\ T^*$. We have therefore proved that $T^{**}T^* \le J^{**}J^*$, and since $J^{**}J^*$ is the smallest positive self-adjoint extension of TT^*, we obtain that $T^{**}T^* = J^{**}J^*$. □

5 The Friedrichs extension

A densely defined positive symmetric operator A on a real or complex Hilbert space $\mathcal{K}$ always has a positive self-adjoint extension. Indeed, the generalized Schwarz inequality

$$|(Ag \mid h)|^2 \le (Ag \mid g)(Ah \mid h), \qquad g, h \in \mathrm{dom}\ A$$

implies that $\mathrm{dom}\ A \subseteq \mathcal{D}_*(A)$ and, therefore, A admits a positive self-adjoint extension according to Theorem 4.1. In particular, by the same theorem, the Kreĭn-von Neumann extension A_N of A

exists. In that case it is known that the so-called Friedrichs extension, that is, the largest positive self-adjoint extension, exists as well. Using the procedure described in Theorem 4.1, we prove the existence of the Friedrichs extension. Our method is very similar to that of Prokaj & Sebestyén (1996a), but somewhat simpler.

Theorem 5.1. *Let A be a densely defined positive symmetric operator in the real or complex Hilbert space $\mathcal{K}$. Then there exists the largest positive self-adjoint extension A_F of A.*

Proof. Let us recall the construction of the proof of Theorem 4.1 and consider the energy Hilbert space $\mathcal{E}$ and the inclusion operator $J_A : \mathcal{E} \supseteq \operatorname{ran} A \to \mathcal{K}$ defined by

$$J_A(Ag) \coloneqq Ag, \qquad g \in \operatorname{dom} A.$$

By (4.5) we have $\operatorname{dom} J_A^* = \mathcal{D}_*(A) \supseteq \operatorname{dom} A$, and therefore we may consider the restriction Q_A of J_A^* to $\operatorname{dom} A$, i.e.,

$$Q_A \coloneqq J_A^*|_{\operatorname{dom} A}.$$

By (4.6),

$$Q_A g = Ag \in \mathcal{E}, \qquad g \in \operatorname{dom} A.$$

On the other hand, from $Q_A \subset J_A^*$ we get $J_A^{**} \subset Q_A^*$ and $Q_A^{**} \subset J_A^*$, whence it follows that $A_F \coloneqq Q_A^* Q_A^{**}$ is a positive self-adjoint extension of A. We claim that A_F is the largest among the set of all positive self-adjoint extensions of A. Indeed, let $B \supset A$ be any positive self-adjoint extension of A. Repeating the above process we apparently have $B = Q_B^* Q_B^{**}$. Then

$$\begin{aligned}\operatorname{dom}(Q_A^* Q_A^{**})^{1/2} &= \operatorname{dom} Q_A^{**} = \operatorname{dom} \overline{Q_A} \\ &= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A,\, k_n \to k,\, (A(k_n - k_m) \,|\, k_n - k_m) \to 0\},\end{aligned}$$

and, accordingly,

$$\begin{aligned}\operatorname{dom}(Q_B^* Q_B^{**})^{1/2} &= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} B,\, k_n \to k,\, (B(k_n - k_m) \,|\, k_n - k_m) \to 0\} \\ &\supseteq \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A,\, k_n \to k,\, (A(k_n - k_m) \,|\, k_n - k_m) \to 0\} \\ &= \operatorname{dom}(Q_A^* Q_A^{**})^{1/2}.\end{aligned}$$

Finally, for $k \in \operatorname{dom}(Q_A^* Q_A^{**})^{1/2} \subseteq \operatorname{dom}(Q_B^* Q_B^{**})^{1/2}$, take $(k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} A$ such that

$$k_n \to k \quad \text{and} \quad (A(k_n - k_m) \,|\, k_n - k_m) \to 0,$$

then $Q_A k_n \to Q_A^{**} k$ in $\mathcal{E}$, and hence

$$\|(A_F)^{1/2} k\|^2 = (Q_A^* Q_A^{**})^{1/2} k\|^2 = \|Q_A^{**} k\|_{\mathcal{E}}^2 = \lim_{n\to\infty} \|Q_A k_n\|_{\mathcal{E}}^2 = \lim_{n\to\infty} (Ak_n \,|\, k_n).$$

Moreover, since $B \supset A$,

$$\|B^{1/2} k\|^2 = \|(Q_B^* Q_B^{**})^{1/2} k\|^2 = \lim_{n\to\infty} (Bk_n \,|\, k_n) = \lim_{n\to\infty} (Ak_n \,|\, k_n).$$

As a consequence we see that $A_F \geq B$, as desired. □

Theorem 5.2. *Let $T : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator satisfying*

$$\operatorname{ran} T \subseteq \operatorname{dom} T^*. \tag{5.1}$$

Then T is closable and the Friedrichs extension of the positive symmetric operator T^*T is equal to T^*T^{**}:

$$(T^*T)_F = T^*T^{**}. \tag{5.2}$$

Proof. Condition (5.1) guarantees, according to Lemma 3.1, that T is closable. Hence, T^{**} exists and T^*T^{**} is a positive self-adjoint extension of T^*T, thanks to von Neumann, see (Schmüdgen, 2012: Proposition 3.18). Our duty is therefore to establish identity (5.2). To this end we need only to prove the domain inclusion

$$\operatorname{dom}(T^*T)_F^{1/2} \supseteq \operatorname{dom}(T^*T^{**})^{1/2}, \tag{5.3}$$

because we know that $\operatorname{dom}(T^*T)_F^{1/2} \subseteq \operatorname{dom}(T^*T^{**})^{1/2}$ and that

$$\|(T^*T)_F^{1/2}k\|^2 = \|(T^*T^{**})^{1/2}k\|^2, \qquad k \in \operatorname{dom}(T^*T)_F^{1/2},$$

see the proof of Theorem 5.1. First we note that

$$\operatorname{dom}(T^*T^{**})^{1/2} = \operatorname{dom} T^{**} = \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} T,\, k_n \to k,\, Tk_n - Tk_m \to 0\}.$$

Recalling the proof of Theorem 5.1, let us denote by $\mathcal{E}$ the "energy space" associated with T^*T, that is, the completion of $\operatorname{ran} T^*T$ endowed with the inner product

$$\langle T^*Tk, T^*Tf\rangle := (Tk \,|\, Tf), \qquad k, f \in \operatorname{dom} T^*T.$$

Consider the operator $Q : \mathcal{K} \to \mathcal{E}$ given by $\operatorname{dom} Q = \operatorname{dom} T^*T = \operatorname{dom} T$,

$$Q(T^*Tk) := T^*Tk \in \mathcal{E}, \qquad k \in \operatorname{dom} T,$$

then we have $(T^*T)_F = Q^*Q^{**}$, again according to the proof of Theorem 5.1. Consequently, the domain $\operatorname{dom}(T^*T)_F^{1/2}$ can be described as follows:

$$\begin{aligned}
\operatorname{dom}(T^*T)_F^{1/2} &= \operatorname{dom}(Q^*Q^{**})^{1/2} = \operatorname{dom} Q^{**} \\
&= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} T,\, k_n \to k,\, \|T^*Tk_n - T^*Tk_m\|_{\mathcal{E}}^2 \to 0\} \\
&= \{k \in \mathcal{K} : \exists (k_n)_{n\in\mathbb{N}} \subset \operatorname{dom} T,\, k_n \to k,\, \|Tk_n - Tk_m\|_{\mathcal{K}}^2 \to 0\} \\
&= \operatorname{dom} T^{**} = \operatorname{dom}(T^*T^{**})^{1/2}.
\end{aligned}$$

This proves identity (5.3) and therefore $(T^*T)_F = T^*T^{**}$, as is claimed. □

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